Cohomology of Flag Manifolds

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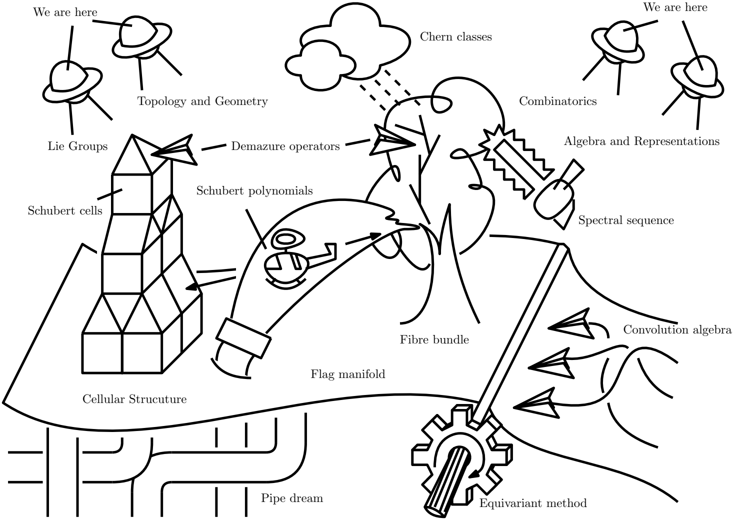
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$\sim$ § <u>Introduction</u> § $\sim$

Flag manifolds

- In this talk, the base field is taken to be $\mathbb{C}$, and $H^\bullet(X) = H^\bullet(X; \Bbbk)$ where the coefficient ring $\Bbbk$ is a field of characteristic zero, for example $\mathbb{Q}$.
- Let V be a finite dimensional vector space of dimension n. A **flag** $\mathcal{F}_\bullet$ is a sequence of subspaces of V,

$$0 = \mathcal{F}^0 \subsetneqq \mathcal{F}^1 \subsetneqq \cdots \subsetneqq \mathcal{F}^n = V,$$

  with $\dim \mathcal{F}^i = i$.
- Denote the set of all such flags to be $\mathcal{F}\ell(V)$, and call it the **flag manifold/variety** (see below).
- Our purpose: compute $H^\bullet(\mathcal{F}\ell(V))$.

Flag manifolds (continued)

- Fix some isomorphism $V = \mathbb{C}^n$, and consider the map

$$\mathrm{span} : \mathrm{GL}(V) \longrightarrow \mathcal{F}\ell(V) \qquad x = \big(v_1, \ldots, v_n\big) \longmapsto \mathcal{F}_x,$$

where

$$\mathcal{F}_x : \quad 0 \quad \subsetneqq \quad \mathbb{C}v_1 \quad \subsetneqq \mathbb{C}v_1 + \mathbb{C}v_2 \subsetneqq \quad \cdots \quad \subsetneqq \quad V \quad .$$

- This map is clearly surjective, and

$$\mathcal{F}_x = \mathcal{F}_y \iff x = y \cdot (\text{an invertible upper triangle matrix}).$$

Flag manifolds (continued)

- Denote $G = \mathrm{GL}_n$, and B the group of invertible upper triangle matrices (the **Borel subgroup**). We have a bijection

$$\mathrm{span} : G/B \xrightarrow{1:1} \mathcal{F}\ell(V).$$

So we can define the topology and smooth/variety structure to be as G/B.

- It is easy to check that

$$\mathrm{U}(n)/\begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \cong G/B$$

thus $\mathcal{F}\ell(V)$ is compact.

$\sim$ § <u>Cellular Structure</u> § $\sim$

Bruhat decomposition

Theorem (Bruhat decomposition)

We have the decomposition

$$G = \bigsqcup_{w \in W} BwB \qquad W = \{\textit{permutation matrices}\} = \mathfrak{S}_n.$$

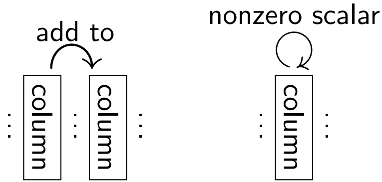
Furthermore, $BwB/B \cong \mathbb{C}^{\ell(w)}$ *with* $\ell(w)$ *the number of inversions, more precisely*

$$\ell(w) = \#\{(i,j) : i < j, w(i) > j\}.$$

The proof

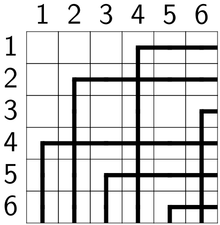
- The action of B on the left can be decomposed into

- The action of B on the right can be decomposed into



The proof (continued)

- For a permutation $w \in \mathfrak{S}_n$, consider the **Rothe diagram** by its "graph" and the space $U_w \subseteq G$, as follows



$$\begin{pmatrix} 1&2&3&4&5&6 \\ 4&2&6&1&3&5 \end{pmatrix}$$

$$U_w = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & 1 & 0 & 0 \\ \mathbb{C} & 1 & 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 & \mathbb{C} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The proof (continued)

Theorem

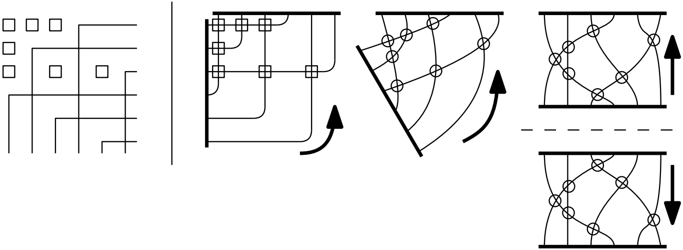
There is a bijection (thus homoemophism)

$$\mathrm{span}: U_w \xrightarrow{1:1} Bw^{-1}B/B.$$

- For any $x \in U_w$, $xB \in Bw^{-1}B$.
  — Dig the hole for each column.
- For any $x \in G$, there is some $b \in B$ such that $xb \in U_w$ for some $w \in \mathfrak{S}_n$.
  — Dig the hole from the last row.
- For any $x, y \in U_w$, if $y \in xB$, then $x = y$.
  — Clearly.

The proof (continued)

- The proof of the dimension follows from the following diagram.



Topology Remind

Theorem

For a CW-complex X, the cohomology group of the complex

$$\cdots \longrightarrow H^\bullet(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \longrightarrow \cdots$$

is isomorphic to $H^\bullet(X)$, and

$$H^\bullet(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \cong \bigoplus_{\dim=\bullet \text{ cell } \Delta} \Bbbk \cdot \Delta.$$

Similar result for homology.

Computation of the cohomology

- In our case, $\{BwB/B : w \in \mathfrak{S}_n\}$ defines a cellular structure of G/B, called the **Schubert cells**.
- But $\dim_{\mathbb{R}} U_w$ are all even dimensional, so the above complex is trivial.

Theorem

The cohomology ring $H^\bullet(\mathcal{F}\ell(V))$ and homology group $H_\bullet(\mathcal{F}\ell(V))$ has only even dimensions. Furthermore,

$$\dim H^{2i}(G/B) = \#\{w : \ell(w) = i\} = \dim H_{2i}(G/B).$$

Remaining Problems

- There remains to answer

$$\text{what is the product structure of } H^*(G/B)?$$

- It is still mysterious

$$\text{what is the Poincaré duality between } H^*(G/B) \text{ and } H_*(G/B)?$$

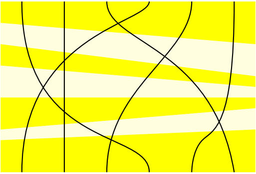
- To answer this, we need to analyse the topological properties of (closed) Schubert cells $Bw^{-1}B/B$ or $\overline{Bw^{-1}B/B}$. It involves some combinatorics of the symmetric groups.

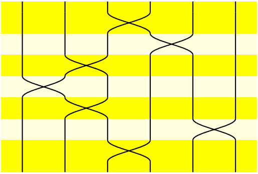
Combinatorics of Symmetric groups

- For any $w \in \mathfrak{S}_n$, $\ell(w)$ is the least length to write w into a product of

$$s_1 = (12),\ s_2 = (23),\ \cdots,\ s_{n-1} = (n-1, n).$$

  Any shortest expression is called a **reduced word**.





- Define the **Bruhat order** $\leq$ to be the order of "sub-reduced word".

Combinatorics of Symmetric groups (continued)

Theorem (Another description of Bruhat order)

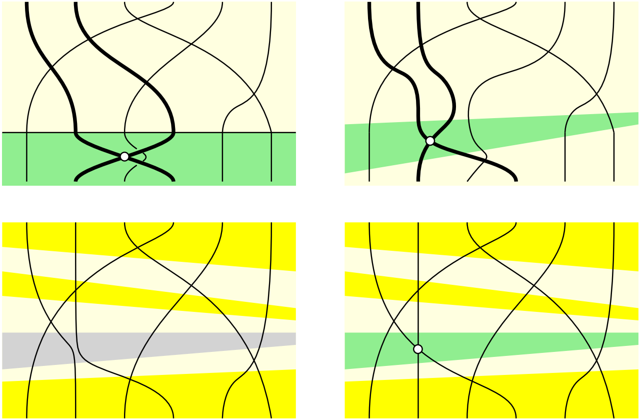
If we write

$$u \lessdot v \iff \ell(u) + 1 = \ell(v), v = su$$

where s is any swap then the Bruhat order

$$u \leq v \iff u = u_0 \lessdot {}^{\exists}u_1 \lessdot \cdots \lessdot {}^{\exists}u_{k-1} \lessdot u_k = v.$$

The proof



Called the **strong exchange property**

Combinatorics of Symmetric groups (continued)

Theorem (Geometric meaning of the Bruhat order)

The Bruhat order can be realized geometrically,

$$BvB \subseteq \overline{BuB} \iff v \leq u.$$

As a result (since $\overline{BuB}$ is also a union of double cosets),

$$BvB/B \subseteq \overline{BuB/B} \iff v \leq u \iff Bv^{-1}B/B \subseteq \overline{Bu^{-1}B/B}$$

The proof

- Firstly, to show $\Leftarrow$, it suffices to show when $v \lessdot u$. When $v = \mathrm{id}$ and u be any of s_i, that is, to check $\overline{Bs_iB} = Bs_iB \sqcup B$. This is easy by checking the SL_2-case.
- Generally, note that

$$BwBs_i \subseteq Bw\overline{Bs_iB} = BwB \sqcup Bws_iB$$

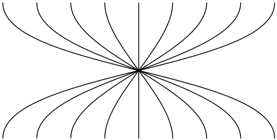
  since only the permutations w and ws_i are in $Bw\overline{Bs_iB}$. So it follows from induction and the description of the Bruhat order.
- To show the $\Rightarrow$, it suffices to show $\bigcup_{v\leq u} BvB$ is already closed. Note that $\bigcup_{v\leq u} BvB$ is product of closed subgroups $P_i = Bs_iB \cup B$. By an induction, from the map between compact space $G/B \to G/P$, we see it is compact.

Combinatorics of Symmetric groups (continued)

- The longest word $w_0 = \binom{1 \cdots n}{n \cdots 1} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is also the only maximal element of $\leq$.
- Actually, due to the **LU decomposition**,

$$Bw_0B = \{x \in \mathrm{GL}_n : \text{sequential principal minor of } x \neq 0\}$$

is a Zariski dense open subset.



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Intersection of Schubert Cells

- Until now, it is hard to find the product of the cells, but at least we can get a pairing. Let $u, v \in \mathfrak{S}_n$ with complement length, i.e. $\ell(u) + \ell(v) = \ell(w_0)$. The trick is, using another cellular structure $\{w_0 B w^{-1} B/B : w \in \mathfrak{S}_n\}$.
- Note that,

$$w_0 B u^{-1} B/B \cap B v^{-1} B/B = \begin{cases} \varnothing, & u w_0 \neq v, \\ \{v^{-1} B\}, & u w_0 = v. \end{cases}$$

by considering the preimage in $w_0 U_u$ and U_v column by column.

Intersection of Schubert Cells (continued)

Theorem

For $u, v \in \mathfrak{S}_n$ with complement length,

$$\overline{w_0 B u^{-1} B}/B \cap \overline{B v^{-1} B}/B = \begin{cases} \varnothing, & u w_0 \neq v, \\ \{v^{-1} B\}, & u w_0 = v. \end{cases}$$

In the last case, they intersect transversally. In particular, we have the pairing of intersection product

$$[\overline{B u^{-1} B/B}] \bullet [\overline{B v^{-1} B/B}] = \begin{cases} 0, & u^{-1} v \neq w_0, \\ [\mathrm{pt}], & u^{-1} v = w_0. \end{cases}$$

The Proof

- A trick is, $\overline{w_0 B u^{-1} B/B} \cap \overline{B v^{-1} B}/B$ is a union of $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \cong (\mathbb{C}^\times)^n$-orbit. Due to compactness of G/B, taking the limit $t \to 0$ will reduce the dimension, so there must be a fixed point.
- But the fixed points of T over G/B are exactly $\{wB : w \in \mathfrak{S}_n\}$. So if they intersect, then some $w \leq w_0 u^{-1}$, and $w \leq v^{-1}$. So $\ell(v) + \ell(u) \geq \ell(w) + \ell(w_0 w) = \ell(w_0)$ with equality only when $v^{-1} = w_0 u^{-1}$, i.e. $u w_0 = v$.
- Since the tangent space is exactly $w_0 U_u$ and U_v, by the same trick, column by column, we see they intersect transversally at $v^{-1}B$.

