Symplectic Calculus

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∼ § Notations of Manifolds § ∼

Let M be a manifold, we denote $\mathcal{C}(M)$ the structure algebra, and

TM	tangent bundle	T^*M	cotangent bundle
T_xM	tangent space at x	T_x^*M	cotangent space at x
$\mathfrak{X}(M)$	global section of TM = all vector fields over M;	$\Omega^1(M)$	global section of T^*M = all 1-forms over M.

We denote $X_x \in T_xM$ the value of $X \in \mathfrak{X}(M)$ at $x \in M$. We denote $\Omega^k(M)$ the space of k-forms over M. There is a natural pairing of

$$\Omega^k(M) \otimes T_xM^{\otimes k} \to \mathbb{k} \quad \text{or} \quad \Omega^k(M) \otimes \mathfrak{X}(M)^{\otimes k} \to \mathcal{C}(M).$$

We will use $\omega(X_1, \ldots, X_k) = \langle \omega, X_1 \otimes \ldots \otimes X_k \rangle$ to denote them.

Over $\Omega^*(M)$, there is

$$\begin{array}{cccll} d & \Omega^k(M) & \to & \Omega^{k+1}(M) & \text{differential} \\ L_X & \Omega^k(M) & \to & \Omega^k(M) & \text{Lie derivative} \\ i_X & \Omega^k(M) & \to & \Omega^{k-1}(M) & \text{inner product} \end{array}$$

where $X \in \mathfrak{X}(M)$. They satisfies **Cartan magic formula**

$$L_X = i_X \circ d + d \circ i_X.$$

Over $\mathfrak{X}(M)$, there is a Lie bracket $[X, Y] = XY - YX$.

If we have a local coordinate

$$(x_1, \ldots, x_n) : {}^{M \supseteq} U \to \mathbb{k}^n.$$

We will denote

$$\begin{aligned}
\frac{\partial}{\partial x_1}\Big|_x, \ldots, \frac{\partial}{\partial x_n}\Big|_x &\in T_x M, \\
\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} &\in \mathfrak{X}(U), \\
dx_1, \ldots, dx_n &\in T_x^* M \text{ or } \Omega^1(U).
\end{aligned}$$

If $M = V$ is a vector space, then there is a natural identification

$$T_x V = V, \qquad T_x^* V = V^*.$$

Under this identification, for any linear functional λ, $d\lambda = \lambda$.

For a morphism $f: X \to Y$, it induces

$$f_*: TX \to TY, \qquad f^*: \Omega^k(Y) \to \Omega^k(X).$$

Note that f_* is often denoted by df. They are adjoint under the pairing,

$$\omega(f_* X_1, \ldots, f_* X_k) = (f^* \omega)(X_1, \ldots, X_k),$$

that is,

$$\langle \omega, f_*(X_1 \otimes \cdots X_k) \rangle = \langle f^* \omega, X_1 \otimes \cdots X_k \rangle.$$

Besides, f^* commutes with d,

$$d(f^* \omega) = f^* d\omega.$$

Let G be a Lie group, denote the space of left invariant vector fields by $\mathfrak{g}$ its Lie algebra. It is also identified with $T_1 G$. For each $x \in G$, the adjoint action

$$\mathrm{Ad}_x : G \longrightarrow G \qquad y \longmapsto xyx^{-1},$$

induces

$$\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

This defines $G \to \mathrm{GL}(\mathfrak{g})$ inducing $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and this defines for $X \in \mathfrak{g}$,

$$\mathrm{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Actually, $\mathrm{ad}_X(Y) = [X, Y]$.

