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The Multiplier for the Ball and Radial Functions

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It is shown that the multiplier for the ball is restricted weak type on radial functions in $L^p(\mathbb{R}^n)$ when $p = 2n/(n+1)$. Interpolation then yields a theorem of Herz.

INTRODUCTION

We wish to examine here the weak behavior of a certain multiplier operator. Let $B = \{\xi \in \mathbb{R}^n, |\xi| \leqslant 1\}$. Let $\hat{}$ denote the Fourier transform. We wish to study the operator $Tf(\xi) = \chi_B(\xi)\hat{f}(\xi)$. A theorem of Herz [3] shows that for $L^p(\mathbb{R}^n)$ radial functions we have,

$$\|Tf\|_p \leqslant c_p \|f\|_p, \qquad \frac{2n}{n+1} < p < \frac{2n}{n-1}.$$

In contrast a celebrated theorem of Fefferman [2] shows that for general functions in $L^p(\mathbb{R}^n)$, the operator T is bounded if and only if $p = 2$. Recently Kenig and Tomas [5] have shown that the operator T is not weak type on $L^p(\mathbb{R}^n)$ radial functions when $p = 2n/(n+1)$. We prove here the following theorem.

THEOREM. *Let $\chi_E(x)$ be radial, then, for $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : |T\chi_E(x)| > \lambda\}| \leqslant (c/\lambda^{2n/(n+1)})\,|E|, \qquad n \geqslant 2.$$

The constant c does not depend on E or λ.

This means that the operator T is restricted weak type at the index $2n/(n+1)$. Using a well-known interpolation result due to Stein and Weiss [6], we may use our theorem above and the trivial estimate $\|Tf\|_2 \leqslant c\|f\|_2$ and by interpolation on the space $(\int_0^\infty |f(r)|^p r^{n-1}\,dr)^{1/p}$ obtain the result of Herz. The method of proof parallels our earlier result on Legendre polynomials [1].

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There is another motivation for our theorem and it comes from restriction phenomena for the Fourier transform. For functions which are compactly supported and for $|x|$ large, one roughly has,

$$Tf(x) \sim c\,\frac{\hat{f}(x/|x|)}{|x|^{(n+1)/2}}.$$

Thus if $f(x)$ is radial, $\hat{f}(x/|x|)$ is a constant and the question of weak type at $2n/(n+1)$ is quickly seen to be connected with the estimate,

$$\|\hat{f}\|_{L^\infty(S^{n-1})} \leqslant C\,\|f\|_{2n/(n+1)},$$

S^{n-1} being the surface of the sphere in n dimensions. It is easy to see that the inequality above fails for arbitrary radial functions thus explaining the result of [5], but does hold when $f(x) = \chi_E(x)$ and $\chi_E(x)$ radial.

Before we begin with the proofs we note that given a set $E \subset \mathbb{R}^n$, such that $\chi_E(x)$ is radial we may consider it to be a set $\tilde{E}$ in $(0, \infty)$ equipped with the measure $r^{n-1}dr$, i.e.,

$$|E| = \int_0^\infty \chi_{\tilde{E}}(r)\, r^{n-1}\, dr.$$

With a slight abuse of notation we shall henceforth denote $\tilde{E}$ by E itself. We also need the following basic estimate for the Bessel functions,

$$|J_m(t)| \leqslant ct^{-1/2}.$$

Here c depends only on the order m. This may be found in [6] or [7].

To prove the theorem we begin with the following lemma.

Lemma. *For any set $E \subset (0, \infty)$,*

$$\int_0^\infty \chi_E(r)\, r^{(n-1)/2}\, dr \leqslant c \left(\int_0^\infty \chi_E(r)\, r^{n-1}\, dr\right)^{(n+1)/2n}.$$

Proof. We rewrite the integral on the left as,

$$\int_0^\infty \chi_E(r)\, r^{-(n-1)/2}\, r^{n-1}\, dr.$$

Now we note that $r^{-(n-1)/2} \in L(2n/(n-1), \infty)$ with respect to the measure $r^{n-1}\, dr$. We may thus apply Theorem (4.5) of [4] to obtain the conclusion of the lemma. Q.E.D.