Intersection of Schubert Cells (continued)

Theorem

Under the Poincaré duality,

$$H^{2\ell(w_0 u)}(G/B) \ni [\overline{Bw_0 uB/B}] \overset{dual}{\longleftrightarrow} [\overline{BuB/B}] \in H_{2\ell(u)}(G/B).$$

$$H^{2\ell(u)}(G/B) \ni [\overline{BuB/B}] \overset{dual}{\longleftrightarrow} [\overline{Bw_0 uB/B}] \in H_{2\ell(w_0 u)}(G/B).$$

- Since $[\overline{Bw_0 uB/B}] \bullet [\overline{BvB/B}] = \delta_{uv}$.

The Schubert cells in terms of flags

Theorem (Geometric Bruhat decomposition)

For two flags $\mathcal{F}_1$ and $\mathcal{F}_2$, one can find an $x \in \mathrm{GL}(V)$ and a permutation $w \in \mathfrak{S}_n$ such that

$$\mathrm{span}(v_1, \ldots, v_n) = \mathcal{F}_1, \qquad \mathrm{span}(v_{w(1)}, \ldots, v_{w(n)}) = \mathcal{F}_2.$$

The permutation is unique.

- This is equivalent to say, for any pair of two cosets xB and yB, we can adjust x, y such that $y = xw^{-1}$ for some w. That is, for any x, $\bigsqcup_w xBwB = G$.

The Schubert cells in terms of flags (continued)

- If we denote the permutation asserted by the theorem by $w(\mathcal{F}_1, \mathcal{F}_2)$. Then

$$\dim \frac{\mathcal{F}_1^{i-1}+\mathcal{F}_2^{j} \quad \cap \mathcal{F}_1^{i}}{\mathcal{F}_1^{i-1}+\mathcal{F}_2^{j-1}\cap \mathcal{F}_1^{i}} = \begin{aligned} &\#\{v_\bullet : \bullet \le i-1\} \cup \{v_\bullet : w(\bullet) \le j \quad\} \cap \{v_\bullet : \bullet \le i\} \\ -&\#\{v_\bullet : \bullet \le i-1\} \cup \{v_\bullet : w(\bullet) \le j-1\} \cap \{v_\bullet : \bullet \le i\} \end{aligned}$$

$$= \begin{cases} 1, w(i)=j, \\ 0, \text{otherwise.} \end{cases}$$

- By the Zassenhaus Butterfly Lemma

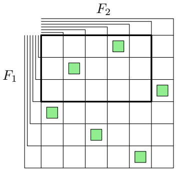
$$\frac{\mathcal{F}_1^{i-1}+\mathcal{F}_2^{j} \quad \cap \mathcal{F}_1^{i}}{\mathcal{F}_1^{i-1}+\mathcal{F}_2^{j-1}\cap \mathcal{F}_1^{i}} \cong \frac{\mathcal{F}_2^{j-1}+\mathcal{F}_1^{i} \quad \cap \mathcal{F}_2^{j}}{\mathcal{F}_2^{j-1}+\mathcal{F}_1^{i-1}\cap \mathcal{F}_2^{j}} \cong \frac{\mathcal{F}_1^{i}\cap \mathcal{F}_2^{j}}{(\mathcal{F}_1^{i-1}\cap \mathcal{F}_2^{j})+(\mathcal{F}_1^{i}\cap \mathcal{F}_2^{j-1})} \cong \frac{(\mathcal{F}_1^{i}+\mathcal{F}_2^{j-1}}{\mathcal{F}_1^{i-1}}$$

- As a result,

$$\dim(\mathcal{F}_1^i \cap \mathcal{F}_2^j) = \#\{\bullet \le i : w(\bullet) \le j\}$$

$$\Updownarrow$$

$$\dim(\mathcal{F}_1^i + \mathcal{F}_2^j) = i + j - \#\{\bullet \le i : w(\bullet) \le j\}$$



The Schubert cells in terms of flags (continued)

- Let $\mathcal{F}_0$ be the standard flag $\mathrm{span}(e_1, \ldots, e_n) \leftrightarrow 1 \cdot B/B$. We have

$$\begin{aligned} Bw^{-1}B/B &\overset{1:1}{\longleftrightarrow} \{\mathcal{F} : w(\mathcal{F}_0, \mathcal{F}) = w\} \\ &= \left\{\mathcal{F} : \dim \frac{\mathcal{F}_0^{i-1} + \mathcal{F}^j \quad \cap \mathcal{F}_0^i}{\mathcal{F}_0^{i-1} + \mathcal{F}^{j-1} \cap \mathcal{F}_0^i} = \begin{cases} 1, w(i)=j, \\ 0, \text{otherwise.} \end{cases}\right\} \\ &= \left\{\mathcal{F} : \dim(\mathcal{F}_0^i \cap \mathcal{F}^j) = \#\{\bullet \le i : w(\bullet) \le j\}\right\}. \\ &= \left\{\mathcal{F} : \dim(\mathcal{F}_0^i + \mathcal{F}^j) = i + j - \#\{\bullet \le i : w(\bullet) \le j\}\right\}. \end{aligned}$$

- Then (they are not equal in general)

$$\begin{aligned} \overline{Bw^{-1}B/B} &\subseteq \left\{\mathcal{F} : \dim(\mathcal{F}_0^i + \mathcal{F}^j) \le i + j - \#\{\bullet \le i : w(\bullet) \le j\}\right\} \\ &= \left\{\mathcal{F} : \dim(\mathcal{F}_0^i \cap \mathcal{F}^j) \ge \#\{\bullet \le i : w(\bullet) \le j\}\right\}. \end{aligned}$$

  The right hand side is closed since $\dim \mathrm{span}(v_1, \ldots, v_n) \le k$ is closed by the description of rank in terms of determinant.

References for this section

- Fulton. Young tableaux. (Last chapter about Schubert cells)
- Hiller. The Geometry of Coxeter groups. (For general Coxeter groups and Schubert cells)
- MacDonald. Notes on Schubert Polynomials. (For the combinatorics of symmetric groups)

∼ § Fibre Bundle Structure § ∼

Lie theory

- We still fix the following notations, say a reductive group with it Borel subgroup, maximal torus and Weyl group,

$$G = \mathrm{GL}_n, \qquad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}, \qquad T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}, \qquad W = \mathfrak{S}_n.$$

- Here list some results,

$$N_G(B) = B, \qquad N_G(T) = W \cdot T, \qquad N_B(T) = N_G(T) \cap B = T.$$

$$T \cong (\mathbb{C}^\times)^n \simeq T^{n-1}, \qquad B/T \cong \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \simeq \mathrm{pt}, \qquad B \simeq T.$$

Topology Remind

Theorem (A variant of Maschke Theorem)

Let G be a finite group and $\mathbb{k}$ be a characteristic zero field. For any G-covering $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$,

$$H^*(B;\mathbb{k}) = H^*(E;\mathbb{k})^G.$$

Theorem

For a space X where G acts freely, then

$$H^*(X/G;\mathbb{k}) \cong H^*(X/T;\mathbb{k})^W$$

for any characteristic zero field $\mathbb{k}$.

- Note that $H^*(X/T;\mathbb{k})^W = H^*(X/N(T))$. Considering the fibre $X/N_G(T) \to X/G$, it suffices to show $G/N_G(T)$ is $\mathbb{k}$-acyclic.
- Then it reduces to show the case when $X = G$. That is, $H^*(G/N_G(T);\mathbb{k}) = H^*(G/T;\mathbb{k})^W = H^*(\mathrm{pt};\mathbb{k})$. Since now, there are only even dimensional stuff, so

$$\begin{aligned} |W| &= \dim H^*(G/T) = \chi(G/T) \\ &= |W| \cdot \chi(G/N_G(T)) = |W| \dim H^*(G/N_G(T)). \end{aligned}$$

Topology Remind

Theorem (Minor)

For any topological group G, there exists a principle G-bundle, called the **universal bundle** $\begin{bmatrix} E_G \\ \downarrow \\ B_G \end{bmatrix}$ *with E_G contractible. Any such principle G-bundle is unique up to homotopy equivalence. It has the following universal property,*

$$\{\textit{principle } G\textit{-bundles over } B\} = [B, B_G]$$

Equivalently, for any principle G-bundle of CW complexes $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, *there exists a map* $\begin{bmatrix} E \to E_G \\ \downarrow \quad\quad \downarrow \\ B \to B_G \end{bmatrix}$ *which is unique up to homotopy.*

Calculations of the classifying spaces

- The classifying space for $\mathbb{C}^\times$ is known to be $\mathbb{C}P^\infty$.
- The classifying space for GL_n is known to be the infinite Grassmanian $\mathcal{G}\mathrm{r}(n, \infty)$.
- As a result,

$$B_T = B_B = (\mathbb{C}P^\infty)^n, \qquad B_G = \mathcal{G}\mathrm{r}(n, \infty).$$

$$H^*(B_T; \mathbb{k}) = \mathbb{k}[X_1, \dots, X_n], \qquad H^*(B_G; \mathbb{k}) = \mathbb{k}[X_1, \dots, X_n]^{\mathfrak{S}_n}.$$

Topology Remind

Theorem (Serre)

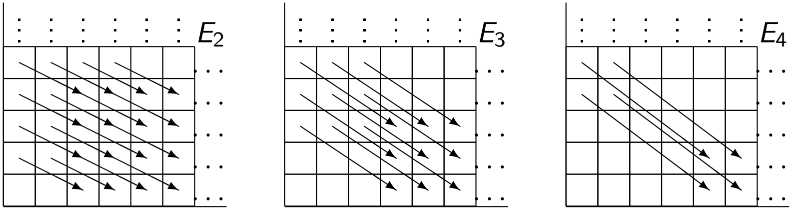
To be simpler, assume B is simply connected, and $\Bbbk$ a field. For any fibre bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ *with fibre F, there is*

- *a set of modules* $\{E_k^{pq} : p, q, k \geq 0\}$,
- *a sequence of differentials* $E_k^{pq} \xrightarrow{d_k^{pq}} E_k^{p+k,q-k+1}$, *and*
- *a series of isomorphisms* $\dfrac{\ker[E_k^{pq} \xrightarrow{d} E_k^{\cdots}]}{\mathrm{im}[E_k^{\cdots} \xrightarrow{d} E_k^{pq}]} \cong E_{k+1}^{pq}$

such that $E_2^{pq} = H^p(B; H^q(F; \Bbbk))$ *such that*

$$H^n(E) = \bigoplus_{p+q=n} E_\infty^{pq}, \qquad E_\infty^{pq} = E_N^{pq} \text{ for } N \gg 0.$$

Spectral Sequences



Computation of the cohomology

Theorem

As an $\mathbb{k}$-algebra,

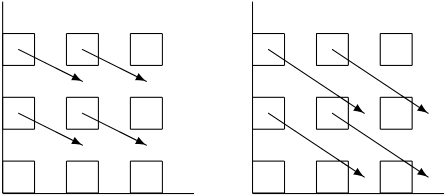
$$H^*(G/B) \cong \frac{\mathbb{k}[X_1, \dots, X_n]}{\langle E_1, \dots, E_n \rangle}.$$

where E_i is the i-th elementary symmetric polynomial.

- Firstly, note that $G/T \to G/B$ has contractible fibre, so $H^*(G/T) = H^*(G/B)$.
- Since B_T can be taken to be E_G/T, there is a fibre bundle $\begin{bmatrix} B_T \\ \downarrow \\ B_G \end{bmatrix}$ with fibre G/T, and $\pi_1(B_G) = \pi_0(G)$ is trivial. So we can apply the Serre spectral sequence.

Computation of the cohomology (continued)

- There are only even dimensional terms, so each d is zero.



- So we have the $H^*(B_G)$-module isomorphism,

$$H^*(B_T) = H^*(B_G) \otimes H^*(G/T).$$

- So $H^*(B_T)/\left\langle H^{\geq 1}(B_G)\right\rangle = H^*(G/T)$.

Remaining Problems

- Now, the multiplication structure is easy to compute, but

  how to express the Schubert Cells?

- Besides,

  what is the meaning of $X_1, \ldots, X_n$?

- A good tool to understand cohomology is the Chern classes. We will find some line bundle over G/T. Since G/B is compact (a projective complex variety), it is better to work beck over G/B.

Topology Remind

- Define the **tautological bundle** over $\mathbb{C}P^1$,

$$\mathcal{O}(-1) = \begin{bmatrix} \{(\ell, x) \in \mathbb{C}P^1 \times \mathbb{C}^2 : x \in \ell\} \\ \downarrow \\ \mathbb{C}P^1 \end{bmatrix}$$

Theorem (1st Chern class)

For any CW-complex B, there is a natural transform between

$$c_1 : \{\textit{Line Bundles over } B\} \to H^2(B)$$

funtorial in B such that $-c_1(\mathcal{O}(-1)) \in H^2(\mathbb{C}P^1)$ dual to $[\mathrm{pt}] \in H_0(\mathbb{C}P^1)$.

Chern Classes

- For any character (i.e. a group homomorphism to $\mathbb{C}^\times$) of B, it defines a representation, denoted by $\mathbb{C}\rho$, of B, say

$$\mathbb{C} \xrightarrow{b} \mathbb{C} \qquad v \mapsto \rho(b)v.$$

This also defines a line bundle $\underline{\mathbb{C}}\rho := \begin{bmatrix} G\times_B \mathbb{C}\rho \\ \downarrow \\ G/B \end{bmatrix}$.

