Cohomology of Flag Manifolds (I)
The Classic Methods

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The Cellular Structure

Flag manifolds

- In this talk, the base field is taken to be $\mathbb{C}$, and $H^\bullet(X) = H^\bullet(X; \mathbb{k})$ where the coefficient ring $\mathbb{k}$ is a field of characteristic zero, for example $\mathbb{Q}$.
- Let V be a finite dimensional vector space of dimension n. A **flag** $\mathcal{F}_\bullet$ is a sequence of subspaces of V,

$$0 = \mathcal{F}_0 \subsetneqq \mathcal{F}_1 \subsetneqq \cdots \subsetneqq \mathcal{F}_n = V,$$

  with $\dim \mathcal{F}_i = i$.
- Denote the set of all such flags to be $\mathcal{F}\ell(V)$, and call it the **flag manifold/variety** (see below).
- Our purpose: compute $H^\bullet(\mathcal{F}\ell(V))$.

The Topology

- Fix some isomorphism $V = \mathbb{C}^n$, and consider the map

$$\mathrm{span} : \mathrm{GL}(V) \longrightarrow \mathcal{F}\ell(V) \qquad x = \big(v_1, \dots, v_n\big) \longmapsto \mathcal{F}_x,$$

  where

$$\mathcal{F}_x : \quad 0 \quad \subsetneq \quad \mathbb{C}v_1 \quad \subsetneq \mathbb{C}v_1 + \mathbb{C}v_2 \subsetneq \quad \cdots \quad \subsetneq \quad V \quad .$$

- This map is clearly surjective, and

$$\mathcal{F}_x = \mathcal{F}_y \iff x = y \cdot (\text{an invertible upper triangle matrix}).$$

The Topology

- Denote $G = \mathrm{GL}_n$, and B the group of invertible upper triangle matrices (the **Borel subgroup**). We have a bijection

$$\text{span} : G/B \xrightarrow{1:1} \mathcal{F}\ell(V).$$

So we can define the topology and smooth structure to be as G/B.

- By the Gauss elimination process, we have (see below)

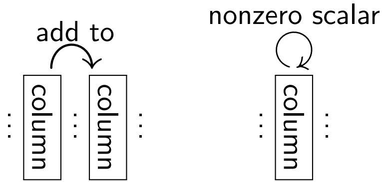
Theorem (Bruhat decomposition)

$$G = \bigsqcup_{w \in W} BwB \qquad W = \{\textit{permutation matrices}\} \cong \mathfrak{S}_n.$$

The action

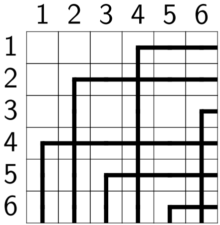
- The action of B on the left can be decomposed into

- The action of B on the right can be decomposed into



The Combinatorics I

- For a permutation $w \in \mathfrak{S}_n$, consider the **Rothe diagram** by its "graph" and the space $U_w \subseteq G$, as follows



$$\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,6\,1\,3\,5 \end{pmatrix}$$

$$U_w = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & 1 & 0 & 0 \\ \mathbb{C} & 1 & 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 & \mathbb{C} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The Combinatorics I

Theorem

There is a bijection (thus homoemophism)

$$\mathrm{span} : U_w \xrightarrow{1:1} Bw^{-1}B/B.$$

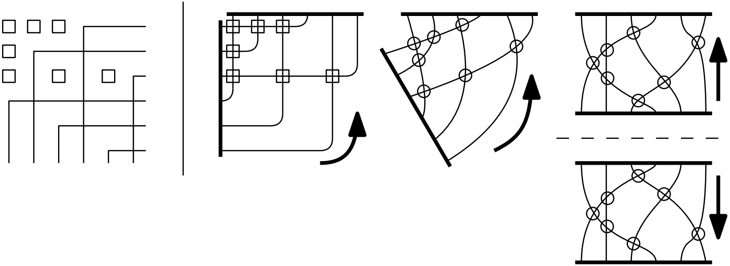
- For any $x \in U_w$, $xB \in Bw^{-1}B$.
  — Dig the hole for each column.
- For any $x \in G$, there is some $b \in B$ such that $xb \in U_w$ for some $w \in \mathfrak{S}_n$.
  — Dig the hole from the last row.
- For any $x, y \in U_w$, if $y \in xB$, then $x = y$.
  — Clearly.

The Combinatorics I

Theorem

The dimension of U_ℓ is the number of inversions, more precisely

$$\dim U_w = \ell(w) = \#\{(i,j) : i < j, w(i) > j\}.$$



Topology remind I

Theorem

For a CW-complex X, the homology group of the complex

$$\cdots \longrightarrow H^\bullet(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \longrightarrow \cdots$$

is isomorphic to $H^\bullet(X)$, and

$$H^\bullet(X_{\dim\leq\bullet}, X_{\dim\leq\bullet-1}) \cong \bigoplus_{\dim=\bullet \text{ cell } \Delta} \Bbbk \cdot \Delta.$$

The cellular structure

- In our case, $\{BwB/B : w \in \mathbb{S}_n\}$ defines a cellular structure of G/B.
- But $\dim_{\mathbb{R}} U_w$ are all even dimensional, so the above complex is trivial.

