Hardy's uncertainty principle

18 February, 2009 in expository, math.CA, math.CV | Tags: Chebyshev polynomials, Fourier transform, Phragmen-Lindelof principle, uncertainty principle

[This post was typeset using a LaTeX to WordPress-HTML converter kindly provided to me by Luca Trevisan.]

Many properties of a (sufficiently nice) function $f : \mathbb{R} \to \mathbb{C}$ are reflected in its Fourier transform $\hat f : \mathbb{R} \to \mathbb{C}$, defined by the formula

$$\hat f(\xi) := \int_{-\infty}^\infty f(x) e^{-2\pi i x \xi}\ dx. \qquad (1)$$

For instance, decay properties of f are reflected in smoothness properties of $\hat f$, as the following table shows:

If f is…	then $\hat f$ is…	and this relates to…
Square-integrable	square-integrable	Plancherel's theorem
Absolutely integrable	continuous	Riemann-Lebesgue lemma
Rapidly decreasing	smooth	theory of Schwartz functions
Exponentially decreasing	analytic in a strip	
Compactly supported	entire and at most exponential growth	Paley-Wiener theorem

Another important relationship between a function f and its Fourier transform $\hat f$ is the *uncertainty principle*, which roughly asserts that if a function f is highly localised in space, then its Fourier transform $\hat f$ must be widely dispersed in space, or to put it another way, f and $\hat f$ cannot both decay too strongly at infinity (except of course in the degenerate case $f = 0$). There are many ways to make this intuition precise. One of them is the Heisenberg uncertainty principle, which asserts that if we normalise

$$\int_{\mathbb{R}} |f(x)|^2\ dx = \int_{\mathbb{R}} |\hat f(\xi)|^2\ d\xi = 1$$

then we must have

$$(\int_{\mathbb{R}} |x|^2 |f(x)|^2\ dx) \cdot (\int_{\mathbb{R}} |\xi|^2 |\hat f(\xi)|^2\ dx) \geq \frac{1}{(4\pi)^2}$$

thus forcing at least one of f or $\hat f$ to not be too concentrated near the origin. This principle can be proven (for sufficiently nice f, initially) by observing the integration by parts identity

$$\langle xf, f' \rangle = \int_{\mathbb{R}} x f(x) \overline{f'(x)}\ dx = -\frac{1}{2} \int_{\mathbb{R}} |f(x)|^2\ dx$$

and then using Cauchy-Schwarz and the Plancherel identity.

Another well known manifestation of the uncertainty principle is the fact that it is not possible for f and $\hat f$ to both be compactly supported (unless of course they vanish entirely). This can be in fact be seen from the above table: if f is compactly supported, then $\hat f$ is an entire function; but the zeroes of a non-zero entire function are isolated, yielding a contradiction unless f vanishes. (Indeed, the table also shows that if one of f and $\hat f$ is compactly supported, then the other cannot have exponential decay.)

On the other hand, we have the example of the Gaussian functions $f(x) = e^{-\pi a x^2}$, $\hat f(\xi) = \frac{1}{\sqrt{a}} e^{-\pi \xi^2/a}$, which both decay faster than exponentially. The classical *Hardy uncertainty principle* asserts, roughly speaking, that this is the fastest that f and $\hat f$ can simultaneously decay:

Theorem 1 (Hardy uncertainty principle) *Suppose that f is a (measurable) function such that $|f(x)| \leq Ce^{-\pi a x^2}$ and $|\hat{f}(\xi)| \leq C' e^{-\pi \xi^2/a}$ for all x, ξ and some $C, C', a > 0$. Then $f(x)$ is a scalar multiple of the gaussian $e^{-\pi a x^2}$.*

This theorem is proven by complex-analytic methods, in particular the Phragmén-Lindelöf principle; for sake of completeness we give that proof below. But I was curious to see if there was a real-variable proof of the same theorem, avoiding the use of complex analysis. I was able to find the proof of a slightly weaker theorem:

Theorem 2 (Weak Hardy uncertainty principle) *Suppose that f is a non-zero (measurable) function such that $|f(x)| \leq Ce^{-\pi a x^2}$ and $|\hat{f}(\xi)| \leq C' e^{-\pi b \xi^2}$ for all x, ξ and some $C, C', a, b > 0$. Then $ab \leq C_0$ for some absolute constant C_0.*

Note that the correct value of C_0 should be 1, as is implied by the true Hardy uncertainty principle. Despite the weaker statement, I thought the proof might still might be of interest as it is a little less "magical" than the complex-variable one, and so I am giving it below.

— 1. The complex-variable proof —

We first give the complex-variable proof. By dilating f by $\sqrt{a}$ (and contracting $\hat{f}$ by $1/\sqrt{a}$) we may normalise $a = 1$. By multiplying f by a small constant we may also normalise $C = C' = 1$.