Proof of Theorem. Let $s = (\sum_{i=1}^n x_i^2)^{1/2}$. Then from [5] or [7, p. 134] we know that for any radial function $f(r)$,

$$Tf(s) = \frac{2\pi}{s^{(n-2)/2}} \int_0^\infty \frac{(sJ_{(n-2)/2}(2\pi r)\, J_{n/2}(2\pi s) - rJ_{(n-2)/2}(2\pi s)\, J_{n/2}(2\pi r))}{s^2 - r^2} \times f(r)\, r^{n/2}\, dr. \tag{1}$$

Let $I_k = \{r: 2^k < r \leqslant 2^{k+1}\}$. We decompose $\chi_E(r)$ as follows. Let $f_{1,k}(r) = \chi_E(r)\,\chi(r < 2^{k-1})$, $f_{2,k}(r) = \chi_E(r)\,\chi(2^{k-1} \leqslant r < 2^{k+2})$, and $f_{3,k}(r) = \chi_E(r)\,\chi(r > 2^{k+2})$. Note now that,

$$\chi_E(r) = f_{1,k}(r) + f_{2,k}(r) + f_{3,k}(r).$$

Now,

$$\{s: |T\chi_E(s)| > \lambda\} = \bigcup_k \{s: |T\chi_E(s)| > \lambda\} \cap I_k. \tag{2}$$

We further decompose (2) by noting that the right side above is contained in,

$$\bigcup_k \left(\left\{ s: |Tf_{1,k}(s)| > \frac{\lambda}{3} \right\} \cup \left\{ s: |Tf_{2,k}(s)| > \frac{\lambda}{3} \right\} \cup \left\{ s: |Tf_{3,k}(s)| > \frac{\lambda}{3} \right\} \right) \cap I_k. \tag{3}$$

We now claim that,

$$\left(\left\{ s: |Tf_{1,k}(s)| > \frac{\lambda}{3} \right\} \cup \left\{ s: |Tf_{3,k}(s)| > \frac{\lambda}{3} \right\} \right) \cap I_k \subset \left\{ s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda \right\}. \tag{4}$$

To see this, we use (1). Thus for $s \in I_k$, we have $s > 2r$, if r is in the support of $f_{1,k}(r)$. Thus,

$$|Tf_{1,k}(s)| \leqslant \frac{c}{s^{(n-2)/2}} \int_{s>2r} \frac{s^{1/2} r^{-1/2} + r^{1/2} s^{-1/2}}{s^2} \chi_E(r)\, r^{n/2}\, dr$$

$$\leqslant \frac{c}{s^{(n+1)/2}} \int_{s>2r} (r^{-1/2} + r^{1/2} s^{-1})\, \chi_E(r)\, r^{n/2}\, dr.$$

But because $s > 2r$, the last expression is bounded by,

$$\frac{c}{s^{(n+1)/2}} \int_0^\infty \chi_E(r)\, r^{(n-1)/2}\, dr.$$

By the lemma, the term above is bounded by,

$$\frac{c}{s^{(n+1)/2}}\left(\int_0^\infty \chi_E(r)\, r^{n-1}\, dr\right)^{(n+1)/2n} = \frac{c\,|E|^{(n+1)/2n}}{s^{(n+1)/2}}.$$

Thus, we have shown that,

$$\{s: |Tf_{1,k}(s)| > \lambda/3\} \cap I_k \subset \left\{s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda\right\}. \tag{5}$$

Now, for $s \in I_k$, we have $2s \leqslant r$, for r in the support of $f_{3,k}(r)$. Thus,

$$\begin{aligned}
|Tf_{3,k}(s)| &\leqslant \frac{c}{s^{(n-2)/2}} \int_{2s\leqslant r} \left(\frac{s^{1/2}r^{-1/2} + r^{1/2}s^{-1/2}}{r^2}\right) \chi_E(r)\, r^{n/2}\, dr. \\
&\leqslant \frac{c}{s^{(n-2)/2}} \int_{2s\leqslant r} \left(\frac{s^{1/2}r^{-1/2}}{s^2} + \frac{r^{1/2}s^{-1/2}}{rs}\right) \chi_E(r)\, r^{n/2}\, dr. \\
&\leqslant \frac{c}{s^{(n+1)/2}} \int_0^\infty \chi_E(r)\, r^{(n-1)/2}\, dr.
\end{aligned}$$

Thus again by the lemma, the expression above is bounded by, $cs^{-(n+1)/2}\,|E|^{(n+1)/2n}$. We have thus proved that,

$$\{s: |Tf_{3,k}(s)| > \lambda/3\} \cap I_k \subset \left\{s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda\right\}. \tag{6}$$

Thus (5) and (6) together prove (4) as claimed.