Theorem

The cohomology ring $H^\bullet(\mathcal{F}\ell(V))$ has only even dimensions. Furthermore,

$$\dim H^{2i}(\mathcal{F}\ell(V)) = \#\{w : \ell(w) = i\}.$$

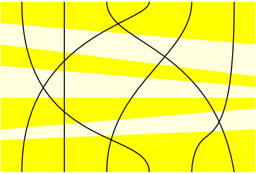
- The problem is, how to describe the product structure?

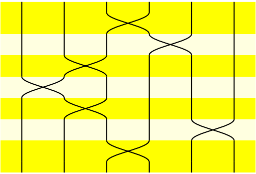
The Combinatorics II

- For any $w \in \mathfrak{S}_n$, $\ell(w)$ is the least length to write w into a product of

$$s_1 = (12),\ s_2 = (23),\ \cdots,\ s_{n-1} = (n-1, n).$$

Any shortest expression is called a **reduced word**.





The Combinatorics II

- For two permutation u, v, we write

$$u \lessdot v \iff \ell(u) + 1 = \ell(v), v = su$$

where s is any swap. We write

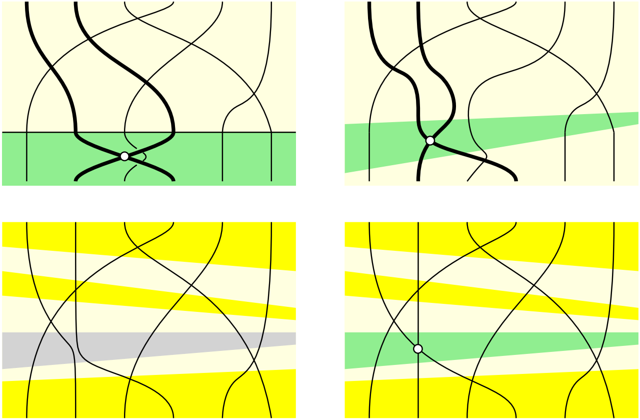
$$u \leq v \iff u = u_0 \lessdot {}^{\exists}u_1 \lessdot \cdots \lessdot {}^{\exists}u_{k-1} \lessdot u_k = v.$$

The following theorem indicates this is a partial order, called the **Bruhat order**.

Theorem

If $u = s_1 \cdots s_k$ a reduced word, then

$$v \lessdot u \iff \exists i, \quad v = s_1 \cdots \widehat{s_i} \cdots s_k \text{ is reduced.}$$



The Combinatorics II

Theorem

The Schubert cell

$$BvB/B \subseteq \overline{BuB/B} \iff v \leq u.$$

Sketch of the proof

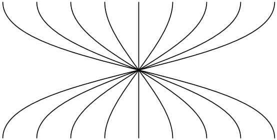
- Note that it suffices to show when $\ell(v) + 1 = \ell(u)$.
- For $v \lessdot u$, one can show that $BvB/B \subseteq \overline{BuB/B}$.
- Otherwise, then one can construct an open subset $U \supseteq BvB/B$ disjoint to Σ_u. The open subset is given by the dimensions intersects the standard flag.

The Combinatorics II

- The longest word $w_0 = \binom{1 \cdots n}{n \cdots 1} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is also the only maximal element of $\leq$.
- Actually, due to the **LU decomposition**,

$$Bw_0B = \{x \in \mathrm{GL}_n : \text{sequential principal minor of } x \neq 0\}$$

is Zariski dense.



wrong figure

The Characteristic Classes

Lie theory

- It is harmless to consider SL rather than GL, since

$$\mathrm{GL}_n /B \cong \mathrm{SL}_n /(B \cap SL_n).$$

  Let us replace G by SL_n, and B by $B \cap \mathrm{SL}_n$.
- We will define $*_{ij}$ for $i < j$ for a lot of symbol $*$, and we will simply write $*_i$ for $*_{i,i+1}$ when $1 \leq i \leq n-1$.
- For example, we define $s_{ij} \in \mathfrak{S}_n$ the exchange of i and j. Then we will denote $s_i = s_{ij}$.