- For example $\rho : \begin{pmatrix} x & * \\ & x^{-1} \end{pmatrix} \mapsto x$, the bundle $\begin{bmatrix} \mathrm{SL}_2 \times_{B_2} \mathbb{C}\rho & & \\ \downarrow & \searrow & \\ \mathrm{SL}_2/B_2 & \cong & \mathbb{C}P^1 \end{bmatrix}$ is isomorphic to $\mathcal{O}(-1)$. Just by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda\right) \mapsto \left(\mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}, \lambda\begin{pmatrix} a \\ c \end{pmatrix}\right)$.

Chern Classes (continued)

Theorem

The Chern class $-c_1(\underline{\mathbb{C}}X_i)$ *is exactly* $X_i \in H^2(G/B)$*, where the character* X_i *is* $\begin{pmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{pmatrix} \mapsto x_i$*. So we have the map*

$$\Bbbk[X_1, \ldots, X_n] \xrightarrow{\psi} H^*(G/B) \qquad \lambda \cdot X_1^{a_1} \cdots X_n^{a_n} \longmapsto \lambda \cdot X_1^{a_1} \cdots X_n^{a_n}$$

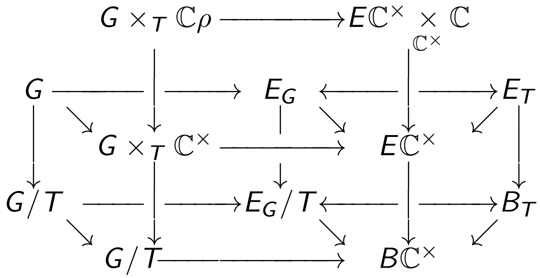
Warning: maybe confusing notation

- Note that $\psi(a_1 X_1 + \cdots + a_n X_n)$ corresponds to the character $\begin{pmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{pmatrix} \longmapsto x_1^{a_1} \cdots x_n^{a_n}$ by the formula for tensor product of line bundles.

The proof

- Still, $G \times_T \mathbb{C}\rho \to G \times_B \mathbb{C}\rho$ has contractible fibre, thus homotopically the same.
- The vector bundle $\begin{bmatrix} E\mathbb{C}^\times \times_{\mathbb{C}^\times} \mathbb{C} \\ \downarrow \\ B\mathbb{C}^\times \end{bmatrix}$ is tautological.
- The map $G/T \to E_G/T$ is the fibre map for $\begin{bmatrix} E_G/T = B_T \\ \downarrow \\ B_G \end{bmatrix}$, where $X_1, \ldots, X_n$ from.
- The map $G/T \to E_G/T$ is also the classifying map for $\begin{bmatrix} G \\ \downarrow \\ G/T \end{bmatrix}$.
- The map $B_T \to B\mathbb{C}^\times$ is induced by ρ.
- The theorem follows from the diagram chasing.

The proof (continued)



The action of Weyl groups

Theorem

The map ψ defined in the previous theorem is $\mathfrak{S}_n$-equivariant. That is, $\psi(wf) = w\psi(f)$; where the action $\mathfrak{S}_n$ acts on polynomial ring by permuting the indices, and on $H^*(G/T)$ induced by conjugation.

- For any character ρ, the action $X_i \xmapsto{w} X_{w(i)}$ is induced by $w\rho(X) = \rho(w^{-1}Xw)$.
- Then chase the following diagram, we know ψ is $\mathfrak{S}_n$-equivariant.

$$\begin{array}{ccc} G \times_T \mathbb{C}\rho & \xrightarrow{(g,v)\mapsto(wgw^{-1},v)} & G \times_T \mathbb{C}w\rho \\ \downarrow & & \downarrow \\ G/T & \xrightarrow[gT\mapsto wgw^{-1}T]{} & G/T \end{array}$$

Fibres in terms of Flags

- Define the i-th tautological bundle is defined to be $\mathbf{F}^i = \begin{bmatrix} E^i \\ \downarrow \\ \mathcal{F}\ell(V) \end{bmatrix}$ where $E^i = \{(\mathcal{F}, v) \in \mathcal{F}\ell(V) \times V : v \in \mathcal{F}^i\}$.
- If we take $V = \mathbb{C}^n$, then we have the short exact sequence

$$0 \to \mathbf{F}^{i-1} \to \mathbf{F}^i \xrightarrow{*} \underline{\mathbb{C}} X_i \to 0.$$

The map $*$ is given by

$$\begin{array}{ccc} E^i & \to & G \times_B \mathbb{C}X_i \\ \downarrow & & \downarrow \\ \mathcal{F}\ell(\mathbb{C}^n) & \to & G/B \end{array} \quad (\mathcal{F}, v) \mapsto (x, \lambda)$$

where $x = (x_1, \ldots, x_n)$, and $v = \lambda x_k + (\text{lower terms})$.

Fibres in terms of Flags (continued)

- As a result, the total Chern class (by the **Whiteney formula**)

$$c(\mathbf{F}^i) = (1 - X_1)(1 - X_2)\cdots(1 - X_i).$$

- In particular, $\mathbf{F}^n = \left[\begin{smallmatrix} \mathcal{F}\ell(V)\times V \\ \downarrow \\ \mathcal{F}\ell(V) \end{smallmatrix}\right]$ is trivial, so

$$c(\mathbf{F}^n) = (1 - X_1)(1 - X_2)\cdots(1 - X_n) = 1$$

this is the geometric reason that E_i lies in the denominates.

Partial Flags

- It is useful to introduce the **partial flags** to be flags of length k, note that
$$\mathcal{F}\ell^k(\mathbb{C}^n) \cong \mathrm{GL}_n \Big/ \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & * & * & \cdots & * \\ & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{pmatrix} := G/P.$$
- To compute it in an inductive way, consider the fibre bundle $\begin{bmatrix} \mathcal{F}\ell^k(\mathbb{C}^n) \\ \downarrow \\ \mathcal{F}\ell^{k-1}(\mathbb{C}^n) \end{bmatrix}$ by truncating the first $k-1$ flags. Its fibre is exactly $\mathbb{P}(\mathbb{C}^n/\mathbb{C}^{k-1}) = \mathbb{C}P^{n-k}$.

Partial Flags (continued)

- Let ρ be a group character of $\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}$, which extends to P by $\rho\begin{pmatrix} x & * \\ & * \end{pmatrix} = \rho(x)$. Then one can also define $\begin{bmatrix} G \times_P \mathbb{C}\rho \\ \downarrow \\ G/P \end{bmatrix}$.
- Inductively, by the Serre spectral sequence or the Harish–Leray argument, one can show that $H^*(\mathcal{F}\ell^k(\mathbb{C}))$ has only even dimensional part, and as $\Bbbk$-graded space

$$H^*(\mathcal{F}\ell^k(\mathbb{C}^n)) \quad = \frac{\Bbbk[X_1]}{\langle X_1^{n-1} \rangle} \otimes \cdots \otimes \frac{\Bbbk[X_k]}{\langle X_k^{n-k} \rangle} = \frac{\Bbbk[X_1, \ldots, X_k]}{\langle X_1^{n-1}, \ldots, X_k^{n-k} \rangle}.$$

where $X_i = -c_1 \begin{bmatrix} G \times_P \mathbb{C}X_i \\ \downarrow \\ G/P \end{bmatrix}$.

Partial Flags (continued)

- One can check that the map induced by $\mathcal{F}\ell^k(\mathbb{C}^n) \to \mathcal{F}\ell(\mathbb{C}^n)$ is compatible and injective, that's to say $X_i \mapsto X_i$.

Theorem

The monomials $X^\lambda = X_1^{\lambda_1} \cdots X_k^{\lambda_k}$ with $\lambda \leq \rho = (n-1, \dots, n-k)$ forms a basis of $H^*(\mathcal{F}\ell^k(\mathbb{C}^n))$.

Theorem

The monomials $X^\lambda = X_1^{\lambda_1} \cdots X_n^{\lambda_n}$ with $\lambda \leq \rho = (n-1, \dots, 1, 0)$ forms a basis of $H^*(G/B)$.

Algebraic Remarks

- It is highly nontrivial to show that there is an isomorphism as graded vector space

$$\mathbb{k}[X_1, \ldots, X_n] = \mathbb{k}[E_1, \ldots, E_n] \otimes \frac{\mathbb{k}[X_1, \ldots, X_n]}{\langle E_1, \ldots, E_n \rangle}$$

  in a pure algebraic way (due to Chevalley for general reflection groups). For example, this implies $(E_1, \ldots, E_n)$ is a regular sequence.
- The Poincaré polynomial for $H^*(G/B)$ is

$$\prod_{k=1}^{n} \frac{1 - t^{2k}}{1 - t^2} = \sum_{i \geq 0} \#\{w : \ell(w) = i\} t^{2i},$$

  also a nontrivial combinatorial identity.

References for this section

- Shintaro Kuroki. An introduction to Torus Equivariant Cohomology. [Youtube] (for the computation by spectral sequneces)
- Hiller. The Geometry of Coxeter groups. (Chapter II for the algebraic result, and the combinatorics)
- Goodman and Wallash. Symmetry, Representations, and Invariants. (Chapter 5 contains the algebraic treatment mentioned)
- Hatcher. Algebraic Topology. (See Page 343 for the computation of cohomology of the partial flags)
- Fulton. Young tableaux. (Appendix for a quick introduction to Chern classes)

∼ § Demazure Operators § ∼

Parabolic Subgroups

- We have the embeddings from SL_2, and the parabolic subgroups

$$\begin{array}{rccc} \kappa_i : & \mathrm{SL}_2 & \longrightarrow & G \\ & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \longmapsto & \begin{pmatrix} \mathbf{1} & & \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \\ & & \mathbf{1} \end{pmatrix} \end{array}, \qquad P_i = \operatorname{im} \kappa_i \cdot B = \begin{pmatrix} \begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{smallmatrix} & \begin{smallmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{smallmatrix} \\ \begin{smallmatrix} & & * \end{smallmatrix} & \begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{smallmatrix} \end{pmatrix}.$$

- Here list some results,

$$\kappa_i : \mathrm{SL}_2 / \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \cong P_i/B$$

$$\mathrm{SL}_2 / \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) = \mathcal{F}\ell(\mathbb{C}^2) = \mathbb{P}(\mathbb{C}^2) \cong \mathbb{C}P^1 = S^2.$$

- The main target of this section is to analyse the fibre bundle $\left[\begin{smallmatrix} G/B \\ \downarrow \\ G/P_i \end{smallmatrix}\right]$ whose fibre is $P/B \cong \mathbb{C}P^1$.

Topology Remind

Theorem (Gysin push forward)

For any fibre bundle $E \xrightarrow{\pi} B$ *whose fibre* F *is a* d*-dimensional with Poincaré duality, there is a well-defined functor called the* **Gysin push forward**

$$H^*(E) \xrightarrow{\pi_*} H^{*-d}(B).$$

In the case when E *and* B *are for which Poincaré duality holds,* π_* *is induced from homology map through duality.*

Topology Remind

Theorem (Gysin sequence)

For any d-dimensional sphere bundle $E \xrightarrow{\pi} B$, there is a long exact sequence

$$\cdots \to H^{i-1+d}(B) \xrightarrow{*} H^i(B) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\pi_*} H^{i-d}(B) \xrightarrow{*} H^{i+1}(B) \to \cdots$$

where π^* is the usual induced cohomology map, π_* is the Gysin push forward, and $*$ is the cup product with the **Euler class** *of π.*

Demazure operator

- The fibre bundle $G/B \xrightarrow{\pi} G/P_i$ has fibre $P_i/B \cong S^2$, thus we have Gysin push forward π_*. We define the **Demazure operator** ∂_i to be

$$\partial_i : H^*(G/B) \xrightarrow{\pi^*} H^{*-2}(G/P) \xrightarrow{\pi^*} H^{*-2}(G/B).$$

From the Gysin sequence, we see that

$$H^{*+2}(G/B) \xrightarrow{\partial_i} H^*(G/B) \xrightarrow{\partial_i} H^{*-2}(G/P)$$

is zero (actually exact since $H^{\text{odd}}(G/P_i) = 0$ see below).

Demazure operator (continued)

Theorem

Denote the cohomology class of $[\overline{BwB/B}]$ (Poincaré dual to homology class $[\overline{Bw_0wB/B}]$) by X_w, then

$$\partial_i X_w = \begin{cases} X_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise}. \end{cases}$$

- Actually, there is a cellular structure over G/P_i, say by

$$G = \bigsqcup_{w:\ell(ws_i)=\ell(w)+1} BwP_i,$$

whose corresponding $\{BwP/P : \ell(ws_i) = \ell(w) + 1\}$ defines a cellular structure over G/P.

The proof

- Note that

$$BwP = BwBs_iB \cup BwB = Bws_iB \sqcup BwB$$

by considering the permutation matrices both sides.

- If $\ell(ws_i) = \ell(w) + 1$ we have a bijection

$$BwB/B \overset{1:1}{\longleftrightarrow} BwP/P,$$

Since $\pi^{-1}(BwP/P) = BwB/B \sqcup Bws_iB/B$, and the fibre of $\begin{bmatrix} \pi^{-1}(BwP/P) \\ \downarrow \\ BwP/P \end{bmatrix}$ at xP is exactly $xP/B = xB/B \sqcup xs_iB/B$, so there is a single point over BwB/B, this proves the bijection.