The super-exponential decay of f allows us to extend the Fourier transform $\hat{f}$ to the complex plane, thus

$$\hat{f}(\xi + i\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} e^{2\pi \eta x}\ dx$$

for all $\xi, \eta \in \mathbb{R}$. We may differentiate under the integral sign and verify that $\hat{f}$ is entire. Taking absolute values, we obtain the upper bound

$$|\hat{f}(\xi + i\eta)| \leq \int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi \eta x}\ dx;$$

completing the square, we obtain

$$|\hat{f}(\xi + i\eta)| \leq e^{\pi \eta^2} \qquad (2)$$

for all ξ, η. We conclude that the entire function

$$F(z) := e^{\pi z^2} \hat{f}(z)$$

is bounded in magnitude by 1 on the imaginary axis; also, by hypothesis on $\hat{f}$, we also know that F is bounded in magnitude by 1 on the real axis. *Formally* applying the Phragmen-Lindelöf principle (or maximum modulus principle), we conclude that F is bounded on the entire complex plane, which by Liouville's theorem implies that F is constant, and the claim follows.

Now let's go back and justify the Phragmén-Lindelöf argument. Strictly speaking, Phragmén-Lindelöf does not apply, since it requires exponential growth on the function F, whereas we have quadratic-exponential growth here. But we can tweak F a bit to solve this problem. Firstly, we pick $0 < \theta < \pi/2$ and work on the sector

$$\Gamma_\theta := \{re^{i\alpha} : r > 0, 0 \leq \alpha \leq \theta\}.$$

Using (2) we have

$$|F(\xi+i\eta)| \leq e^{\pi\xi^2}.$$

Thus, if $\delta > 0$, and θ is sufficiently close to $\pi/2$ depending on δ, the function $e^{i\delta z^2}F(z)$ is bounded in magnitude by 1 on the boundary of Γ_θ. Then, for any sufficiently small $\epsilon > 0$, $e^{-i\epsilon e^{i\epsilon} z^{2+\epsilon}} e^{i\delta z^2} F(z)$ (using the standard branch of $z^{2+\epsilon}$ on Γ_θ) is also bounded in magnitude by 1 on this boundary, and goes to zero at infinity in the interior of Γ_θ, so is bounded by 1 in that interior by the maximum modulus principle. Sending $\epsilon \to 0$, and then $\theta \to \pi/2$, and then $\delta \to 0$, we obtain F bounded in magnitude by 1 on the upper right quadrant. Similar arguments work for the other quadrants, and the claim follows.

— 2. The real-variable proof —

Now we turn to the real-variable proof of Theorem 2, which is based on the fact that polynomials of controlled degree do not resemble rapidly decreasing functions.

Rather than use complex analyticity $\hat f$, we will rely instead on a different relationship between the decay of f and the regularity of $\hat f$, as follows:

Lemma 3 (Derivative bound) *Suppose that $|f(x)| \leq Ce^{-\pi a x^2}$ for all $x \in \mathbb{R}$, and some $C, a > 0$. Then $\hat f$ is smooth, and furthermore one has the bound $|\partial_\xi^k \hat f(\xi)| \leq \frac{C}{\sqrt{a}} \frac{k! \pi^{k/2}}{(k/2)! a^{(k+1)/2}}$ for all $\xi \in \mathbb{R}$ and every even integer k.*

Proof: The smoothness of $\hat f$ follows from the rapid decrease of f. To get the bound, we differentiate under the integral sign (one can easily check that this is justified) to obtain

$$\partial_\xi^k \hat f(\xi) = \int_{\mathbb{R}} (-2\pi i x)^k f(x) e^{-2\pi i x \xi}\ dx$$

and thus by the triangle inequality for integrals (and the hypothesis that k is even)

$$|\partial_\xi^k \hat f(\xi)| \leq C \int_{\mathbb{R}} e^{-\pi a x^2} (2\pi x)^k\ dx.$$

On the other hand, by differentiating the Fourier analytic identity

$$\frac{1}{\sqrt{a}} e^{-\pi \xi^2/a} = \int_{\mathbb{R}} e^{-\pi a x^2} e^{-2\pi i x \xi}\ dx$$

k times at $\xi = 0$, we obtain

$$\frac{d^k}{d\xi^k} \left(\frac{1}{\sqrt{a}} e^{-\pi\xi^2/a}\right)|_{\xi=0} = \int_{\mathbb{R}} e^{-\pi a x^2} (2\pi i x)^k\ dx;$$

expanding out $\frac{1}{\sqrt{a}} e^{-\pi\xi^2/a}$ using Taylor series we conclude that

$$\frac{k!}{\sqrt{a}} \frac{(-\pi/a)^{k/2}}{(k/2)!} = \int_{\mathbb{R}} e^{-\pi a x^2} (2\pi i x)^k\ dx$$

□

Using Stirling's formula $k! = k^k (e + o(1))^{-k}$, we conclude in particular that

$$|\partial_\xi^k \hat f(\xi)| \leq \left(\frac{\pi e}{a} + o(1)\right)^{k/2} k^{k/2} \qquad (3)$$

for all large even integers k (where the decay of $o(1)$ can depend on a, C).