For any open subset $U \subseteq \mathbb{k}$, a morphism $s : U \to M$ will be called a **(parameterized) curve**. For $t \in U$, we denote the **derivative** of s by $\dot{s}(t) = s_*(\frac{\partial}{\partial t}\big|_t)$, where t is the coordinate of $U \to \mathbb{k}$.
Equivalently, for $f \in \mathcal{C}(M)$,

$$\dot{s}(t) \cdot f = \lim_{\Delta t \to 0} \frac{f(s(t + \Delta t)) - f(s(t))}{\Delta t} = \frac{\partial}{\partial t} f(s(t)).$$

We call a map f defines over an open neighborhood of $M = 0 \times M \subseteq \Bbbk \times M$ to M an **infinitesimal homoemorphism** of M if f restricting over M is identity and

$$f(x, t_1) = f(y, t_2) \Longrightarrow f_*\big(\tfrac{\partial}{\partial t}\big|_{(x,t_1)}\big) = f_*\big(\tfrac{\partial}{\partial t}\big|_{(y,t_2)}\big).$$

We identify two infinitesimal homoemorphism if they are equal in a neighborhood of $0 \times M$. Actually, we have a bijection

$$\{\text{infinitesimal homoemorphism of } M\} = \mathfrak{X}(M).$$

By the image of $\frac{\partial}{\partial t}$. The converse is given by the theory of ordinary differential equations. For a vector field $X \in \mathfrak{X}(M)$, we usually call the correspondent infinitesimal homoemorphism the **one-parameter group**, or the **flow generated** by X.

There is a differential morphism called **exponential map**

$$\exp : \mathfrak{g} \longrightarrow G$$

such that

$$\mathbb{k} \times G \to G \qquad (t, x) \longmapsto x \cdot \exp tX$$

is the infinitesimal homoemorphism correspondent to X.
Equivalently, for any $X \in \mathfrak{g}$, the map

$$e : \mathbb{k} \to G \qquad t \longmapsto \exp(tX),$$

is a group homomorphism with $\dot{e}(t) = e_*\left(\frac{\partial}{\partial t}\Big|_t\right) = X_{e(t)}$.

Assume the manifold M is acted by G smoothly. Then it defines a Lie algebra homomorphism (up to a minus due to left-right reason)

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M),$$

such that

$$\mathbb{k} \times M \to M \qquad (t, x) \longmapsto \exp tX \cdot x$$

is the infinitesimal homoemorphism correspondent to the image of X.

≫ *Questions?* ≪

∼ § Symplectic Manifolds § ∼

Definitions

- Let M be a manifold, we call $\omega \in \Omega^2(M)$ a symplectic form, if it is closed and nondegenerate at each point.
- That is, $d\omega = 0$, and as an anti-symmetric bilinear form over T_xM at each point $x \in M$, it is nondegenerate.
- In particular, $\dim M$ is even.

First Example — Symplectic Vector Spaces

- A symplectic vector space is a vector space with a nondegenerate anti-symmetric bilinear form ω.
- By linear algebra, we can take a set of basis such that

$$\omega(e_i, f_j) = \delta_{ij}, \qquad \omega(e_i, e_j) = 0, \qquad \omega(f_i, f_j) = 0.$$

- We denote $p_1 e_1 + \cdots + p_n e_n + q_1 f_1 + \cdots + q_n f_n$ by $(p_1, \ldots, p_n, q_1, \ldots, q_n)$. Then a fortiori,

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

Since formally p_i, q_j are dual basis for e_i, f_j

Second Example — Cotangent bundles

- Let M be a manifold, then there is a natural symplectic structure over T^*M.
- Firstly, there is a 1-form $\lambda \in \Omega^1(T^*M)$ called the **tautological form** with the following universal property

For any $\alpha \in \Omega^1(M)$, $\alpha^*\lambda = \alpha$.

$$\begin{array}{ccccc} \alpha \in & \Omega^1(M) & \overset{\alpha^*}{\leftarrow} & \Omega^1(T^*M) & \ni \lambda \\ & M & \underset{\alpha}{\rightarrow} & T^*M & \end{array}$$

- Then we define $\omega = d\lambda \in \Omega^2(T^*M)$ to be the symplectic form.

Second Example — Cotangent bundles (continued)

- Locally, if $(q_1, \ldots, q_n)$ is a local coordinate over $U \subseteq M$, then there is a natural coordinate $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ of T^*M, presenting

$$p_1 dq_1 + \cdots + p_n dq_n \in T^*_{(q_1,\ldots,q_n)}(U).$$

- By the universal property, λ locally must be of the form

$$p_1 dq_1 + \cdots + p_n dq_n.$$

So $\omega = d\lambda$ locally looks like

$$dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n,$$

which is nondegenerate.