The proof (continued)

- So $G/B \xrightarrow{\pi} G/P_i$ is cellular, i.e. $\pi(G/B_{\dim\leq\bullet}) \subseteq (G/P_i)_{\dim\leq\bullet}$, thus it is easy to compute the induced homology map,

$$\begin{array}{ccc} [\overline{BwB/B}] & \leftarrow\!\shortmid [\overline{BwP/P}] \\ \uparrow & \uparrow \\ \pi^*[\overline{BwP/P}] & \leftarrow\!\shortmid [\overline{BwP/P}] \end{array} \quad \begin{array}{c} \Leftarrow \ell(ws_i) = \ell(w)+1 \\ \ell(ws_i) = \ell(w)-1 \Rightarrow \end{array} \quad \begin{array}{ccc} 0 & \leftarrow\!\shortmid [\overline{Bws_iP/P}] \\ \uparrow & \uparrow \\ \pi^*[\overline{Bws_iP/P}] & \leftarrow\!\shortmid [\overline{Bws_iP/P}] \end{array}$$

So

$$\begin{aligned} \pi^*: \quad H^*(G/P) &\longrightarrow H^*(G/B) \\ [\overline{BwP/P}] &\longmapsto \begin{cases} [\overline{BwB/B}], & \ell(ws_i) = \ell(w)+1. \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The proof (continued)

- Then consider the cellular map $\begin{array}{ccc} \pi^{-1}(\overline{BwP_i/P_i}_{\dim\leq\bullet}) & \to & \overline{BwP_i/P_i}_{\dim\leq\bullet} \\ \downarrow & & \downarrow \\ G/B & \to & G/P_i \end{array}$ induced the Gysin push forward for cellular cohomology, thus

$$\begin{array}{ccc} [\overline{Bws_iB/B}] & \mapsto & [\overline{BwP/P}] \\ \updownarrow & & \updownarrow \\ [\overline{Bws_iB/B}] & \mapsto & \pi_*[\overline{Bws_iB/B}] \end{array} \quad \Leftarrow \ell(ws_i) = \ell(w) + 1 \Rightarrow \quad \begin{array}{ccc} [\overline{BwB/B}] & \mapsto & 0 \\ \updownarrow & & \updownarrow \\ [\overline{BwB/B}] & \mapsto & \pi_*[\overline{BwB/B}] \end{array}$$

So

$$\begin{aligned} \pi_* : \quad H^*(G/B) &\longrightarrow H^{*-2}(G/P) \\ [\overline{Bws_iB/B}] &\longmapsto \begin{cases} [\overline{BwP/P}], & \ell(ws_i) = \ell(w) + 1. \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The proof (continued)

We can also compute the Gysin push forward by the Poincaré duality.

- Firstly, we can also work in homology, where Gysin "pull back" $H_*(G/P_i) \to H_{*+2}(G/B)$ can be described as "taking preimage" (from the functorial assertion). This is based on the Poincaré duality we computed in the first section.
- Secondly, we can also find the Poincaré duality for G/P_i as G/B, then the push forward can be computed yet. The duality is exactly $[BwP/P] \leftrightarrow [Bw_0ws_iP_i/P_i]$, where $\ell(ws_i) = \ell(w) + 1$. The technique of transversal intersection still works.

Demazure operator (continued)

Theorem

Recall the algebra homomorphism $\psi : \mathbb{k}[X_1, \dots, X_n] \to H^*(G/B)$, *we have*

$$\psi(\partial_i f) = \partial_i(\psi f),$$

where the **Demazure operator** *is defined over* $\mathbb{k}[X]$ *by*

$$\partial_i f(X) = \frac{f(\cdots, X_i, X_{i+1}, \cdots) - f(\cdots, X_{i+1}, X_i, \cdots)}{X_i - X_{i+1}}.$$

Topology Remind

Theorem (Harish–Leray)

For a fibre bundle $E \to B$, if each fibre F_x has free cohomology, and there is a set $\{\alpha\} \subseteq H^\bullet(E)$ present the bases restricting each fibre. Then

$$H^*(B) \otimes H^*(F) \longrightarrow H^*(E) \qquad \beta \times i_*\alpha \longrightarrow \pi_*\beta \smile \alpha$$

is an isomorphism between $H^*(B)$ modules.
Furthermore, the map is funtorial in $(E \to B, \{\alpha\})$ with fixed fibre F.

The proof

- Consider the character

$$\omega_i : B \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{pmatrix} \longmapsto x_1 \cdots x_i .$$

We have the following bundle

$$\begin{array}{ccccc} \mathrm{SL}_2 \times_{B_2} \mathbb{C}(\omega_i \circ \kappa_i) & \longrightarrow & G \times_B \mathbb{C}\omega_i & \longleftarrow & \mathrm{SL}_2 \times_{B_2} \mathbb{C}(\omega_i \circ \kappa_j) \\ \mathcal{O}(-1) \Big\downarrow & & \Big\downarrow \underline{\mathbb{C}}\omega_i \quad (j \neq i) & & \Big\downarrow \text{trivial} \\ \mathrm{SL}_2 / B_2 & \xrightarrow{\kappa_i} & G/B & \xleftarrow{\kappa_j} & \mathrm{SL}_2 / B_2 \end{array}$$

The proof (continued)

- In conclusion

$$\kappa_j^*(-c_1(\underline{\mathbb{C}}\omega_i)) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \in \mathbb{Z} \cong H^2(\mathrm{SL}_2 / B_2).$$

- As a result, we have the $H^*(G/B)$-isomorphism,

$$H^*(G/B) \cong H^*(G/P_i)[\omega_i] / \left\langle \omega_i^2 \right\rangle,$$

where $\omega_i = -c_1(\underline{\mathbb{C}}\omega_i)$ over G/B.

- I claim for $\alpha \in H^{*-2}(G/P_i), \beta \in H^*(G/P_i)$, the Demazure operator,

$$\begin{array}{ccccc} H^*(G/B) & \longrightarrow & H^{*-2}(G/P_i) & \longrightarrow & H^{*-2}(G/B) \\ \alpha\omega_i + \beta & \longmapsto & \alpha & \longrightarrow & \alpha \end{array}$$

The proof (continued)

- One can compute the Gysin push forward by the Poincaré duality, for $\alpha \in H^{*-2}(G/P_i), \beta \in H^*(G/P_i)$,

$$\begin{aligned} & \pi_*((\alpha\omega_i + \beta) \frown [G/B]) \\ = {} & \pi_*(\alpha \frown (\omega_i \cap [G/B])) + \pi_*(\beta \frown [G/B]) && \because {\scriptstyle (\alpha\cup\beta)\cap\Gamma = \alpha\cap(\beta\cap\Gamma)} \\ = {} & \pi_*(\pi_*\alpha \frown (\omega_i \cap [G/B])) + \pi_*(\pi_*\beta \frown [G/B]) && \because {\scriptstyle \begin{cases}\alpha=\pi^*\alpha \\ \beta=\pi^*\beta\end{cases}} \\ = {} & \alpha \frown \pi_*(\omega_i \frown [G/B]) + \beta \cap \pi_*[G/B]. && \because {\scriptstyle f_*(f^*\alpha\cap\Gamma) = \alpha\cap f_*\Gamma} \\ = {} & \alpha \frown \pi_*(\omega_i \frown [G/B]) && \because {\scriptstyle \dim G/P_i<\dim G/B,\ \pi_*[G/B]=0} \end{aligned}$$

By the dimension reason, one can write $\pi_*(\omega_i \frown [G/B]) = \lambda[G/P_i]$ for some $\lambda \in \mathbb{Z}$. Actually $\lambda = 1$, by considering $\begin{matrix} P_i/B & \to & \text{pt} \\ \downarrow & & \downarrow \\ G/B & \to & G/P_i \end{matrix}$.

The proof (continued)

- Besides, note that s_i also acts on G/P_i by $x \mapsto s_i x s_i^{-1} P = s_i x P$, and
$$\begin{array}{ccc} G/T \to G/B \to G/P_i \\ \downarrow\ s_i \qquad s_i\ \downarrow \\ G/T \to G/B \to G/P_i \end{array}$$
commutes. But since G/P_i is path-connected, left multiplication is homotopically trivial, so s_i acts on $H^*(G/P_i)$ is trivial.
- As a result, the Demazure operators

$$\frac{1-s_i}{\psi(X_i - X_{i+1})}(\alpha\omega_i + \beta) = \frac{\omega_i - \omega_i \circ s_i}{\psi(X_i - X_{i+1})}\alpha$$
$$= \frac{\psi(X_1 + \cdots + X_i) - \psi(X_1 + \cdots + X_{i-1} + X_{i+1})}{\psi(X_i - X_{i+1})}\alpha = \alpha$$

does the same work as the Demazure operators on polynomials.

Computation of the cohomology

Theorem

The algebra homomorphism $\psi : \mathbb{k}[X_1, \ldots, X_n] \to H^*(G/B)$, has its kernel

$$\ker \psi = \left\{ f : \begin{array}{l} \text{the constant term of } f \text{ acted by any iterated} \\ \text{Demazure operator } \partial_* \cdots \partial_* \text{ is zero.} \end{array} \right\}$$

for any coefficient group $\mathbb{k}$. In particular, when $\mathbb{k}$ is a field of characteristic zero, it is generated by the elementary symmetric polynomials.

- The trick is, ψ is isomorphic on the degree 0 part, this shows $\subseteq$.
- Due our description of the action on Demazure operators on cells, the constant term $\partial_* \cdots \partial_* \alpha$ is the coefficient of one Schubert cell of α, this shows $\supseteq$.

Computation of the cohomology (continued)

Theorem

In our case, $G = \mathrm{GL}_n$, the map $\psi : \mathbb{Z}[X_1, \ldots, X_n] \to H^*(\mathcal{F}\ell(\mathbb{C}^n); \mathbb{Z})$ is surjective, and $\ker \psi$ generated by the elementary symmetric polynomials. By the universal coefficient theorem, the same for any coefficient group $\Bbbk$.

- Apply the spectral sequence in the last section. Since $B_G = \mathcal{G}\mathrm{r}(n, \infty)$ has a cellular structure (the Schuber cells), so $H^{\mathrm{odd}}(B_G) = 0$, thus everything runs perfectly.

Theorem

Actually,

$$H^*(\mathcal{G}\mathrm{r}(n, \infty)) = \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} = \mathbb{Z}[E_1, \ldots, E_n].$$

$$H^*(\mathcal{F}\ell(\mathbb{C}^n)) = \mathbb{Z}[X_1, \ldots, X_n]_{\mathfrak{S}_n} := \frac{\mathbb{Z}[X_1, \ldots, X_n]}{\langle E_1, \ldots, E_n \rangle}.$$

Demazure operators in terms of Flags

- Since the Gysin push forward is functorial, thus we can also work on the fibre product $\begin{bmatrix} G/B \times_{G/P_i} G/B & \to & G/B \\ \downarrow & & \downarrow \\ G/B & \to & G/P_i \end{bmatrix}$. Then the Demazure operator can be described by

$$\partial_i : H^*(G/B) \longrightarrow H^*(G/B \times_{G/P} G/B) \overset{\text{push forward}}{\longrightarrow} H^{*-2}(G/B).$$

- Consider

$$Z_i = \{(\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{F}\ell(V) \times \mathcal{F}\ell(V) : j \neq i \Rightarrow \mathcal{F}_1^j = \mathcal{F}_1^j\}$$

then it is clear $Z_i \cong G/B \times_{G/P} G/B$ and compatible with two projections to $\mathcal{F}\ell(V) \cong G/B$.

Demazure operators in terms of Flags (continued)

- Note that, the fibre at $\mathcal{F} \in \mathcal{F}\ell(V)$ is exactly the choice of spaces between $\mathcal{F}^{i-1} \subseteq \mathcal{F}^{i+1}$, i.e. $\mathbb{P}(\mathcal{F}^{i+1}/\mathcal{F}^{i-1})$, homeomorphic to $\mathbb{C}P^1$.
- As we expected,

$$p_1^{-1}(Bw^{-1}B/B) = Bw^{-1}B/B \times (Bw^{-1}B/B \sqcup Bw^{-1}sB/B).$$

$$p_2(p_1^{-1}(\overline{Bw^{-1}B/B})) = \begin{cases} \overline{Bw^{-1}sB/B}, & \ell(w^{-1}s_i) = \ell(w^{-1}) - \\ \text{higher dimension cells} & \text{otherwise.} \end{cases}$$

- Using the same argument on cells in this section, we get again

$$\partial_i X_w = \begin{cases} X_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Summary

- We will denote the algebra generated by the operator ∂_i by NH_n, called the **nil-Heck algebra** (or **nil-coxeter algebra**) (see next section).

Demazure operator ∂_i

$$
\begin{array}{rccccl}
 & \Bbbk[X] & \longrightarrow & \Bbbk[X] & & f \longmapsto \frac{f - s_i f}{x_i - x_{i+1}} \\
\text{Chern classes } \psi & \downarrow & & \downarrow & & \\
 & H^*(G/B) & \longrightarrow & H^{*-2}(G/B) & & \alpha\omega_i + \beta \longmapsto \alpha \\
\text{Schubert polynomials } \mathfrak{S}_w & \| & & \| & & \\
 & H^*(G/B) & \longrightarrow & H^{*+2}(G/B) & & X_w \mapsto \begin{cases} X_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases} \\
\text{Schubert cells } X_{w_0} & \uparrow & & \uparrow & & \\
 & \mathrm{NH} & \longrightarrow & \mathrm{NH} & & \Delta \longmapsto \partial_i \circ \Delta
\end{array}
$$

References

- Kac. Torion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups.
- Fulton. Young tableaux. (Last chapter for the Demazure operator)

∼ § Schubert Polynomials § ∼

Schubert Polynomials

Theorem

For any permutation $w \in \mathfrak{S}_n$, denote X_w the cohomology class of BwB/B, there is a unique polynomial $\mathfrak{S}_w$ such that

$$X_w = \mathfrak{S}_w(X_1, \ldots, X_n) \in H^*(G/B; \Bbbk)$$

with each monomial $X^\lambda \leq X_1^{n-1} \cdots X_{n-1}$ in $\mathfrak{S}_w$. Such polynomial is called the **Lascoux and Schützenberger's Schubert polynomial**.