Now consider $Tf_{2,k}(s)$. We rewrite it as follows:

$$\begin{aligned}
Tf_{2,k}(s) &= \frac{2\pi}{s^{(n-2)/2}} \int_0^\infty \frac{(sJ_{(n-2)/2}(2\pi r)\, J_{n/2}(2\pi s) - rJ_{(n-2)/2}(2\pi s)\, J_{n/2}(2\pi r))}{s-r} \\
&\quad \times \left(\frac{r}{s+r} - \frac{1}{2}\right) f_{2,k}(r)\, r^{(n/2)-1}\, dr \\
&\quad + \frac{\pi}{s^{(n-2)/2}} \int_0^\infty \frac{sJ_{(n-2)/2}(2\pi r)\, J_{n/2}(2\pi s)\, f_{2,k}(r)\, r^{(n/2)-1}}{s-r}\, dr \\
&\quad - \frac{\pi}{s^{(n-2)/2}} \int_0^\infty \frac{rJ_{(n-2)/2}(2\pi s)\, J_{n/2}(2\pi r) f_{2,k}(r)\, r^{(n/2)-1}}{s-r}\, dr \\
&\equiv A_k(s) + B_k(s) + C_k(s).
\end{aligned}$$

Now because $r/(s+r)-\frac{1}{2}=(r-s)/2(r+s)$,

$$|A_k(s)| \leqslant \frac{c}{s^{(n-2)/2}} \int_{2^{k-1}}^{2^{k+2}} \frac{(s^{1/2}r^{-1/2}+r^{1/2}s^{-1/2})}{r+s} \chi_E(r)\, r^{(n/2)-1}\, dr.$$

Thus for $s \in I_k$,

$$|A_k(s)| \leqslant \frac{c}{s^{(n+1)/2}} \int_0^\infty \chi_E(r)\, r^{(n-1)/2}\, dr \leqslant c\, \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}}. \tag{7}$$

We now denote the Hilbert transform by H. Then,

$$B_k(s) = \frac{\pi s J_{n/2}(2\pi s)}{s^{(n-2)/2}} H(f_{2,k}(r)\, r^{(n/2)-1} J_{(n-2)/2}(2\pi r))(s),$$

$$C_k(s) = \frac{-\pi J_{(n-2)/2}(2\pi s)}{s^{(n-2)/2}} H(f_{2,k}(r)\, r^{n/2} J_{n/2}(2\pi r))(s).$$

Thus from (3), (4),

$$\begin{aligned} |\{s: |T\chi_E(s)| > \lambda\}| \leqslant c \Bigg(\left| \left\{ s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda \right\} \right| &+ \sum_{k=-\infty}^{\infty} |\{s \in I_k : |A_k(s)| > c\lambda\}| \\ &+ \sum_{k=-\infty}^{\infty} |\{s \in I_k : |B_k(s)| > c\lambda\}| \\ &+ \sum_{k=-\infty}^{\infty} |\{s \in I_k : |C_k(s)| > c\lambda\}| \Bigg). \end{aligned}$$

In view of (7) and because the I_k's are disjoint, the right side above is bounded by,

$$\begin{aligned} &\left| \left\{ s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda \right\} \right| + \sum_{k=-\infty}^{\infty} |\{s \in I_k : |B_k(s)| > c\lambda\}| \\ &\quad + \sum_{k=-\infty}^{\infty} |\{s \in I_k : |C_k(s)| > c\lambda\}|. \end{aligned} \tag{8}$$