We can combine (3) with Taylor's theorem with remainder, to conclude that on any interval $I \subset \mathbb{R}$, we have an approximation

$$\hat{f}(\xi) = P_I(\xi) + O(\frac{1}{k!}(\frac{\pi e}{a} + o(1))^{k/2} k^{k/2} |I|^k)$$

where $|I|$ is the length of I and P_I is a polynomial of degree less than k. Using Stirling's formula again, we obtain

$$\hat{f}(\xi) = P_I(\xi) + O((\frac{\pi}{ea} + o(1))^{k/2} k^{-k/2} |I|^k) \qquad (4)$$

Now we apply a useful bound.

Lemma 4 (Doubling bound) *Let P be a polynomial of degree at most k for some $k \geq 1$, let $I = [x_0 - r, x_0 + r]$ be an interval, and suppose that $|P(x)| \leq A$ for all $x \in I$ and some $A > 0$. Then for any $N \geq 1$ we have the bound $|P(x)| \leq (CN)^k A$ for all $x \in NI := [x_0 - Nr, x_0 + Nr]$ and for some absolute constant C.*

Proof: By translating we may take $x_0 = 0$; by dilating we may take $r = 1$. By dividing P by A, we may normalise $A = 1$. Thus we have $|P(x)| \leq 1$ for all $-1 \leq x \leq 1$, and the aim is now to show that $|P(x)| \leq (CN)^k$ for all $-N \leq x \leq N$.

Consider the trigonometric polynomial $P(\cos\theta)$. By de Moivre's formula, this function is a linear combination of $\cos(j\theta)$ for $0 \leq j \leq k$. By Fourier analysis, we can thus write $P(\cos\theta) = \sum_{j=0}^k c_j \cos(j\theta)$, where

$$c_j = \frac{1}{\pi} \int_{-\pi}^{\pi} P(\cos\theta) \cos(j\theta)\ d\theta.$$

Since $P(\cos\theta)$ is bounded in magnitude by 1, we conclude that c_j is bounded in magnitude by 2. Next, we use de Moivre's formula again to expand $\cos(j\theta)$ as a linear combination of $\cos(\theta)$ and $\sin^2(\theta)$, with coefficients of size $O(1)^k$; expanding $\sin^2(\theta)$ further as $1 - \cos^2(\theta)$, we see that $\cos(j\theta)$ is a polynomial in $\cos(\theta)$ with coefficients $O(1)^k$. Putting all this together, we conclude that the coefficients of P are all of size $O(1)^k$, and the claim follows. □

Remark 1 *One can get slightly sharper results by using the theory of Chebyshev polynomials. (Is the best bound for C known? I do not know the recent literature on this subject. I think though that even the sharpest bound for C would not fully recover the sharp Hardy uncertainty principle, at least with the argument given here.)*

We return to the proof of Theorem 2. We pick a large integer k and a parameter $r > 0$ to be chosen later. From (4) we have

$$\hat{f}(\xi) = P_r(\xi) + O(\frac{r^2}{ak})^{k/2}$$

for $\xi \in [-r, 2r]$, and some polynomial P_r of degree k. In particular, we have

$$P_r(\xi) = O(e^{-br^2}) + O(\frac{r^2}{ak})^{k/2}$$

for $\xi \in [r, 2r]$. Applying Lemma 4, we conclude that

$$P_r(\xi) = O(1)^k e^{-br^2} + O(\frac{r^2}{ak})^{k/2}$$

for $\xi \in [-r, r]$. Applying (4) again we conclude that

$$\hat f(\xi) = O(1)^k e^{-br^2} + O(\frac{r^2}{ak})^{k/2}$$

for $\xi \in [-r, r]$. If we pick $r := \sqrt{\frac{k}{cb}}$ for a sufficiently small absolute constant c, we conclude that

$$|\hat f(\xi)| \leq 2^{-k} + O(\frac{1}{ab})^{k/2}$$

(say) for $\xi \in [-r, r]$. If $ab \geq C_0$ for large enough C_0, the right-hand side goes to zero as $k \to \infty$ (which also implies $r \to \infty$), and we conclude that $\hat f$ (and hence f) vanishes identically.

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18 February, 2009 at 11:07 am
sjt

This new LaTeX-to-HTML converter doesn't make alt text for its images, which is very unfriendly.