Schubert Polynomials (continued)

Theorem

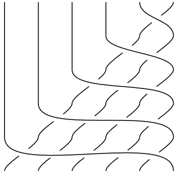
We can compute Schubert polynomials by

$$\mathfrak{S}_{w_0} = X_1^{n-1} \cdots X_{n-1} \qquad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases}$$

The proof

- The second condition is clear, and what we know is $X_e = [\mathrm{pt}] = 1$.
- Firstly, by dimension reason, $\mathfrak{S}_{w_0} = \lambda X_1^{n-1} \cdots X_{n-1}$ for some $\lambda \in \mathbb{Z}$. So, if we pick a reduced word of w_0, for example,

$$\omega_0 = \begin{array}{cccc} s_{n-1} & \cdots & \cdots & s_1 \\ & s_{n-1} & \cdots & s_2 \\ & & \ddots & \vdots \\ & & & s_{n-1}, \end{array}$$



the corresponding

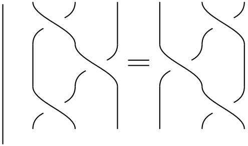
$$\begin{array}{cccc} \partial_{n-1} & \cdots & \cdots & \partial_1 \\ & \partial_{n-1} & \cdots & \partial_2 \\ & & \ddots & \vdots \\ & & & \partial_{n-1}, \end{array} \lambda X_1^{n-1} \cdots X_{n-1} = \lambda$$

So $X_e = \lambda = 1$.

Nil-Hecke algebra

- Note that the operators ∂_i (on polynomials) satisfy the nil-braid relation

$$\begin{gathered}\partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1},\\ |i-j| \geq 2, \quad \partial_i\partial_j = \partial_j\partial_i,\\ \partial_i^2 = 0.\end{gathered}$$



Theorem

the nil-Hecke algebra

$$\mathrm{NH}_n = \Bbbk \left\langle \partial_i \right\rangle_{1\leq i\leq n-1} \Big/ \left\langle \begin{gathered}\partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1},\\ |i-j| \geq 2, \quad \partial_i\partial_j = \partial_j\partial_i,\\ \partial_i^2 = 0.\end{gathered} \right\rangle$$

The proof

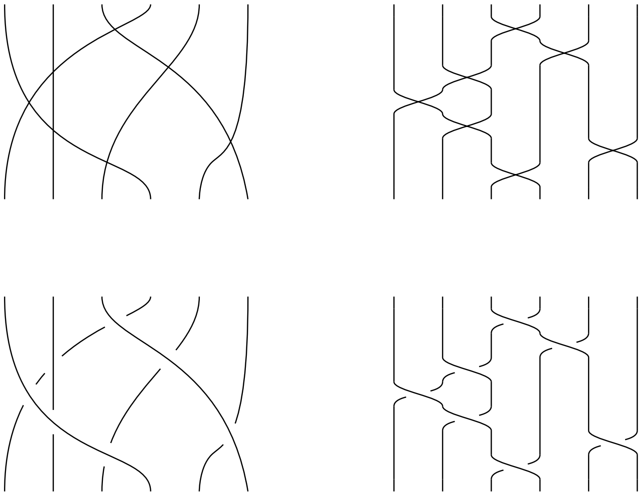
- Let $v \in \mathfrak{S}_n$ be any permutation, and $v = s_{i(1)} \cdots s_{i(k)}$ a reduced word, we define the operator

$$\partial_v = \partial_{i(1)} \cdots \partial_{i(k)}, \qquad \text{(reducing degree by } \ell(v)\text{)}$$

this does not depend on the choice of the reduced word (called **Matsumoto's theorem**).

- Warning: maybe confusing notation $\partial_{s_{ij}}$ is not ∂_{ij} in general for example $\partial_{s_{13}} = \partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.

Matsumoto's theorem



The proof (continued)

- Then, $\partial_v \partial_u = \begin{cases} \partial_{vu}, & \ell(vu) = \ell(v) + \ell(u), \\ 0, & \text{otherwise.} \end{cases}$. It suffices to show when $v = s_i$, and the otherwise case. Express $u = s_* \cdots s_*$, then by strong exchange property (i.e. our description of Bruhat order) $s_i u = s_* \cdots \widehat{s_*} \cdots s_*$. So $u = s_1 s_* \cdots \widehat{s_*} \cdots s_*$ is a reduced word, then $\partial_i \partial_u = \partial_i \partial_i \cdots = 0$.
- As a result, $\{\partial_w : w \in \mathfrak{S}_n\}$ forms a basis of the right hand side algebra.
- So it rests to show $\{\partial_w\}$ is linearly independent as operators, just composing with $\partial_{w_0 w^{-1}}$ and acting on $\mathfrak{S}_{w_0} = X_1^{n-1} \cdots X_{n-1}$.

Nil-Hecke algebra (continued)

Theorem

The Schubert polynomials

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} X_1^{n-1} \cdots X_{n-1},$$

and there is a perfect pairing

$$\mathrm{NH}_n \times H^*(G/B) \xrightarrow{\langle \cdot, \cdot \rangle} \Bbbk \qquad (\partial_w, \alpha) \mapsto \textit{constant term of } \partial_w \alpha$$

with $\langle \partial_u, \mathfrak{S}_v \rangle = \delta_{uv}$.

A generating function for Schubert polynomials

Theorem (Fomin and Stanley 1993)

The coefficient of ∂_w of

$$\mathfrak{S}(x) = \begin{matrix} (1+x_1\partial_{n-1}) & \cdots & (1+x_1\partial_2) & (1+x_1\partial_1) \\ & \ddots & \vdots & \vdots \\ & & (1+x_{n-2}\partial_{n-1}) & (1+x_{n-2}\partial_{n-2}) \\ & & & (1+x_{n-1}\partial_{n-1}) \end{matrix}$$

is the Schubert polynomial $\mathfrak{S}_w(x)$.

- This is definitely true for $w = w_0$. So it rests to show the induction formula.

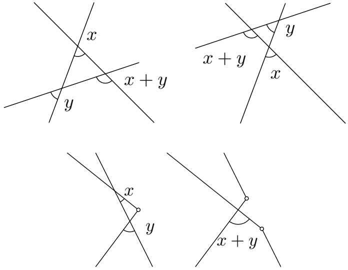
The proof (due to Fomin and Kirrilov)

- Note that

$$(1+x\partial_i)(1+(x+y)\partial_{i+1})(1+y\partial_i) = (1+y\partial_{i+1})(1+(x+y)\partial_i)(1+x\partial_{i+1})$$

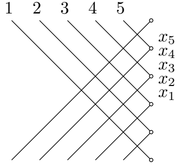
$$|i-j| \geq 2, \quad (1+x\partial_i)(1+y\partial_j) = (1+y\partial_j)(1+x\partial_i)$$

$$(1+x\partial_i)(1+y\partial_i) = (1+(x+y)\partial_i)$$



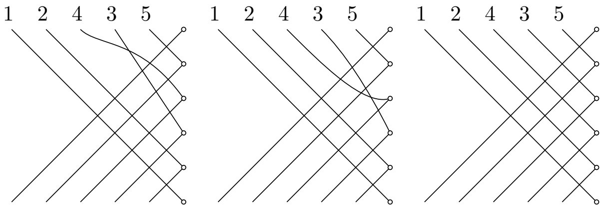
- As a result,

$$\mathfrak{S}(x) = \begin{matrix} (1+x_1\partial_{n-1}) \cdots & (1+x_1\partial_1) \\ \ddots & \vdots \\ & (1+x_{n-1}\partial_{n-1}) \end{matrix}$$



The proof (continued)

- The following diagram



shows $\mathfrak{S}(x)(1+(x_{i+1}-x_i)\partial_i) = s_i\mathfrak{S}(x)$, that is,

$$\mathfrak{S}(x)\partial_i = \frac{\mathfrak{S}(x) - s_i\mathfrak{S}(x)}{x_i - x_{i+1}}$$

As a result, $\mathfrak{S}(x)$ runs the Demazure operator mechanically.

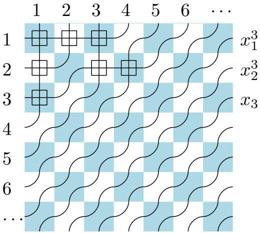
The Pipe Dream

- It is suggested to use the **pipe dream** to expend the brackets above. A **pipe dream** for w is a filling of the board with pipes $+$ and  connected left i to upper $w(i)$ such that no pair of pipes cross twice.
- For a pipe dream π, define its weight

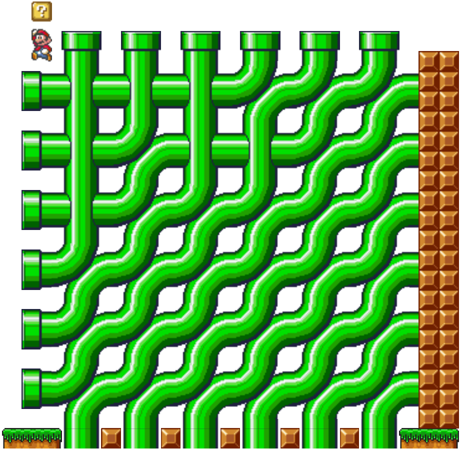
$$\mathtt{wt}(\pi) = \prod_{+\in\pi} x_{\text{the row number of the } +}$$

$$w = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,6\,1\,3\,5 \end{pmatrix}$$

$$\mathtt{wt} = x_1^3 x_2^3 x_3$$



Here we go!



The Pipe Dream (continued)

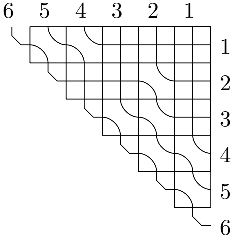
Theorem (Bergeron and Billey, 1993)

For a permutation $w \in \mathfrak{S}_\infty$,

$$\mathfrak{S}_w(x) = \sum_{\text{pipe dream } \pi \text{ for } w} \mathtt{wt}(\pi).$$

In particular, the coefficient of Schubert polynomials is positive.

$$\begin{array}{ccccc} 1 & 1 & x_1\partial_3 & x_1\partial_2 & x_1\partial_1 \\ & x_2\partial_5 & x_2\partial_4 & 1 & x_2\partial_2 \\ & & 1 & 1 & x_3\partial_3 \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$



Computation of Schubert polynomials

Theorem

$$\partial_{w_0} f(X_1, \dots, X_n) = \frac{1}{\prod_{i<j}(X_i - X_j)} \sum_{\sigma} (-1)^{\sigma} f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

- It is easy to see that ∂_{w_0} has the form $\frac{1}{\prod_{i<j}(X_i - X_j)} \sum_\sigma c_\sigma \sigma$ with $c_e = 1$. But $\partial_i \partial_{w_0} = 0$, the $c_\sigma = (-1)\sigma c_1$ is alternative, thus it follows.

Theorem

The Schubert polynomial are stable under the recognization of $\mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}$, so it is well-defined for $\mathfrak{S}_\infty = \bigcup_{n \geq 1} \mathfrak{S}_n$.

Grassmannians

- Let $\mathcal{G}\mathrm{r}(\mathbb{C}^n, k)$ be the set of k-subspace in $\mathbb{C}^k$.
- Note that

$$\mathcal{G}\mathrm{r}(\mathbb{C}^n, k) = \mathrm{GL}_n \Big/ \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \\ & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{pmatrix} =: G/P$$

  All of them can be computed by both cell method and the fibre bundle method.
- As what we did last section

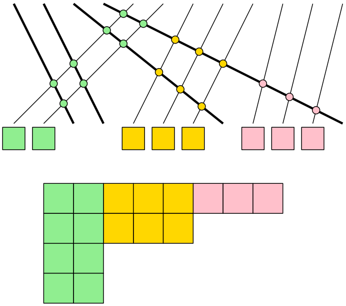
$$H^*(\mathcal{G}\mathrm{r}(\mathbb{C}^n, k)) = \left\{ [X_w] : \begin{matrix} \ell(w) \text{ is minimal among} \\ w\mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{matrix} \right\}$$

  we choose the minimal element w due to $BwB/B \cong BwP/P$, therefore $\ell(w)$ gives the right dimension.