But,

$$\left| \left\{ s: \frac{|E|^{(n+1)/2n}}{s^{(n+1)/2}} > c\lambda \right\} \right| \leqslant c \int_0^{|E|^{1/n} \lambda^{-2/(n+1)}} s^{n-1}\, ds = \frac{c\,|E|}{\lambda^{2n/(n+1)}}.$$

Now,

$$\sum_{k=-\infty}^{\infty} |\{s \in I_k : |B_k(s)| > c\lambda\}|$$

$$\leqslant \sum_{k=-\infty}^{\infty} \left| \left\{ s \in I_k : \frac{|H(f_{2,k}(r) r^{(n/2)-1} J_{(n-2)/2}(2\pi r))(s)|}{s^{(n-3)/2}} > c\lambda \right\} \right|$$

$$\leqslant \sum_{k=-\infty}^{\infty} |\{s \in I_k : |H(f_{2,k}(r)\, r^{(n/2)-1} J_{(n-2)/2}(2\pi r))(s)| > c 2^{k(n-3)/2} \lambda\}|$$

$$\leqslant \frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{-kn(n-3)/(n+1)} \int_{I_k}$$

$$\times |H(f_{2,k}(r)\, r^{(n/2)-1} J_{(n-2)/2}(2\pi r))(s)|^{2n/(n+1)} s^{n-1}\, ds$$

$$\leqslant \frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{k(n-1-n(n-3)/(n+1))} \int_{I_k}$$

$$\times |H(f_{2,k}(r) r^{(n/2)-1} J_{(n-2)/2}(2\pi r))(s)|^{2n/(n+1)}\, ds.$$

Using the M. Riesz inequality for the Hilbert transform the last term is bounded by,

$$\frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{k(n-1-n(n-3)/(n+1))} \int |f_{2,k}(r)|^{2n/(n+1)}\, r^{n(n-3)/(n+1)}\, dr.$$

But the expression above is bounded by,

$$\frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} \int |f_{2,k}(r)|^{2n/(n+1)}\, r^{n-1}\, dr \leqslant \frac{c\,|E|}{\lambda^{2n/(n+1)}}.$$

The last inequality follows because the supports of $f_{2,k}(r)$ have bounded overlaps and lie in the sets $\{r: 2^{k-1} < r < 2^{k+2}\}$.

We now estimate $\sum_{k=-\infty}^{\infty} |\{s \in I_k : |C_k(s)| > c\lambda\}|$ in a similar fashion. Now,

$$\sum_{k=-\infty}^{\infty} |\{s \in I_k : |C_k(s)| > c\lambda\}|$$

$$\leqslant \sum_{k=-\infty}^{\infty} |\{s \in I_k : |H(f_{2,k}(r)\, r^{n/2} J_{n/2}(2\pi r))(s)| > c 2^{k(n-1)/2} \lambda\}|$$

$$\leqslant \frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{-kn(n-1)/(n+1)} \int_{I_k}$$

$$\times |H(f_{2,k}(r)\, r^{n/2} J_{n/2}(2\pi r))(s)|^{2n/(n+1)} s^{n-1}\, ds$$

$$\leqslant \frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{k(n-1-n(n-1)/(n+1))} \int_{\mathbb{R}}$$
$$\times |H(f_{2,k}(r)\, r^{n/2}\, J_{n/2}(2\pi r))(s)|^{2n/(n+1)}\, ds.$$

Thus by the M. Riesz inequality the expression above is bounded by,

$$\frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} 2^{k(n-1-n(n-1)/(n+1))} \int_{\mathbb{R}} |f_{2,k}(r)|^{2n/(n+1)} r^{n(n-1)/(n+1)}\, dr$$
$$\leqslant \frac{c}{\lambda^{2n/(n+1)}} \sum_{k=-\infty}^{\infty} \int |f_{2,k}(r)|^{2n/(n+1)}\, r^{n-1}\, dr.$$

By bounded overlaps again, the last term is bounded by, $c|E|\lambda^{-2n/(n+1)}$. Thus all three terms in (8) may be bounded by $c\,|E|\,\lambda^{-2n/(n+1)}$. This proves the theorem. Q.E.D.

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