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18 February, 2009 at 11:50 am
luca

It's WordPress that has stopped providing alt text for the latex formulas. (The alt text is missing from the other posts, and in all other WordPress.com blogs too.)

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19 February, 2009 at 6:13 am
Stones Cry Out - If they keep silent… » Things Heard: e55v4

[...] Uncertainty. [...]

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19 February, 2009 at 8:47 am
sjt

luca: “It’s WordPress that has stopped providing alt text for the latex formulas.” [not the new LaTeX-to-HTML converter]

My mistake. (Why on earth would they do that?!)

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19 February, 2009 at 8:50 am
dweebydoofus

Hi Dr. Tao

do you think that Luca Trevisan would be interested in making this LaTeX to WordPress-HTML converter available to more users?

Thanks

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19 February, 2009 at 9:39 am
dweebydoofus

By the way: Lemma 4 has a formula that doesn’t parse.

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21 February, 2009 at 4:28 am
Pedro Lauridsen Ribeiro

Dear Prof. Tao,

There’s a related result, which seems to go back to Schrödinger (1926): the only minimisers of Heisenberg’s uncertainty principle are precisely the Gaussians (up to scaling, translation and modulation; I’m not sure if Schrödinger proved the “only” part). In quantum mechanics, these are called coherent states, although this name came much later (I think it was given by Klauder in the sixties).

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21 February, 2009 at 7:02 am
Tom Jones

Dear Terry,

I was just wondering: What is your favorite movie? Because in the Tomb Raider post you mentioned that you played this game when you were younger. But do you like the movies today?

Thanks

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Reply

21 February, 2009 at 1:15 pm
Fabrice Planchon

you may want to have a look at Bruno Demange’s PhD thesis, which has some interesting generalizations of Hardy’s theorem (building up on a theorem by Beurling). Still complex analysis, tho. A PDE oriented perspective may be found in a recent paper by Kenig-Ponce-Vega, and this is real analysis ;-)

http://www.univ-orleans.fr/mapmo/publications/theses/DemangeTHE.pdf

http://arxiv.org/abs/0802.1608

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21 February, 2009 at 1:59 pm
Converting LaTeX to WordPress « in theory [...] Tao has tested it on a couple of posts. Thanks to his feedback, the current version, while surely bug-filled and very limited, is stable [...]

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23 February, 2009 at 12:36 am
Philippe Jaming

Hi Terry,

thanks for this post. If I remember well, there was a similar proof of Hardy's Uncertainty Principle (UP) in Thangavelu's book on the subject. A real variable proof would also help to get some progress on versions of Hardy's UP for (say) the Heisenberg group, that would characterize the heat kernel (the natural analogue of gaussians). This was, I believe,
one of Thangavelu's motivation.

I would also like to mention that there is a proof of Hardy's UP du to B. Demange that works in the distributional setting (if f is a tempared distribution such that $e^{\pi x^2} f$ and $e^{\pi \xi^2} \hat{f}$ are also tempered distributions, then $f = P(x)e^{-\pi x^2}$ where P is a polynomial). This relies on transforming the problem into a complex analysis one via the Bargman transform and Phragmen Lindelöf. Demange's result generalizes several extensions of Hardy's UP most notably Beurling-Hörmander's version of the UP (and the extension thereoff by Bonami-Demange and myself in 2003). He's proof
should appear soon in Memoires de la Société Math. Française
(as far as I know).

Finally, there is a series of recent papers by L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega (sorry if I forget somebody) on the UP from the PDE
point of view (Shrödinger equation) that mainly relies on real analysis too.

Best regards
Philippe

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26 March, 2009 at 7:17 pm
Well, there's a comparison I didn't expect to see… « Feed The Bears [...] Tao, who amongst other things is a harmonic analyst, wrote a post a few weeks ago thinking about a similar question. Of course he thinks about it in a much deeper way, with much more advanced technology, and actually [...]

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6 April, 2010 at 6:52 am
Kaveh Khodjasteh

I keep coming back to this post because I think it must have to do with a problem that has occupied my mine for a long time. In particular, what triggered my attention was Lemma 3. Unfortunately I cannot use it because it holds only (?) for even order derivatives. Is that the case?

The problem I am talking here appears here:

http://mathoverflow.net/questions/16771/lower-bounds-on-truncated-fourier-transform-of-functions-of-constant-modulus-an

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7 April, 2010 at 10:09 am
Yashar

Good to see you at Terry's blog Kaveh!

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25 June, 2010 at 6:44 pm
The uncertainty principle « What's new 

[...] formalisations of this principle, most famously the Heisenberg uncertainty principle and the Hardy uncertainty principle – but in many situations, it is the heuristic formulation of the principle that is more [...]

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