Grassmannians (continued)

- Such permutation is so-called a **shuffle**, and determined by a partition

$$U_w = \begin{bmatrix}
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & 1 & - & - & - & - & - & - & - \\
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mid & 1 & - & - & - & - & - & - \\
1 & - & - & - & + & + & - & - & - & - & - & - \\
\mid & 1 & - & - & + & + & - & - & - & - & - & - \\
\mid & \mid & \mathbb{C} & \mathbb{C} & \mid & \mid & 1 & - & - & - & - & - \\
\mid & \mid & \mathbb{C} & \mathbb{C} & \mid & \mid & \mid & 1 & - & - & - & - \\
\mid & \mid & \mathbb{C} & \mathbb{C} & \mid & \mid & \mid & \mid & 1 & - & - & - \\
\mid & \mid & 1 & - & + & + & + & + & + & - & - & - \\
\mid & \mid & \mid & \mathbb{C} & \mid & \mid & \mid & \mid & \mid & 1 & - & - \\
\mid & \mid & \mid & \mathbb{C} & \mid & \mid & \mid & \mid & \mid & \mid & 1 & - \\
\mid & \mid & \mid & \mathbb{C} & \mid & \mid & \mid & \mid & \mid & \mid & \mid & 1 \\
\mid & \mid & \mid & 1 & + & + & + & + & + & + & + & +
\end{bmatrix}$$



- Then we can define the Schubert cells

$$\Sigma_\lambda = \{V \in \mathcal{G}r(k,n) : \dim(V \cap \mathcal{F}_0^{n+i-\lambda_i}) = i\}.$$

Grassmannians (continued)

- It is clear, for u, v two shuffles,

$$u \leq v \iff \text{the corresponding partition } \lambda_u \subseteq \lambda_v.$$

$$\cdots \leq \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ 1 & - & - & - & \cdots \\ \mid & 1 & - & - & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & 1 & - & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & 1 & \cdots \end{bmatrix} \leq \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ 1 & - & - & - & \cdots \\ \mid & 1 & - & - & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & 1 & - & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & 1 & \cdots \end{bmatrix} \leq \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ 1 & - & - & - & \cdots \\ \mid & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & 1 & - & - & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & \mathbb{C} & \mathbb{C} & \cdots \\ \mid & \mid & 1 & - & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & \mathbb{C} & \cdots \\ \mid & \mid & \mid & 1 & \cdots \end{bmatrix} \leq \cdots$$

Grassmannians (continued)

Theorem

$$\overline{\Sigma_\lambda} = \{V \in \mathcal{G}\mathrm{r}(k,n) : \dim(V \cap \mathcal{F}_0^{n+i-\lambda_i}) \geq i\}.$$

- If $u' \leq v'$, and u, v the corresponding shuffle, then $u \leq v$. Since $u \leq u'$, and $v' = v(v^{-1}v')$ is a reduced word, then u as a sub-reduced word must be a sub-reduced word of v.
- The rest follows from reading the partitions.

Grassmannians (continued)

- Next, the map $G/B \to G/P$ is cellular, and thus induces injective algebra homomorphism

$$H^*(G/P) \to H^*(G/B),$$

with $[BwP/P] \mapsto [BwB/B]$ if $\ell(w)$ is minimal among $w\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. So the Schubert polynomial helps to decide the ring structure.

Theorem

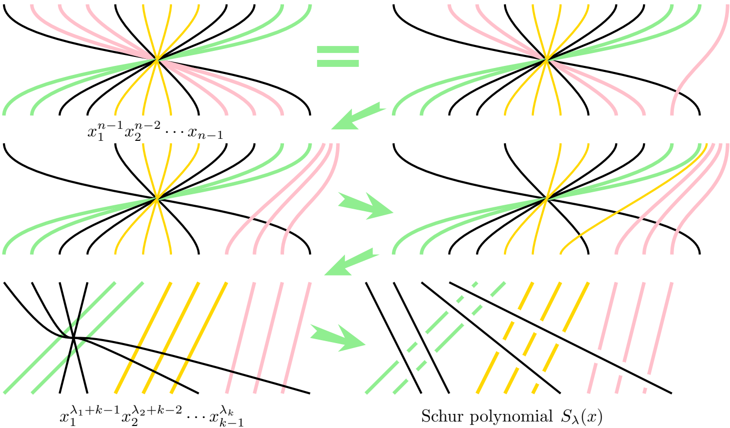
For a shuffle w, corresponding to the partition λ,

$$\mathfrak{S}_w(X_1, \ldots, X_n) = S_\lambda(X_1, \ldots, X_k),$$

where S_λ is the Schur polynomial.

The proof

- $\partial_i(\cdots X_i^d X_{i+1}^{d-1}\cdots) = \cdots X_i^{d-1} X_{i+1}^{d-1}\cdots$.
- $\partial_{w_0\in\mathfrak{S}_k} = \frac{1}{\prod_{1\le i<j\le k}(X_i - X_j)} \sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sigma$.



Grassmannians (continued)

Theorem

There is a surjective algebra homomorphism

$$k[X_1, \ldots, X_k]^{\mathfrak{S}_k} \longrightarrow H^*(\mathcal{G}\mathrm{r}(k, n)),$$

with

$$S_\lambda \mapsto \begin{cases} [\overline{\Sigma_\lambda}] & \text{if } \operatorname{length} \lambda \le k, \operatorname{width} \lambda \le n-k, \\ 0, & \text{otherwise.} \end{cases}$$

References

- Fulton. Young tableaux.
- Knutson. Schubert polynomials, pipe dreams, equivariant classes, and a co-transition formula.
- Fomin and Stanley. Schubert polynomials and Nil-coxeter algebras.
- Bergeron and Billey. RC-Graphs and Schubert Polynomials.
- Fomin and Kirillov. Yang-Baxter equation, symmetric functions, and Schubert polynomials.
- MacDonald. Notes on Schubert Polynomials.

∼ § Equivariant Cohomology § ∼

Actions on Homogenous Manifolds

- Still, we fix the notations, called a reductive group with it Borel subgroup, maximal torus and Weyl group,

$$G = \mathrm{GL}_n, \qquad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}, \qquad T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}, \qquad W = \mathfrak{S}_n.$$

- The group T or B acts on G/B and G/T by left multiplication.

$$G/B \xrightarrow{g} G/B \qquad xB \mapsto gxB.$$

- The Weyl group W acts on G and T by conjugation, thus acts on G/T.

$$G/T \xrightarrow{w} G/T \qquad xB \mapsto wxw^{-1}B.$$

Note that the two actions do not commute.

Actions on Homogenous Manifolds (continued)

- Denote $K = \mathrm{U}_n$, due to the QR decomposition (Iwasawa decomposition), $G = K \cdot B$, and the choice are up to an $K \cap B = K \cap T =: T_K$ element. Thus $G/B \cong K/T_K$.
- Since the Weyl group acts on K/T_K, so it also acts on G/B.
- The fibre $\left[\begin{smallmatrix} G/T \\ \downarrow \\ G/B \end{smallmatrix}\right]$ has contractible fibre $B/T \cong \left(\begin{smallmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} 0 & \cdots & * \\ & \ddots & \vdots \\ & & 0 \end{smallmatrix}\right)$, and

$$G/B \to K/T_K \to G/T$$

is an W-section, so the action of W on $H^*(G/B) = H^*(G/T) = H^*(K/T_K)$ coincides.

Topology Remind

- For a topological group G, and a G-space X, the **equivariant cohomology** is defined to be

$$H_G^*(X) = H^*(E_G \times_G X).$$

- The following two maps

$$E_G \times_G X \to E_G \times_G \mathrm{pt} = B_G, \qquad E_G \times X \to E_G \times_G X$$

makes $H_G^*(X)$ an $H_G^*(\mathrm{pt}) = H^*(B_G)$ algebra equipped with the augment map $H_G^*(X) \to H^*(X)$.

Topology Remind

- For a space X with G acts freely,

$$H_G^*(X) = H^*(X/G).$$

- For a space X with G acts trivially,

$$H_G^*(X) = H^*(B_G \times X).$$

- When G is discrete, and $X = K(\pi, n)$,

$$H_G^*(X) = H^n(G; \pi),$$

the group cohomology.

Calculations of Equivariant Cohomology

- Here is some easy case

$$\begin{aligned} H^*_G(G) &= \Bbbk \\ H^*_G(G/T) &= H^*(E_G \times_G G/T) = H^*(E_G/T) \\ &= H^*(B_T) = \Bbbk[X_1, \dots, X_n] \\ H^*_T(G) &= H^*(T\backslash G) \\ H^*_T(\mathrm{pt}) &= H^*(B_T) = \Bbbk[t_1, \dots, t_n] \end{aligned}$$

Theorem (Borel)

As an $\Bbbk$-algebra,

$$H^*_T(G/T) = \frac{\Bbbk[t_i, X_i : 1 \le i \le n]}{\langle E_i(t) - E_i(X) : 1 \le i \le n \rangle}.$$

where E_i is the i-th elementary symmetric polynomial.

The proof

$$\begin{array}{ccccc} & G/T & = & G/T & \\ & \downarrow & & \downarrow & \\ B_T = & E_G \times_G G/T & \leftarrow & E_G \times_T G/T & \\ & \downarrow & & \downarrow & \\ B_G = & E_G \times_G \mathrm{pt} & \leftarrow & E_G \times_T \mathrm{pt} & \end{array}$$

$$\begin{array}{ccccc} & H^*(G/T) & = & H^*(G/T) \\ & \uparrow & & \uparrow \\ H^*(B_T) = & H^*_G(G/T) & \rightarrow & H^*_T(G/T) \\ & \uparrow & & \uparrow \\ H^*(B_G) = & H^*_G(\mathrm{pt}) & \rightarrow & H^*_T(\mathrm{pt}) \end{array}$$

The proof (continued)

- The first column is the fibre we used to calculate $H^*(G/T)$, and the augment map
$$H_G^*(G/T) \to H^*(G/T)$$
is known to be surjective.
- By the Harish-Leray theorem, the algebra map
$$H_G^*(G/T) \otimes_{H_G^*(\mathrm{pt})} H_T^*(\mathrm{pt}) \to H_T^*(G/T)$$
is surjective.
- Note that we have seen $H_T^*(\mathrm{pt}) = H_G^*(\mathrm{pt}) \otimes H(G/T)$ as $H_G^*(\mathrm{pt})$ module, so by comparison of dimensions, this is an isomorphism.

Connection with Chern classes

Theorem

We have the following

$$X_i = -c_1 \begin{bmatrix} E_T \times_T G \times_T \mathbb{C}X_i \\ \downarrow \\ E_T \times_T G/T \end{bmatrix} \qquad t_i = -c_1 \begin{bmatrix} E_T \times_T (G/T \times \mathbb{C}X_i) \\ \downarrow \\ E_T \times_T G/T \end{bmatrix}$$

where the character X_i *is* $\begin{pmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{pmatrix} \mapsto x_i$.

- The two arguments are both nearly the definition. The first is as we did for normal cohomology, the second is just to note that it is the pull back of $\begin{bmatrix} E_T \times_T \mathbb{C}\rho \\ \downarrow \\ B_T \end{bmatrix}$.

Topology Remind

Theorem

For a G-complex X, the cohomology group of the complex

$$\cdots \longrightarrow H_G(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \longrightarrow \cdots$$

is isomorphic to $H_G^\bullet(X)$, and

$$H_G(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \cong \bigoplus_{\dim=\bullet \text{ cell } \Delta} H_G(\mathrm{pt})\cdot\Delta.$$

- The terminology G-complex means all the cells are G-subspaces.

Cellular Structure

- Note that the Schubert cells BwB/B are all T-invariant, so it defines

$$[\overline{BwB/B}] \in H_T^*(G/B).$$

and $\{[\overline{BwB/B}] : w \in W\}$ forms a $H_T^*(\mathrm{pt})$ basis of $H_T^*(G/B)$.

- Question

How to express $[\overline{BwB/B}]$ in terms of Equivariant Cohomology?

We want to establish the equivariant version of the theory before.

Demazure operator

- Since $\begin{bmatrix} E_T \times_T G/B \\ \downarrow \\ E_T \times_T G/P_i \end{bmatrix}$ has the same fibre as $\begin{bmatrix} G/B \\ \downarrow \\ G/P_i \end{bmatrix}$, so we can define the Gysin push forward as well

$$\partial_i : H_T^*(G/B) \to H_T^{*-2}(G/P_i) \to H_T^{*-2}(G/B).$$

Theorem

Denote the equivariant cohomology class of $[\overline{BwB/B}]$ by X_w, then

$$\partial_i X_w = \begin{cases} X_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Demazure operator (continued)

Theorem

For the algebra homomorphism

$$\psi : \Bbbk[t_1, \dots, t_n, X_1, \dots, X_n] \to H^*(G/B)$$

we have

$$\psi(\partial_i f) = \partial_i(\psi f),$$

where the **Demazure operator** *is defined over* $\Bbbk[X]$ *by*

$$\partial_i f(t, X) = \frac{f(t, \cdots, X_i, X_{i+1}, \cdots) - f(t, \cdots, X_{i+1}, X_i, \cdots)}{X_i - X_{i+1}}.$$

Note that we do not permute $t_\bullet$*'s.*

The proof

- The similar reason,

$$H_T^*(G/B) = H_T^*(G/P)[\omega_i]/\langle\omega_i\rangle\,.$$

- By an finite dimensional approximation of $E_T \times_T G/B$, we see for $\alpha \in H_T^{*-2}(G/P_i), \beta \in H_T^*(G/P_i)$, the Demazure operator,

$$\begin{array}{ccccc} H_T^*(G/B) & \longrightarrow & H_T^{*-2}(G/P_i) & \longrightarrow & H_T^{*-2}(G/B) \\ \alpha\omega_i + \beta & \longmapsto & \alpha & \longrightarrow & \alpha \end{array}$$

Schubert Polynomials

- Denote X_w the equivariant cohomology class of $\overline{BwB/B}$, then there is a unique polynomial $\mathfrak{S}_w$ homogenous in $X_1, \ldots, X_n$ and $t_1, \ldots, t_n$

$$X_w = \mathfrak{S}_w(X_1, \ldots, X_n, t_1, \ldots, t_n)$$

  with the degree in X no more than $X_1^{n-1} \cdots X_{n-1}$, called the **double Schubert polynomials**.