Lie theory

- Consider the natural map for $i < j$

$$\kappa_{ij}: \quad \mathrm{SL}_2 \longrightarrow \mathrm{SL}_n \qquad\qquad P_{ij} = \operatorname{im}\kappa_{ij} \cdot B$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{1} & & & & \\ & a & & b & \\ & & \mathbf{1} & & \\ & c & & d & \\ & & & & \mathbf{1} \end{pmatrix} \qquad = \begin{pmatrix} \cdots & & & \ast \\ & \cdots & & \\ & \ast & \cdots & \\ & & & \cdots \end{pmatrix}.$$

- Denote $B_2 = \{\begin{pmatrix} \ast & \ast \\ & \ast \end{pmatrix} \in \mathrm{SL}_2\}$,

$$\kappa_{ij}: \mathrm{SL}_2/B_2 \xrightarrow{1:1} P_{ij}/B$$

- We also have the following homoemorphisms

$$SL_2/B_2 = \mathcal{F}\ell(\mathbb{C}^2) = \mathbb{P}(\mathbb{C}^2) = \mathbb{C}P^1 \cong S^2.$$

Topology remind II

Theorem (Gysin sequence)

For a d-dimensional sphere bundle $E \xrightarrow{\pi} B$, there is a long exact sequence

$$\cdots \to H^{i-1+d}(B) \xrightarrow{*} H^i(B) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\pi_*} H^{i-d}(B) \xrightarrow{*} H^{i+1}(B) \to \cdots.$$

Where

- π^* *is the usual induced cohomology map;*
- *when* E, B *are for which Poincaré duality holds,* π_* *is induced from homology through duality;*
- *the* $*$ *is the cup product with the* **Euler class** *of* π*.*

Demazure operator I

- The natural map $G/B \xrightarrow{\pi} G/P$ has the fibre $P/B \cong S^2$, thus it is a sphere bundle. So we can apply the **Gysin sequence**

$$\cdots \to H^{2i}(G/P) \xrightarrow{\pi^*} H^{2i}(G/B) \xrightarrow{\pi_*} H^{2i-2}(G/P) \to \cdots .$$

Define the composition for $P = P_{ij}$

$$\partial_{ij} : H^{2i}(G/B) \xrightarrow{\pi_*} H^{2i-2}(G/P) \xrightarrow{\pi^*} H^{2i-2}(G/B)$$

to be the **Demazure operator**.

Topology remind III

- Define the **tautological bundle** over $\mathbb{C}P^1$,

$$\mathcal{O}(-1) = \left[\begin{array}{c} \{(\ell, x) \in \mathbb{C}P^1 \times \mathbb{C}^2 : x \in \ell\} \\ \downarrow \\ \mathbb{C}P^1 \end{array} \right]$$

Theorem (1st Chern class)

For any CW-complex B, there is a natural transform between

$$c_1 : \{Line\ Bundles\ over\ B\} \to H^2(B)$$

funtorial in B such that $-c_1(\mathcal{O}(-1)) \in H^2(\mathbb{C}P^1)$ dual to $[\mathrm{pt}] \in H_0(\mathbb{C}P^1)$.

Lie theory

- For any character (i.e. a group homomorphism to $\mathbb{C}^\times$) of B, it defines a representation, denoted by $\mathbb{C}\rho$, of B, say

$$\mathbb{C} \xrightarrow{b} \mathbb{C} \qquad v \mapsto \rho(b)v.$$

This also defines a line bundle $\underline{\mathbb{C}}\rho := \begin{bmatrix} G\times_B \mathbb{C}\rho \\ \downarrow \\ G/B \end{bmatrix}$.

- For $\rho : \begin{pmatrix} x & * \\ & x^{-1} \end{pmatrix} \mapsto x$, the bundle $\begin{bmatrix} \mathrm{SL}_2 \times_{B_2} \mathbb{C}\rho & & \\ \downarrow & \searrow & \\ \mathrm{SL}_2/B_2 & \cong & \mathbb{C}P^1 \end{bmatrix}$ is isomorphic to $\mathcal{O}(-1)$, just by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda\right) \mapsto \left(\mathbb{C}\binom{a}{c}, \lambda\binom{a}{c}\right)$.

- Consider the character

$$\omega_i : B \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} x_1 & * & * \\ & \ddots & * \\ & & x_n \end{pmatrix} \longmapsto x_1 \cdots x_i.$$

We have the following bundle

$$\begin{array}{ccccc} \mathrm{SL}_2 \times_{B_2} \mathbb{C}(\omega_i \circ \kappa_i) & \longrightarrow & G \times_B \mathbb{C}\omega_i & \longleftarrow & \mathrm{SL}_2 \times_{B_2} \mathbb{C}(\omega_i \circ \kappa_j) \\ \mathcal{O}(-1) \Big\downarrow & & \Big\downarrow \underline{\mathbb{C}}\omega_i \quad (j \neq i) & & \Big\downarrow \text{trivial} \\ \mathrm{SL}_2 / B_2 & \xrightarrow{\kappa_i} & G/B & \xleftarrow{\kappa_j} & \mathrm{SL}_2 / B_2 \end{array}$$

- As a result,

$$\kappa_j^*(-c_1(\underline{\mathbb{C}}\omega_i)) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \in \mathbb{Z} \cong H^2(\mathrm{SL}_2 / B_2).$$

Topology remind IV

Theorem (Harish–Leray)

For a fibre bundle $E \to B$, if each fibre F_x has free cohomology, and there is a set $\{\alpha\} \subseteq H^\bullet(E)$ present the bases restricting each fibre. Then

$$H^*(B) \otimes H^*(F) \longrightarrow H^*(E) \qquad \beta \times i_*\alpha \longrightarrow \pi_*\beta \smile \alpha$$

is an isomorphism between $H^*(B)$ modules.
Furthermore, the map is funtorial in $(E \to B, \{\alpha\})$ with fixed fibre F.