Tips — Why it satisfies the universal property?

- Firstly, it is unique if exists since

$$\bigcap_{\alpha\in\Omega^1(M)} \ker[\Omega^1(T^*M)\xrightarrow{\alpha^*}\Omega^1(M)] = 0.$$

  This follows the intuitively trivial fact that for each point $x\in M$ and $\alpha_0\in T_x^*M$ $\sum_{\alpha|_x=\alpha_0}\operatorname{im}[\alpha_* : T_xM\to T_{\alpha_0}(T^*M)] = T_{\alpha_0}(T^*M)$.
- Stare at the following diagram

$$\begin{array}{rcccl} p_1dq_1+\ldots+p_ndq_n\in & T^*U & = & T^*U & \ni \lambda|_* \\ & \downarrow & & \downarrow & \\ p_1dq_1+\ldots+p_ndq_n\in & T^*U' & = & T^*U' & \ni \lambda|_* \\ & \downarrow & & \downarrow & \\ (p_1,\ldots,p_n,q_1,\ldots,q_n)\in & \mathbb{k}^n\times U' & \to & T^*U' & \ni \lambda|_* \end{array}$$

  where $U'\subseteq\mathbb{k}^n$ is the image of U under the coordinate.

Second Example — Cotangent bundles (continued)

- It is also suggested to do some computation.
- Consider the following map

$$\begin{array}{ccc} T^*M & \overset{\text{bundle projection } \rho}{\longleftarrow} & T(T^*M) \\ \text{bundle projection } \pi \downarrow & & \downarrow \pi_* \\ M & \overset{\text{bundle projection}}{\longleftarrow} & TM \end{array}$$

  Then we define $\lambda(X) = \langle \rho(X), \pi_*(X) \rangle$.
- By Cartan magic formula, for $X, Y \in \mathfrak{X}(T^*M)$,

$$\omega(X, Y) = d\lambda(X, Y) = X\lambda(Y) - Y\lambda(X) - \lambda([X, Y]).$$

Tips — Why it follows from diagram chasing?

- Stare at the following diagram

$$
\begin{array}{ccc}
M & \overset{\text{bundle projection}}{\longleftarrow} & TM \\
\alpha \downarrow & & \downarrow \alpha_* \\
T^*M & \overset{\text{bundle projection } \rho}{\longleftarrow} & T(T^*M) \\
\text{bundle projection } \pi \downarrow & & \downarrow \pi_* \\
M & \overset{\text{bundle projection}}{\longleftarrow} & TM
\end{array}
$$

$$\alpha^*\lambda(\xi) = \lambda(\alpha_*\xi) = \langle \rho(\alpha_*\xi), \pi_*(\alpha_*\xi) \rangle = \langle \alpha_x, \xi \rangle$$

$\gg$ *Questions?* $\ll$

$\sim$ § Poisson Structure § $\sim$

Hamiltonians

- Let $H \in \mathcal{C}(M)$. Since ω is nondegenerate, there is an $X_H \in \mathfrak{X}(M)$ called the **Hamiltonian vector field** of H, such that
$$\omega(-, X_H) = dH.$$
Or, equivalently, $dH = -i_{X_H}\omega$.
- We say a vector field $X \in \mathfrak{X}(M)$ is **symplectic** if $L_X\omega = 0$. Or equivalently, the infinitesimal homoemorphism preserving ω.
- We say a vector field $X \in \mathfrak{X}(M)$ is **Hamiltonian** if there exists $H \in \mathcal{C}(M)$, such that $X_H = X$.

Poisson Bracket

- For $f, g \in \mathcal{C}(M)$, define the **Poisson Bracket**

$$\{f, g\} = \omega(X_f, X_g) \in \mathcal{C}(M).$$

- Note that

$$\omega(X_f, X_g) = dg(X_f) = X_f g \qquad \omega(X_f, X_g) = -\omega(X_g, X_f) = -X_g f.$$

- In particular,

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

  A commutative ring with a Lie algebra structure with this property is called a **Poisson algebra**.

