求 min $J=\int_a^b F(x, f(x), f'(x))dx$

Add 一个扰动，$g_\varepsilon = f(x)+\varepsilon\cdot\eta(x)$，where $\eta(a)=\eta(b)=0$

$$J_\varepsilon=\int_a^b F(x, g_\varepsilon(x), g_\varepsilon'(x))dx$$

$$\frac{dJ_\varepsilon}{d\varepsilon}=\frac{d}{d\varepsilon}\int_a^b F(x, g_\varepsilon(x), g_\varepsilon'(x))dx=\int_a^b \frac{d}{d\varepsilon}F(x, g_\varepsilon(x), g_\varepsilon'(x))dx$$

We know that,

$\frac{d}{d\varepsilon}=\frac{\partial}{\partial\varepsilon}+\frac{dg_i}{d\varepsilon}\cdot\frac{\partial}{\partial g_i}$, where g_i is reference to g_ε and g_ε',

$$\frac{dF}{d\varepsilon}=\frac{dx}{d\varepsilon}\cdot\frac{\partial F}{\partial\varepsilon}+\frac{dg_\varepsilon}{d\varepsilon}\cdot\frac{\partial F}{\partial g_\varepsilon}+\frac{dg_\varepsilon'}{d\varepsilon}\cdot\frac{\partial F}{\partial g_\varepsilon'}$$

Because 不存在单独的 ε，即：ε 都包含在函数 g_ε 或者 g_ε' 中，所以 $\frac{dx}{d\varepsilon}=0$

$$\frac{dJ_\varepsilon}{d\varepsilon}=\int_a^b[\eta(x)\cdot\frac{\partial F}{\partial g_\varepsilon}+\eta'(x)\cdot\frac{\partial F}{\partial g_\varepsilon'}]dx, \text{ and } g_\varepsilon(\varepsilon=0)=f,$$

$$\text{So } \frac{dJ_\varepsilon}{d\varepsilon}(0)=\int_a^b[\eta(x)\cdot\frac{\partial F}{\partial f}+\eta'(x)\cdot\frac{\partial F}{\partial f'}]dx=0$$

由于 snake model，即

$$\int[E_{\text{int}}(C(s))+E_{ext}(C(s))]d\ s$$
$$=\int[\frac{1}{2}(\alpha(s)\cdot\|C'(s)\|^2+\beta(s)\cdot\|C''(s)\|^2)+I(C(s))]d,$$

that is to say,

$$J=\int_a^b F(x, f(x), f'(x), f''(x))d$$

若 $J=\int_a^b F(x, f(x), f'(x), f''(x))d$ ，则同理，有

$$\frac{dJ_\varepsilon}{d\varepsilon}(0)=\int_a^b[\eta(x)\cdot\frac{\partial F}{\partial f}+\eta'(x)\cdot\frac{\partial F}{\partial f'}+\eta''(x)\cdot\frac{\partial F}{\partial f''}]dx=0$$

且对 $\eta(x)$ 还有要求，

$$\eta'(a)=\eta'(b)=0$$

下面采用分步积分法

$$\partial(ab)=a\cdot\partial b+b\cdot\partial a$$

于是，

$$\eta'(x)\cdot\frac{\partial F}{\partial f'}\xrightarrow{a=\frac{\partial F}{\partial f'},b=\eta(x)}\frac{d}{dx}[\eta\cdot\frac{\partial F}{\partial f'}]-\eta\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f'}$$

$$\eta''(x)\cdot\frac{\partial F}{\partial f''}\xrightarrow{a=\frac{\partial F}{\partial f''},b=\eta'(x)}\frac{d}{dx}[\eta'\cdot\frac{\partial F}{\partial f''}]-\eta'\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f''}$$

$$\xrightarrow{a=\frac{d}{dx}\cdot\frac{\partial F}{\partial f''},b=\eta(x)}\frac{d}{dx}[\eta'\cdot\frac{\partial F}{\partial f''}]-[\frac{d}{dx}(\eta\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f''})-\eta\cdot\frac{d^2}{dx^2}\cdot\frac{\partial F}{\partial f''}]$$

$$=\frac{d}{dx}[\eta'\cdot\frac{\partial F}{\partial f''}-\eta\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f''}]+\eta\cdot\frac{d^2}{dx^2}\cdot\frac{\partial F}{\partial f''}$$

于是，

$$\frac{dJ_\varepsilon}{d\varepsilon}(0)=\int_a^b[\eta(x)\cdot\frac{\partial F}{\partial f}-\eta(x)\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f'}+\eta(x)\cdot\frac{d^2}{dx^2}\cdot\frac{\partial F}{\partial f''}]dx+$$

$$[\eta\cdot\frac{\partial F}{\partial f'}]\Big|_a^b+[\eta'\cdot\frac{\partial F}{\partial f''}-\eta\cdot\frac{d}{dx}\cdot\frac{\partial F}{\partial f''}]\Big|_a^b=0$$

$$0\equiv\int_a^b[\frac{\partial F}{\partial f}-\frac{d}{dx}\cdot\frac{\partial F}{\partial f'}+\frac{d^2}{dx^2}\cdot\frac{\partial F}{\partial f''}]\cdot\eta(x)dx$$

而 $\eta(x)\neq 0$，so

$$\frac{\partial F}{\partial f}-\frac{d}{dx}\cdot\frac{\partial F}{\partial f'}+\frac{d^2}{dx^2}\cdot\frac{\partial F}{\partial f''}=0$$

所以，对于目标函数

$$\min \quad J=\int[\frac{1}{2}(\alpha(s)\cdot\|C'(s)\|^2+\beta(s)\cdot\|C''(s)\|^2)+I(C(s))]d$$

此时，

$$F=[\frac{1}{2}(\alpha(s)\cdot\|C'(s)\|^2+\beta(s)\cdot\|C''(s)\|^2)+I(C(s))]$$

$$x=s,\ f=C(s),\ f'=C'(s),\ f''=C''(s)$$

于是，

$$\frac{\partial F}{\partial f}=\frac{\partial F}{\partial C}=\nabla\cdot I(C(s))=\nabla E_{ext}(C(s))$$

$$\frac{\partial F}{\partial f'}=\frac{\partial F}{\partial C'}=\alpha(s)\cdot C'(s)$$

$$\frac{\partial F}{\partial f''}=\frac{\partial F}{\partial C''}=\beta(s)\cdot C''(s)$$

$$\Rightarrow\nabla E_{ext}(C(s))-\left(\alpha(s)\cdot C'(s)\right)'+\left(\beta(s)\cdot C''(s)\right)'=0$$ 得证。