The fibre structure

- The natural map $G/B \xrightarrow{\pi} G/P$ has the fibre $P/B \cong \mathbb{C}P^1$.
- When $P = P_i$, then it satisfies the condition of the Harish–Leray theorem (by the Chern class of $\underline{\mathbb{C}}\omega_i$).
- As a result, we have the $H^*(G/B)$-isomorphism,

$$H^*(G/B) \cong H^*(G/P)[\omega_i]/\left\langle \omega_i^2 \right\rangle,$$

where $\omega_i = -c_1(\underline{\mathbb{C}}\omega_i)$ over G/B.

Demazure operator II

- We should compute

$$\begin{aligned}\pi_*((\alpha\omega_i+\beta)\cap[G/B]) &= \pi_*(\alpha\cap(\omega_i\cap[G/B]))+\pi_*(\beta\cap[G/B])\\ &= \alpha\cap\pi_*(\omega_i\cap[G/B])+\underbrace{\beta\cap\pi_*[G/B]}_{=0}.\end{aligned}$$

Now $\pi_*(\omega_i\cap[G/B])=\lambda[G/P]$ for some $\lambda\in\mathbb{Z}$. Consider the diagram $\begin{bmatrix} P/B \to & \text{pt} \\ \downarrow & \downarrow \\ G/B \to & G/P \end{bmatrix}$, we see $\lambda=1$.

- So the Demazure operator

$$\begin{array}{ccccc} H^*(G/B) & \longrightarrow & H^{*-2}(G/P) & \longrightarrow & H^{*-2}(G/B) \\ \alpha\omega+\beta & \longmapsto & \alpha & \longrightarrow & \alpha \end{array}$$

The coinvariant map

- Define

$$\Bbbk[X_1,\ldots,X_n] \xrightarrow{\psi} H^*(G/B) \qquad \lambda\cdot X_1^{a_1}\cdots X_n^{a_n} \longmapsto \lambda\cdot \rho_1^{a_1}\cdots\rho_n^{a_n}$$

where $\rho_i = -c_1(\underline{\mathbb{C}X_i})$ where $X_i : \begin{pmatrix} x_1 & * & * \\ & \ddots & * \\ & & x_n \end{pmatrix} \longmapsto x_i$.

- Warning: maybe confusing notation Note that $\psi(a_1X_1+\cdots+a_nX_n)$ corresponds to the character $\begin{pmatrix} x_1 & * & * \\ & \ddots & * \\ & & x_n \end{pmatrix} \longmapsto x_1^{a_1}\cdots x_n^{a_n}$ by the formula for tensor product of line bundles.
- Define the **Demazure operator**

$$\partial_i f(X_1,\cdots,X_n) = \frac{f(\ldots,X_i,X_{i+1},\ldots)-f(\ldots,X_{i+1},X_i,\ldots)}{X_i - X_{i+1}}.$$

Demazure operator III

Theorem

For a polynomial $f \in \Bbbk[X]$, $\psi(\partial_i f) = \partial_i \psi(f)$.

Sketch of the proof.

- The action of conjugation by a permutation matrix induces an action of $\mathfrak{S}_n$ over $H^*(G/T) = H^*(G/B)$. By the naturality of Chern Classes, ψ is an $\mathfrak{S}_n$-equivariant map.
- Note that s_i lies in P_i, so the conjugation of s_i over G/P is trivial. Then it is easy to see that ∂_i and ∂_i do the same work

$$\frac{1-s_i}{\psi(X_i - X_{i+1})}(\alpha\omega_i + \beta) = \frac{\omega_i - \omega_i \circ s_i}{\psi(X_i - X_{i+1})}\alpha$$

$$= \frac{\psi(X_1 + \cdots + X_i) - \psi(X_1 + \cdots + X_{i-1} + X_{i+1})}{\psi(X_i - X_{i+1})}\alpha = \alpha = \partial_i(\alpha\omega_i + \beta).$$

The multiplication structure

Theorem

When the characteristic of $\mathbb{k}$ is zero, then ψ is surjective, and

$$\ker\psi = \textit{ideal generated by the symmetric polynomials of } \deg > 0.$$

Sketch of the proof for the kernel.

- Firstly, $H^*(G/B;\mathbb{Z})^{\mathfrak{S}_n} = H^0(G/B;\mathbb{Z})$.
- Secondly, we see that $f \in \ker\psi$ if and only if f acted by any iterated Demazure operator $\partial_* \cdots \partial_*$ has zero constant term. The trick is, ψ is isomorphism over zero degree part.
- In characteristic zero case, it coincides the ideal described. (We will see another proof without using the Demazure operator in next lecture, but only work for characteristic zero).