- Warning One may raise such question,

  Does $\mathfrak{S}_w$ purely coincide the usual Schubert polynomials?

  The answer is not, if so, $H_T^*(G/T) = H_T^*(\mathrm{pt}) \otimes H^*(G/T)$ as $H_T^*(\mathrm{pt})$-algebra, but we know it is not (it is as $H_T^*(\mathrm{pt})$-module, a subtle difference).

Schubert Polynomials (continued)

- Warning One may argue that by an approximation of diagonal, the cup product is reflected over T-equivariant cellular complex. But the map induced is not generally an $H_T^*(\mathrm{pt})$-map if the approximation is not T-equivariant.
- Warning Note that the cell $[\overline{BwB/B}]$ and $[\overline{w_0 BwB/B}]$ gives difference equivariant cohomology class, even through it gives the same class in usual cohomology theory. The reason is, the homotopy over G/B making the equivalence of the cell $[\overline{BwB/B}]$ and $[\overline{w_0 BwB/B}]$ are not T-equivariant. We will see from the method below how to compute the difference (by localization).

The trick of Localization

- Note that the fixed point of T on G/B is exactly W. If xB is fixed, then $xTx^{-1} \subseteq B$. But B has only one such subgroup $\cong T$, thus x normalize T, so $x \in N(T)$, so x/B can be presented by a permutation matrix.
- Generally, for a G-space X, and a fixed point $a \in X$, we call

$$\cdot|_a : H_G^*(X) \longrightarrow H_T^*(\mathrm{pt}) \quad \text{induced by } E_T \times_T \{a\} \to E_T \times_T X$$

the **localization**.

The trick of Localization (continued)

Theorem

For a permutation $w \in W$, the localization map

$$\cdot|_{wT} : H_T^*(G/T) \to H_T^*(\mathrm{pt})$$

sends X_i to $t_{w(i)}$.

- For a permutation $w \in W$, consider the pull back square

$$\begin{array}{ccc} E_T \times_T wT \times_T \mathbb{C}\rho & \to & E_T \times_T G \times_T \mathbb{C}\rho \\ \downarrow & & \downarrow \\ E_T \times_T \{wT\} & \to & E_T \times_T G/T \end{array}.$$

The action of $t \in T$ acts on its fibre $\mathbb{C}\rho$ by $\rho(s)$ with $tw = ws$, i.e. $\rho(w^{-1}tw)$.

The trick of Localization (continued)

Theorem

$$\mathfrak{S}_{w_0}(X,t) = \prod_{i+j\le n} (X_i - t_j)$$

- Since $X_w \in H_T(X_{\le w}, X_{<w})$, X_w restricts at any other $u < w$ is trivial,

$$\begin{array}{ccccccc} & & H_T(G/B) & = & H_T(G/B) & & \\ & & \downarrow & & \downarrow & & \\ \cdots\to & H_T(X_{\le w}, X_{<w}) & \to & H_T(X_{\le w}) & \to & H_T(X_{<w}) & \to\cdots \\ & & \downarrow & & \downarrow & & \\ & & H_T(\{uT\}) & = & H_T(\{uT\}) & & \end{array}$$

The proof

- Consider the pairing of $H_T(\mathrm{pt})$-module,

$$\mathrm{NH}_n[t] \times H_T(G/T) \qquad (\partial_i, \alpha) \mapsto \partial_i \alpha|_{eT}.$$

it is perfect, with dual basis $\partial_i \leftrightarrow \mathfrak{S}_i(X, t)$. So it suffices to show

$$\partial_w \prod_{i+j\le n} (X_i - t_j)|_{eT} = \begin{cases} 0, & w \neq w_0, \\ 1, & w = w_0, \end{cases}$$

- Denote $\operatorname{supp} f = \{\sigma \in \mathfrak{S}_n : f|_{\sigma T} \neq 0\}$. Then it is trivial to see that $\operatorname{supp} \partial_i f \subseteq \operatorname{supp} f \cup (\operatorname{supp} f) s_i$. Since $\operatorname{supp} \prod_{i+j\le n}(X_i - t_j) = w_0$, and $\partial_{w_0} \prod_{i+j\le n}(X_i - t_j) = 1$, the proof is complete.

Equivariant Pipe Dream

Theorem

The coefficient of ∂_w of

$$\mathfrak{S}(x,y) = \begin{matrix} (1+(x_1-y_{n-1})\partial_{n-1}) & \cdots & (1+(x_1-y_1)\partial_1) \\ & \ddots & \vdots \\ & & (1+(x_{n-1}-y_1)\partial_{n-1}) \end{matrix}$$

is the double Schubert polynomial $\mathfrak{S}_w(x,y)$.

- The proof is completely the same.

Equivariant Pipe Dream (continued)

- For a pipe dream π, define its equivariant weight

$$\mathtt{wt}(\pi) = \prod_{+\in\pi} \left(x_{\text{the row number of the }+} - y_{\text{the column number of the }+}\right)$$

Theorem

For a permutation $w \in \mathfrak{S}_\infty$,

$$\mathfrak{S}_w(x,y) = \sum_{\textit{pipe dream } \pi \textit{ for } w} \mathtt{wt}(\pi).$$

Toric varieties

Theorem

The sum of restrictions

$$H_T^*(G/B) \to \bigoplus_{w\in\mathfrak{S}_n} H_T^*(\{wT\})$$

is injection.

- Since the Schubert polynomials forms a $H_T^*(\mathrm{pt})$-basis, and by induction of Bruhat order, and a simple computation, $w \in \operatorname{supp}\mathfrak{S}_w$, and $u \notin \operatorname{supp}\mathfrak{S}_w$ when $u < w$.
- This is a special case of the **Localization theorem**.

Topology Remind

Theorem (Localization, Borel)

Assume M is a compact connected manifold with a torus T acted with $H_T^*(M)$ free as $H_T^*(\mathrm{pt})$-module, then the following restriction map is injective

$$H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(M^T;\mathbb{Q}) = \bigoplus_{x \in M^T} H_T^*(x;\mathbb{Q})$$

Theorem (Białynicki–Birula, 1973)

Let X be a smooth algebraic variety over $\mathbb{C}$ equipped with an algebraic action of torus $T = (\mathbb{C}^\times)^n$. If X has discrete fixed points, then X admits an affine cellular structure each of them contains a fixed point. In particular, $H_T^*(X)$ is free as $H_T^*(\mathrm{pt})$-module.

Topology Remind

Theorem (Goresky, Kottwitz, Macpherson, 1998)

Let X be a smooth projective variety over $\mathbb{C}$ equipped with an algebraic action of torus $T = (\mathbb{C}^\times)^n$. If X has finite fixed points and finitely invariant $\mathbb{C}P^1$ between fixed points, then the image of

$$H_T^*(X;\mathbb{Q}) \longrightarrow \bigoplus_{x \in X^T} H_T^*(x;\mathbb{Q})$$

is (α_x) with $\omega_p \mid \alpha_x - \alpha_y$ for all $\mathbb{C}P^1$ connecting $x \xrightarrow{p} y$, and ω_p is the character of the action on the affine space $\mathbb{C}P^1 \setminus \infty = p \setminus \{y\}$.

- In our case, the cellular structure of G/B is parameterized by the set of fixed points, say the Weyl group. For w and ws_i with $\ell(ws_i) = \ell(w) + 1$ are connected with $wP_i/B \cong \mathbb{C}P^1$, the character is $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_{w(i)}$.

References

- Brion, Vergne. On the localization theorem in equivariant cohomology. [arXiv] (for localization theorem)
- Kuroki. Torus Equivariant Cohomology. [YouTube] (For equivariant cohomology)
- Wu Yi Hsiang. Cohomology Theory of Topological Transformation Groups. (For equivariant cohomology)
- Representation theory and complex geometry. (For the discussion of Białynicki–Birula decomposition at the second chapter)
- Goresky, Kottwitz, and Macpherson. Equivariant cohomology, Koszul duality, and the localization theorem.
- Knutson. Schubert polynomials, pipe dreams, equivariant classes, and a co-transition formula.

∼ § <u>Convolution Algebra</u> § ∼

Double Flags

- Note that G-orbits of $G/B \times G/B$ are one to one correspondent to B-orbits of G/B, since

$$\begin{aligned} G\backslash(G/B \times G/B) &= \mathrm{pt} \times_G (G/B \times G/B) \\ &\xlongequal{(x,y)\mapsto(x,x^{-1}y)} \mathrm{pt} \times_G G \times_B G \times_B \mathrm{pt} \\ &= \mathrm{pt} \times_B G \times_B \mathrm{pt} = \mathrm{pt} \times_B G/B \\ &= B\backslash(G/B) \end{aligned}$$

- The correspondence is

$$\begin{array}{ccc} BwB/B & \longleftrightarrow & \{(xB, yB) : x^{-1}y \in BwB\} \\ \updownarrow & & \updownarrow \\ \{\mathcal{F} : w(\mathcal{F}, \mathcal{F}_0) = w\} & \longleftrightarrow & \{(\mathcal{F}_1, \mathcal{F}_2) : w(\mathcal{F}_2, \mathcal{F}_1) = w\} \end{array}$$

Denote $\Omega_w = \{(xB, yB) : x^{-1}y \in BwB\} \in G/B \times G/B$.

Double Flags (continued)

- Note that under the first projection $\begin{bmatrix} G/B\times G/B \\ \downarrow \\ G/B \end{bmatrix} \cong \begin{bmatrix} G\times_B G/B \\ \downarrow \\ G/B \end{bmatrix}$,

$$\begin{array}{ccc} \Omega_w & \cong & \{(x \underset{B}{\times} zB) : z \in BwB\} \\ \downarrow & & \downarrow \\ G/B & = & G/B \end{array}$$

  is a fibre bundle with fibre BwB/B. In particular, $\dim \Omega_w = 2\dim G/B + 2\ell(w)$.
- The decomposition of G/B decomposes Ω_w into affine cells, so $[\overline{\Omega_w}]$ as the closure of fibre over Bw_0B is defined in the cohomology group $H^*(G/B \times G/B)$.

Topology Remind

- For space C, B, A with B holding Poincaré duality, we can define the convolution

$$
\begin{array}{rcl}
 & H^*(C \times B) \times H^*(B \times A) & \\
\text{cohomology map} & \downarrow \; p^* \times q^* & \\
 & H^*(C \times B \times A) \times H^*(C \times B \times A) & \\
\text{cup product} & \downarrow \; \smile & \\
 & H^*(C \times B \times A) & \\
\text{Gysin push forward} & \downarrow \; r_* & \\
 & H^{*-\dim B}(C \times A) &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & C \times B \times A & & \\
 & q \swarrow & \Big| \, r & \searrow p & \\
C \times B & & \downarrow & & B \times A \\
 & & C \times A & &
\end{array}
$$

Theorem

The convolution is associative, with the diagonal $[\Delta] \in H^{\dim B}(B \times B)$ as the identity.

Topology Remind

- The convolution is natural in A and C.
- The convolution is associative.
- The diagonal $[\Delta] \in H^{\dim B}(B \times B)$ is the unit element.
- So $H^*(X \times X)$ forms an associative algebra with unite.
- Let $A = \mathrm{pt}$, we have

$$H^*(C \times B) \times H^*(B) \to H^{*-\dim B}(C)$$

When $B = C$, $H^*(B)$ forms an $H^*(B \times B)$-module.

Topology Remind

- If Poincaré duality holds for all of A, B, C, then we can transfer the convolution to the homology group to be

$$H_*(C \times B) \times H_*(B \times A) \to H_{*-\dim B}(C \times A).$$

Then for two cells (cycles) $\Omega_1 \subset B \times A$ and $\Omega_2 \subset C \times B$,

$$\Omega_2 * \Omega_1 = r_*((\Omega_2 \otimes [A]) \bullet ([C] \otimes \Omega_1)),$$

where $\bullet$ the intersection product.

Computation of convolution algebra

Theorem

There is an algebra embedding $\mathrm{NH}_n \to H^*(G/B \times G/B)$ *with* ∂_w *mapping to the Poincaré dual of* $[\overline{\Omega_w}]$.

- It suffices to show the map of homology $\mathrm{NH}_n \to H_*(G/B \times G/B)$ with $\partial_w \mapsto [\overline{\Omega_w}]$ is an algebra homomorphism.
- Note that the projection of Ω_w to any factor is surjective, so $\overline{\Omega_v} \times G/B$ and $G/B \times \overline{\Omega_u}$ always intersects transversally. Geometrically, the intersection is exactly taking direct product of fibres at each points.