Sketch of the proof for the surjectivity.

- So the image of ψ is

$$k[X_1, \dots, X_n] / \langle E_1, \cdots, E_n \rangle$$

where E_i is the i-th fundamental symmetric polynomial.

- One can show that E_{k+1} is not a zero divisor in $k[X_1, \dots, X_n] / \langle E_1, \dots, E_k \rangle$ (so-called **regular sequence**).
- By a standard trick of computation of Poincaré polynomial,

$$0 \to \frac{k[X]}{\langle E_i \rangle_{i \le k}} \xrightarrow{E_{k+1}} \frac{k[X]}{\langle E_i \rangle_{i \le k}} \to \frac{k[X]}{\langle E_i \rangle_{i \le k+1}} \to 0$$

we get $P(t) = \prod_{k=1}^{n} \frac{1-t^{2k}}{1-t^2}$ the Poincaré polynomial of image of ψ. Since $P(1) = n!$, so ψ is surjective.

The multiplication structure

- Actually, as graded linear spaces,

$$\Bbbk[X_1, \ldots, X_n] \cong \Bbbk[E_1, \ldots, E_n] \otimes \frac{\Bbbk[X_1, \ldots, X_n]}{\langle E_1, \ldots, E_n \rangle}$$

due to the dimension reason.

Theorem

The cohomology ring

$$H^*(\mathcal{F}\ell(V)) \cong \frac{\Bbbk[X_1, \ldots, X_n]}{\langle E_1, \ldots, E_n \rangle}.$$

where E_i is the i-th fundamental symmetric polynomial.

- The problem is, how to express the cells in term of the Chern classes?

The Schubert Polynomials

Topology remind V

Theorem

For a locally trivial fibre bundle $E \xrightarrow{\pi} B$ with fibre, base space and total space compact and smooth, let X be a cell of B, then the map induced by dual and cohomology map

$$H_*(B) \to H_{*+\dim F}(E)$$

maps $[X]$ to $[\pi^{-1}(X)]$.

- This fact can be proven by the Serre–Leray spectral sequence. The algebraic version can be proven by the Borel–Moore homology.

Schubert cells again

- Note that $\pi^{-1}(BwP/P) = BwsB/B \cup Bw^{-1}B/B$.
- We can compute

$$\begin{aligned}
\pi^{-1}(\overline{BwP/P}) &= \pi^{-1}\left(\bigcup_{v\leq w} BwP/P\right) \\
&= \bigcup_{v\leq w} \pi^{-1}(BwP/P) \\
&= \bigcup_{v\leq w} BwB/B \cup Bws_iB/B \\
&= \begin{cases} Bws_iB/B, & \ell(ws_i) = \ell(w)+1, \\ \text{lower dimensional subset}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Demazure operator IV

- As a result,

$$\begin{array}{ccccc} H_{*-2}(G/B) & \longrightarrow & H_{*-2}(G/P) & \longrightarrow & H_*(G/B) \\ [\overline{BwB/B}] & \longmapsto & [\overline{BwP/P}] & \longrightarrow & \begin{cases} [\overline{Bws_iB/B}], & \ell(ws_i)=\ell(w)+1, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

- If we denote the $[X_w]$ for the image of the cell $\overline{Bw_0wB/B}$ under the Poincaré duality, that is, $\left\langle [X_v], [\overline{Bu^{-1}B/B}] \right\rangle = \begin{cases} 1, & vu = w_0, \\ 0 & \text{otherwise.} \end{cases}$ then

$$\partial_i[X_w] = \begin{cases} [X_{ws_i}], & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Demazure operator IV

Theorem

The connection between two description of $H^*(G/B)$ are connected by

$$[X_{w_0}] = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1} \mod \langle E_1, \ldots, E_n \rangle .$$

Sketch of the proof.

- Firstly $[X_{w_0}] = \lambda x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}$.
- Then for any reduced expression for ω_0, for example

$$\omega_0 = (s_n \cdots s_1)\cdot(s_{n-1}\cdots s_2)\cdot\cdots\cdot(s_{n-1}s_{n-2})\cdot s_{n-1},$$

the corresponding

$$(\partial_n \cdots \partial_1)\cdot(\partial_{n-1}\cdots\partial_2)\cdot\cdots\cdot(\partial_{n-1}\partial_{n-2})\cdot\partial_{n-1} = \frac{\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\sigma}{\prod_{i<j}(X_i - X_j)}$$

maps $x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}$ to 1. Then $[X_e] = 1 = \lambda$.

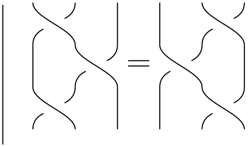
Schubert polynomials

- For each permutation $w \in \mathfrak{S}_n$, there is a unique polynomial $\mathfrak{S}_w$ such that
$$[X_w] = \mathfrak{S}_w(X_1, \dots, X_n)$$
with each monomial of $\mathfrak{S}_w$ appearing strictly lower than $x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}$. This is called the **Lascoux and Schützenberger's Schubert polynomial**.
- Then $\deg \mathfrak{S}_w = \ell(w)$, and
$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i}, & \ell(ws_i) = \ell(w) - 1, \\ 0, & \text{otherwise.} \end{cases}$$
- The $\mathfrak{S}_w$ is stable under the recognization of $\mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}$, so it is well-defined for $\mathfrak{S}_\infty = \bigcup_{n \geq 1} \mathfrak{S}_n$.

Schubert polynomials

- Note that the operator ∂_i satisfies the nil-braid relation

$$\begin{gathered}\partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1}, \\ |i-j| \geq 2, \quad \partial_i\partial_j = \partial_j\partial_i, \\ \partial_i^2 = 0.\end{gathered}$$

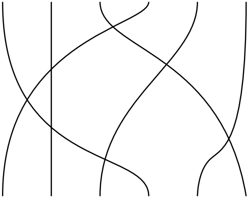


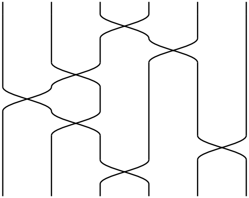
- Let $v \in \mathfrak{S}_n$ be any permutation, and $v = s_{i(1)} \cdots s_{i(k)}$ a reduced word, we define the operator

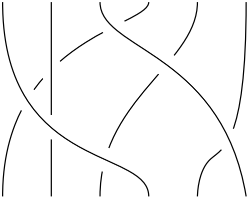
$$\partial_v = \partial_{i(1)} \cdots \partial_{i(k)}, \qquad \text{(reducing degree by } \ell(v))$$

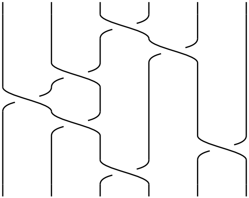
this does not depend on the choice of the reduced word.

- Warning: maybe confusing notation $\partial_{s_{ij}}$ is not ∂_{ij} in general for example $\partial_{s_{13}} = \partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.









Nil-Hecke algebra

- Let us introduce the nil-Hecke algebra

$$\mathrm{NH}_n = \Bbbk \langle \partial_i \rangle_{1\le i\le n-1} \Big/ \left\langle \begin{array}{c} \partial_i\partial_{i-1}\partial_i = \partial_{i-1}\partial_i\partial_{i-1}, \\ |i-j|\ge 2, \quad \partial_i\partial_j = \partial_j\partial_i, \\ \partial_i^2 = 0. \end{array} \right\rangle$$

- It acts on $\Bbbk[X_1,\dots,X_n]$, and nearly equivalent to the definition, $\mathfrak{S}_w$ is given by

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}\mathfrak{S}_{w_0}, \qquad \mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}.$$

A generating function for Schubert polynomials

Theorem (Fomin and Stanley 1993)

The coefficient of ∂_w of

$$\mathfrak{S}(x) = \begin{matrix} (1+x_1\partial_{n-1}) & \cdots & (1+x_1\partial_2) & (1+x_1\partial_1) \\ & \ddots & \vdots & \vdots \\ & & (1+x_{n-2}\partial_{n-1}) & (1+x_{n-2}\partial_{n-2}) \\ & & & (1+x_{n-1}\partial_{n-1}) \end{matrix}$$

is the Schubert polynomial $\mathfrak{S}_w(x)$.

Sketch of the proof.

- By analysing $A_i(x) = (1+x\partial_{n-1})\cdots(1+x\partial_i)$, and induction on ℓ.

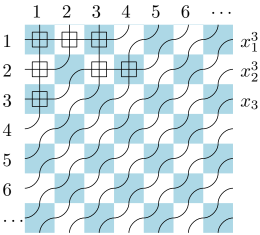
The Combinatorics III

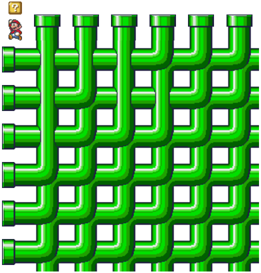
- It is suggested to use the **pipe dream** to expend the brackets above. A **pipe dream** for w is a filling of the board with pipes $+$ and  connected left i to upper $w(i)$ such that no pair of pipes cross twice.
- For a pipe dream π, define its weight

$$\mathtt{wt}(\pi) = \prod_{+\in\pi} x_{\text{the row number of the } +}$$

$$w = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,6\,1\,3\,5 \end{pmatrix}$$

$$\mathtt{wt} = x_1^3 x_2^3 x_3$$





The Combinatorics III

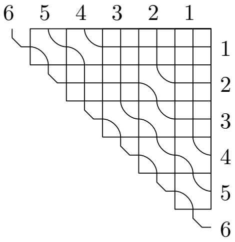
Theorem (Bergeron and Billey, 1993)