The proof

- So to compute the intersection product, it suffices to compute the set-theoretic intersection,

$$
\begin{aligned}
&(\overline{\Omega_v} \times G/B) \cap (G/B \times \overline{\Omega_u}) \\
=& \{(xB, yB, zB) : x^{-1}y \in \overline{BvB}, y^{-1}z \in \overline{BuB}\}
\end{aligned}
$$

- Then the pull forward of the above intersection is exactly

$$
\begin{aligned}
&\{(xB, zB) : \exists y, x^{-1}y \in \overline{BvB}, y^{-1}z \in \overline{BuB}\} \\
=& \{(xB, zB) : x^{-1}z \in \overline{BvB} \cdot \overline{BuB}\} \\
=& \begin{cases} \overline{BvuB}, & \ell(vu) = \ell(v) + \ell(u), \\ \text{lower dimensional stuff}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Computation of convolution algebra (continued)

Theorem

The action of the convolution algebra $H^*(G/B \times G/B)$ on $H^*(G/B)$

$$\mathrm{NH}_n \times H^*(G/B) \to H^*(G/B \times G/B) \times H^*(G/B) \to H^*(G/B)$$

is given by $(\partial_w, \alpha) \mapsto \partial_w \alpha$, the Demazure operator.

Theorem

As a corollary, $[\overline{\Omega}_w]$ acts on $H^*(G/B \times G/B)$ as the action of the Demazure operator ∂_w on the first factor.

The proof

- By the same computation, in homology,

$$\begin{aligned}[\overline{\Omega_v}] * [\overline{BuB/B}] &= \{(xB, yB) : x^{-1}y \in \overline{BvB}, y \in \overline{BuB}\} \\ &= \{(xB, yB) : y \in \overline{BuB} \cdot \overline{Bv^{-1}B}\} \\ &= \begin{cases} [\overline{Buv^{-1}B/B}], & \ell(uv^{-1}) = \ell(u) + \ell(v^{-1}), \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

  Note that the intersection of projections on the middle factor is transversal.

- And we know under the Poincaré duality $[\overline{BwB/B}] \leftrightarrow [\overline{Bw_0wB/B}]$, so in cohomology,

$$\begin{aligned}[\overline{\Omega_v}] * [\overline{BuB/B}] &= \begin{cases} [\overline{Buv^{-1}B/B}], & \ell(uv^{-1}) = \ell(u) - \ell(v^{-1}), \\ 0, & \text{otherwise.} \end{cases} \\ &= \partial_v[\overline{BuB/B}].\end{aligned}$$

Computation of convolution algebra (continued)

- Denote the Poincaré duality of $[\overline{\Omega_w}]$ in cohomology by the symbol ∂_w.
- Recall $H^*(G/B)$ is a quotient ring of $\Bbbk[X]$; abuse of notation, denote $f(X) \smile \partial_e = f(X)$, where $H^*(G/B)$ acts through the first projection.

Theorem

The convolution algebra $H^*(G/B \times G/B)$ is isomorphic to $H^*(G/B) \otimes \mathrm{NH}_n$ with a twisted product

$$\partial_i X_j - X_{s_i(j)} \partial_i = \delta_{i,j} - \delta_{i+1,j} = \begin{cases} 1, \text{if } i=j, \\ -1, \text{if } i+1=j,. \\ 0, \text{otherwise.} \end{cases}$$

The action of X_i on $H^*(G/B)$ is just multiplication by X_i, and the action of ∂_i is the Demazure operator.

The proof

- Firstly,

$$\begin{aligned} X_i * X_j &= (X_i \smile \partial_e) * (X_j \smile \partial_e) \\ &= X_i \smile (\partial_e * (X_j \smile \partial_e)) \\ &= X_i \smile X_j \smile \partial_e = X_i X_j. \end{aligned}$$

- It is an $\mathbb{k}$-isomorphism follows from the standard argument of the Harish–Leray. Note that the restriction of $\{\partial_w : w \in W\}$ at fibre forms a basis.

The proof (continued)

- To be clear, denote Δ_w for the Demazure operator in the proof.
- Note that we have the following twisted Leibniz rule

$$\Delta_i(fg) = \Delta_i f \cdot g + s_i f \cdot \Delta_i g.$$

- As a result,

$$\begin{aligned} \partial_i * X_j &= \partial_i * (X_j \smile \partial_e) \\ &= (\partial_i * X_j) \smile \partial_e + s_i X_j \smile (\partial_i * \partial_e) \\ &= \Delta_i(X_j)\partial_e + X_{s_i(j)}\partial_i \\ &= (\delta_{i,j} - \delta_{i+1,j})\partial_e + X_{s_i(j)}\partial_i \end{aligned}$$

the desired relation.

Computation of convolution algebra (continued)

Theorem

In summary, $H^*(G/B \times G/B)$ is isomorphic to the algebra generated in $\mathrm{End}_{\Bbbk}(H^*(G/B))$ by

$$\text{left multiplication by } X_i, \qquad \text{Demazure operators } \partial_i, \qquad 1 \le i \le n.$$

In a more explicit way, the cohomology group $H^*(G/B \times G/B)$

$$\frac{\Bbbk\left\langle X_i, \partial_j \right\rangle_{\substack{1\le i\le n\\ 1\le j\le n-1}}}{\left\langle E_i \right\rangle_{1\le i\le n}} \Big/ \left\langle \begin{matrix} \partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1}, & X_iX_j = X_jX_i, \\ |i-j|\ge 2, \quad \partial_i\partial_j = \partial_j\partial_i, & \partial_i X_j - X_{s_i(j)}\partial_i \\ \partial_i^2 = 0. & = \delta_{i,j} - \delta_{i+1,j}. \end{matrix} \right\rangle$$

again, E_i the i-th elementary symmetric polynomial.

Equivariant version

- As we computed last section, it is more modern to consider the equivariant version.
- Since the projection $G/B \times G/B \times G/B \to G/B \times G/B$ is G-equivariant, so we can also define the convolution in the G-equivariant cohomology.

$$H_G^*(G/B \times G/B) \times H_G^*(G/B \times G/B) \to H_G^{*-\dim G/B}(G/B \times G/B).$$

$$H_G^*(G/B \times G/B) \times H_G^*(G/B) \to H_G^{*-\dim G/B}(G/B).$$

Equivariant version (continued)

- In this case, we need to use the equivariant homology slightly. By definition, the equivariant homology of G-space X is

$$H_*^G(X) = H_*(E_G \times_G X).$$

- As what we stated last section, if there is a G-cellular structure, then the homology group of the complex

$$\cdots \to H^G(X_{\dim \leq \bullet}, X_{\dim < \bullet}) \to \cdots$$

is exactly $H^G(X)$. Besides, the pairing of complex with

$$\cdots \to H_G(X_{\dim \leq \bullet}, X_{\dim < \bullet}) \to \cdots$$

coincides with the pairing of $H^G(X)$ and $H_G(X)$.

Equivariant version (continued)

- We can define the G-cell $[\overline{\Omega_w}] \in H_*^G(G/B \times G/B)$ and in $H_G^*(G/B \times G/B)$.
- Warning Even though $E_G \times_G \Omega_w = E_G \times_B BwB/B$, the B-cell in $H_B^*(G/B)$ and G-cell in $H_G^*(G/B)$ are different (for example, in dimensions).

$$\begin{aligned} H_G^*(G/B \times G/B) &= H^*(E_G \times_G G/B \times G/B) \\ &= H^*(E_G \times_G G \times_B G/B) \\ &= H^*(E_G \times_B G/B) \\ &= H_B^*(G/B) \quad \text{(computed, but we won't use)} \\ H_G^*(G/B) &= H^*(E_G \times_G G/B) = H_B^*(\mathrm{pt}) = \Bbbk[X_1, \ldots, X_n]. \end{aligned}$$

Equivariant version (continued)

- Since $\begin{bmatrix} G/B \\ \downarrow \\ G/P \end{bmatrix}$ is G-equivariant, we can also define the equivariant Demazure operator

  $$\partial_i : H_G^*(G/B) \longrightarrow H_G^{*-2}(G/B)$$

  it is exactly the Demazure operator we defined over polynomials. The proof is completely the same to nonequivariant case.

Equivariant version (continued)

- Since we work in equivairiant cohomology, there is no proper Poincaré duality, but we still denote the symbol by pairing

$$\partial_w = \left\langle [\overline{\Omega_{w_0}}]^{\in H_G^*(G/B\times G/B)}, [\overline{\Omega_w}]_{\in H_*^G(G/B\times G/B)} \right\rangle \in H_G^*(G/B\times G/B),$$

  of degree $2\ell(w_0) - \ell(w)$.
- Take the canonic isomorphism $H_G^*(G/B) \cong \Bbbk[X_1, \dots, X_n]$, denote $X_i = X_i \smile \partial_e$.

Theorem

There is an algebra embedding $\mathrm{NH}_n \to H_G^*(G/B\times G/B)$ *maps* ∂_w *to* ∂_w*, with its action on* $H_G^*(G/B)$ *the Demazure operator* ∂_w*.*

The proof

- The trick is, the augment map commutes with convolution product (follows from definition).
- Since the Schubert cells are not G-cell, so we pass to the T-equivariant case. It is harmless, since $H^*_G(G/B) \to G^*_T(G/B)$ is injective.
- Note that when we compute the intersection product, everything is T-equivariant, so by an approximation of E_T, the same argument holds for T-equivariant case.

$$\begin{array}{ccc}
H^*_G(G/B \times G/B) \times H^*_G(G/B) & \to & H_G^{*-\dim G/B}(G/B) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \\
H^*_T(G/B \times G/B) \times H^*_T(G/B) & \to & H_T^{*-\dim G/B}(G/B) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \\
H_*(G/B \times G/B) \times H^*(G/B) & \to & H^{*-\dim G/B}(G/B)
\end{array}$$

Equivariant version (continued)

Theorem

The convolution algebra $H_G^*(G/B \times G/B)$ is isomorphic to $H_G^*(G/B) \otimes \mathrm{NH}_n$ with a twisted product

$$\partial_i X_j - X_{s_i(j)} \partial_i = \delta_{i,j} - \delta_{i+1,j} = \begin{cases} 1, \text{ if } j = i, \\ -1, \text{ if } j = i+1, \\ 0, \text{ otherwise.} \end{cases}$$

Besides, the action of $H_G^*(G/B \times G/B)$ on $H_G^*(G/B)$ is an $H_G^*(\mathrm{pt})$-homomorphism.

The proof

- The proof of the isomorphism is the same. So it remains to show it is an $H^*_G(\mathrm{pt})$-homomorphism.
- Of course, the Demazure operator is clearly an $H^*_G(\mathrm{pt}) = \Bbbk[X]^W$-homomorphism, so is the left multiplication.
- Geometrically, clearly, $H^*_G(G/B \times \mathrm{pt})$ is an $H^*_G(G/B \times G/B)$-$H^*_G(\mathrm{pt} \times \mathrm{pt})$ bimodule,

$$H^*_G(G/B \times G/B) \curvearrowright H^*_G(G/B \times \mathrm{pt}) \curvearrowleft H^*_G(\mathrm{pt} \times \mathrm{pt})$$

Note that the convolution

$$H^*_G(G/B \times \mathrm{pt}) \times H^*_G(\mathrm{pt} \times \mathrm{pt}) \to H^*_G(G/B \times \mathrm{pt})$$

is exactly the cup product, thus it follows from the associativity of convolution.

Equivariant version (continued)

Theorem

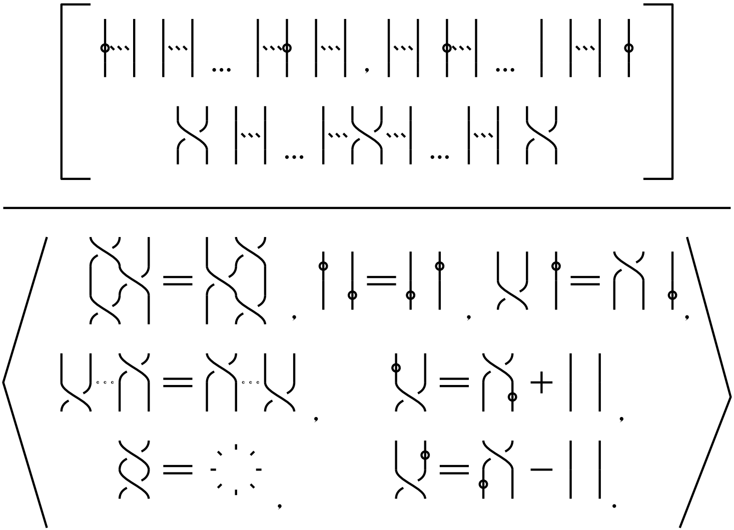
As a result, $H_G^*(G/B \times G/B)$ *is isomorphic to the algebra generated in* $\mathrm{End}_{H_G^*(\mathrm{pt})}(H_G^*(G/B))$ *by*

$$\text{left multiplication by } X_i, \qquad \text{Demazure operators } \partial_i, \qquad 1 \leq i \leq n.$$

In a more explicit way, the equivariant cohomology group $H_G^*(G/B \times G/B)$

$$\Bbbk \langle X_i, \partial_j \rangle_{\substack{1\leq i\leq n \\ 1\leq j\leq n-1}} \Big/ \left\langle \begin{array}{cc} \partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1}, & X_iX_j = X_jX_i, \\ |i-j| \geq 2, \quad \partial_i\partial_j = \partial_j\partial_i, & \partial_i X_j - X_{s_i(j)}\partial_i \\ \partial_i^2 = 0. & = \delta_{i,j} - \delta_{i+1,j}. \end{array} \right\rangle$$

Diagrammatic presentation



References

- Neil, Victor. Representation theory and Complex Geometry. (for definition of convolution)
- Khovanov, Lauda. A diagrammatic approach to categorification of quantum groups. [arXiv] (for the diagrammatic presentation)

$\sim$ § <u>THANKS</u> § $\sim$