For a permutation $w \in \mathfrak{S}_\infty$,

$$\mathfrak{S}_w(x) = \sum_{\text{pipe dream } \pi \text{ for } w} \mathtt{wt}(\pi).$$

The proof.

$$\begin{array}{ccccc} 1 & 1 & x_1\partial_3 & x_1\partial_2 & x_1\partial_1 \\ & x_2\partial_5 & x_2\partial_4 & 1 & x_2\partial_2 \\ & & 1 & 1 & x_3\partial_3 \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$



Remarks, Applications and References

Remarks

- For any reductive groups G, we have the same clue to compute $H^*(G/B)$ for B the Borel subgroup. Actually,

$$G/B = \{\text{Borel subalgebra } \mathfrak{b} \subseteq \mathfrak{g} = \mathrm{Lie}(G)\}.$$

- For positive characteristic field $\mathbb{k}$, it can be shown that the kernel of ψ is still generated by a regular sequence, and $H^*(G/B)$ is freely over the image of ψ (due to Kac).

Remarks

- One can consider the set of the k partial flags, i.e. the length k flags in dimension n space $\mathcal{F}\ell_k(\mathbb{C}^n)$, and the Grassmanians $\mathcal{G}r(\mathbb{C}^n, k)$.
- Note that

$$\mathcal{F}\ell_k(\mathbb{C}^n) = \mathrm{GL}_n \Big/ \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & * & * & \cdots & * \\ & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{pmatrix}$$

$$\mathcal{G}r(\mathbb{C}^n, k) = \mathrm{GL}_n \Big/ \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \\ & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \end{pmatrix}$$

All of them can be computed by both cell method and the fibre bundle method.

Grassmanian

- Especially, for the Grassmanian case, consider the map

$$\mathcal{F}\ell(\mathbb{C}^n) \to \mathcal{G}r(\mathbb{C}^n, k),$$

Clearly, $BwpB/B$ is mapped to $BwpP/P = BwP/P$, where $p \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$.

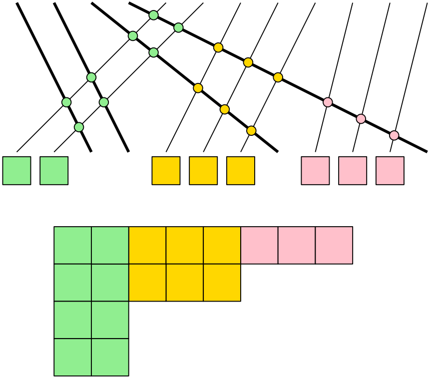
- So the homology map is surjective and splits, with

$$H_*(\mathcal{G}r(\mathbb{C}^n, l)) = \left\{ [\overline{Bw^{-1}B/B}] : \begin{matrix} \ell(w^{-1}) \text{ is maximal among} \\ w^{-1}\mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{matrix} \right\}.$$

$$\begin{aligned} H^*(\mathcal{G}r(\mathbb{C}^n, l)) &= \left\{ [X_w] : \begin{matrix} \ell(w) \text{ is minimal among} \\ w\mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{matrix} \right\} \\ &= \left\{ \mathfrak{S}_w : \begin{matrix} \ell(w) \text{ is minimal among} \\ w\mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{matrix} \right\}. \end{aligned}$$

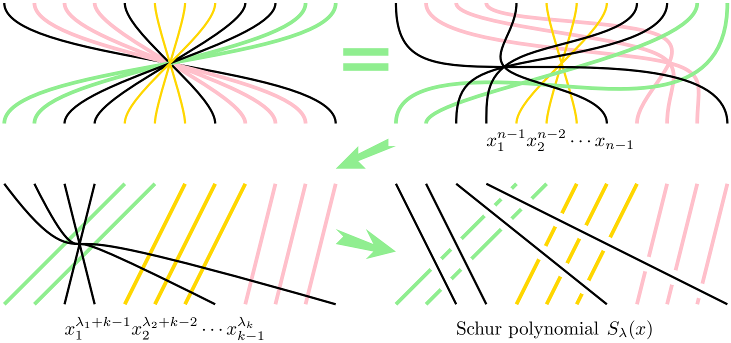
Grassmanian

- Such permutation is so-called a **shuffle**, and determined by a partition (geometrically, the corresponding Schubert cell)



Grassmaninan

- $\partial_i(\cdot\cdot\cdot X_i^d X_{i+1}^{d-1}\cdot\cdot\cdot) = \cdot\cdot\cdot X_i^{d-1} X_{i+1}^{d-1}\cdot\cdot\cdot.$
- $\partial_{w_0\in\mathfrak{S}_k} = \frac{1}{\prod_{1\le i<j\le k}(X_i - X_j)}\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma\sigma.$



Demazure operator V

- We can also define the push forward for $G/B \xrightarrow{\pi} G/P_i$ (in algebraic sense). For a locally trivial bundle $\mathcal{L}$, we can define the **Demazure operator** on $\mathcal{L}$
$$\partial_i \mathcal{L} = \pi^*(\pi_*(\mathcal{L})).$$
- For any B-equivariant bundle $\mathcal{L}$,
$$\partial_i(G \times_B \Gamma(\Sigma_w, \mathcal{L})) = G \times_B \Gamma(\Sigma_{s_i w}, \mathcal{L}),$$
where $\Sigma_w = \overline{Bw^{-1}B/B}$.

Demazure character formula

Theorem (Borel–Weil)

For $\underline{\mathbb{C}}\lambda = \begin{bmatrix} G\times_B\mathbb{C}\lambda \\ \downarrow \\ G/B \end{bmatrix}$, $\Gamma(\underline{\mathbb{C}}\lambda) = \begin{cases} L_{w_0\lambda}, & w_0\lambda \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases}$ *where* L_λ *is the irreducible representation of maximal weight* λ.

Theorem (Demazure)

For $\mathcal{L} = \underline{\mathbb{C}}w_0\lambda$, *denote* $D_w = \Gamma(\Sigma_w, \mathcal{L})$. *Then the character of* T *can be described by (written as a function on* $\mathfrak{t}$*)*

$$\mathrm{ch}(D_{s_i w}) = \frac{\mathrm{ch}(D_w) - e^{x_i - x_{i+1}} s_i \cdot \mathrm{ch}(D_w)}{1 - e^{x_i - x_{i+1}}} \quad \text{if } \ell(s_i w) = \ell(s) + 1.$$

In particular, costing some computation, $\Gamma(\mathcal{L}) = D_{w_0}$ *has the same expression as the Weyl character formula.*

- Here is basic calculus for SL_2, $e = e^{x_1}$, then $e^{x_1 - x_2} = e^{2x_1}$ since $x_1 + x_2 = 0$.

$$\begin{aligned} e^{-n} &\xrightarrow{d} \frac{e^{-n} - e^2 e^n}{1 - e^2} \\ &= e^{-n} + \cdot \ \cdot \ \cdot + e^n \\ e^{-n} + \cdot \ \cdot \ \cdot + e^n &\xrightarrow{d} \frac{1 - e^2}{1 - e^2}(e^{-n} + \cdot \ \cdot \ \cdot + e^n) \\ &= e^{-n} + \cdot \ \cdot \ \cdot + e^n \end{aligned}$$

$$100\cdot \Big\vert \cdot 000 \xrightarrow{d} 111\cdot \Big\vert \cdot 111$$

$$111\cdot \Big\vert \cdot 111 \xrightarrow{d} 111\cdot \Big\vert \cdot 111$$

- Then we can do the calculation for SL_3,

$$\begin{array}{c} 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 0\ 0 \end{array} \xrightarrow{d_1} \begin{array}{c} 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 1 \end{array} \xrightarrow{d_2} \begin{array}{c} 1\ 0\ 0 \\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ 1\ 1\ 1\ 1\ 1\ 1 \end{array} \xrightarrow{d_1} \begin{array}{c} 1\ 1\ 1 \\ 1\ 2\ 2\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 2\ 3\ 3\ 2\ 1 \\ 1\ 2\ 3\ 3\ 3\ 2\ 1 \\ 1\ 2\ 3\ 3\ 3\ 3\ 2\ 1 \\ 1\ 2\ 2\ 2\ 2\ 2\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}$$

Summary

$H^*(\mathcal{F}\ell) = ?$

- cellular strucure — the basis
- $\updownarrow$ Schubert polynomials
- Chern class — multiplication

$\ggg \star$ <u>DEMAZURE OPERATORS</u> $\star \lll$

pull-push, on Chern class, on cells, on polynomials, on vector bundles

(Geometry△□) $H^*(\mathcal{G}r)$—two applications—$\chi(L_\lambda)$ (1=0 Algebra)

Questions motivating next lecture

- It turns out to be very useful to consider the equivariant (co)homology. How to compute it for G/B? More exactly,

$$H_B^*(G/B) = H^*(E_B \times_B G/B), \quad H_G^*(G/B) = H^*(E_G \times_G G/B),$$

where E_* the classifying space for $*$.

- There is a perfect pairing

$$\mathrm{NH}_n \times H^*(G/B) \to \Bbbk \qquad (\partial_u, \mathfrak{S}_v) \longmapsto \partial_u \mathfrak{S}_v(0) = \delta_{uv}.$$

So $\mathrm{NH}_n = H_*(G/B)$ with $\partial_w \leftrightarrow [\overline{Bw_0w^{-1}B/B}]$. It indicates that there will be a product over $H_*(G/B)$. On the other hand, the Hecke algebra in philosophy "should" be interpolated as a convolution. How to realize it geometrically?

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- Bergeron and Billey. RC-Graphs and Schubert Polynomials.
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THANKS.

$\sim$ § **Next Lecture would be More Interesting** § $\sim$