Theorem

We have the following exact sequence of Lie algebra

$$0 \to H^0(M) \to C^\infty(M) \overset{H \mapsto X_H}{\to} \mathfrak{X}^\omega(M) \overset{X \mapsto i_X\omega}{\to} H^1(M) \to 0,$$

where $\mathfrak{X}^\omega(M)$ is the space of symplectic vector space. The space $H^0(M)$ and $H^1(M)$ has the trivial Lie algebra structure.

The proof

- Firstly, $L_X\omega = d \circ i_X\omega + i_X \circ d\omega = d \circ i_X\omega$. So for $H \in \mathcal{C}(M)$, the Hamiltonian vector field X_H is symplectic; a vector field $X \in \mathfrak{X}(M)$ is symplectic if and only if $i_X\omega$ is closed.
- Secondly, $i_{X_H}\omega = -dH$. So a vector field $X \in \mathfrak{X}(M)$ is Hamiltonian if and only if $i_X\omega$ is exact.
- Since ω is nondegenerate, $X \mapsto i_X\omega$ is surjective.
- For a symplectic vector field X, $d(Xf) = L_X df = L_X\omega(-, X_f) = \omega(-, [X, X_f])$. In particular, $X_{\{f,g\}} = [X_f, X_g]$.

Example

- Let M be a manifold, and T^*M with symplectic structure as we stated.
- Pick a local coordinate $(p_1, \dots, p_n, q_1, \dots, q_n)$. Under $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$,

$$X_{p_i} = \frac{\partial}{\partial q_i}, \qquad X_{q_i} = -\frac{\partial}{\partial p_i}.$$

- The Poisson bracket of $f, g \in \mathcal{C}(T^*M)$ is locally given by

$$\{f, g\} = \sum \left(\frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Since $\begin{cases} df = \sum \frac{\partial f}{\partial p_i} dp_i + \sum \frac{\partial f}{\partial q_i} dq_i \\ dg = \sum \frac{\partial g}{\partial p_i} dp_i + \sum \frac{\partial g}{\partial q_i} dq_i \end{cases}.$

Digression

- Consider the space of differential operators over M, that is, locally with coordinate $(q_1, \ldots, q_n)$ over U, like

$$\mathrm{Diff}(U) = \left\{ \sum_{\mathbf{k}=(k_1,\cdots,k_n)} f_{\mathbf{k}}(q) \frac{\partial^{|\mathbf{k}|}}{\partial q_1^{k_1} \cdots \partial q_n^{k_n}} \text{ (finite sum)} \right\}$$

- We can define for $D = \sum_{|\mathbf{k}|} f_{\mathbf{k}}(q) \frac{\partial^{|\mathbf{k}|}}{\partial q^{\mathbf{k}}} \in \mathrm{Diff}(U)$ its symbol $\sigma(D) \in \mathcal{C}(T^*U)$ by

$$\sigma(D) = \sum_{\mathbf{k}} f_{\mathbf{k}}(q) p^{\mathbf{k}},$$

simply changing $\frac{\partial}{\partial q_i}$ to p_i.

Digression

- Let $\mathrm{Diff}^{\leq n}$ of the differential operators with order $\leq n$, then there is a Poisson algebra structure over $\mathrm{gr}\,\mathrm{Diff}^{\bullet}(M)$, induced by commutator,

$$\mathrm{gr}^k \mathrm{Diff}^{\bullet}(M) \times \mathrm{gr}^h \mathrm{Diff}^{\bullet}(M) \longrightarrow \mathrm{gr}^{k+h-1} \mathrm{Diff}^{\bullet}(M).$$

- Actually, through σ, it coincides with the Poisson structure over $\mathcal{C}(T^*M)$. Since due to Leibniz rule, it suffices to check the generator. Under $\omega = \sum dp_i \wedge dq_i$, $X_{p_i} = \frac{\partial}{\partial q_i}$. It is easy to check now

$$\begin{aligned} \{\sigma(\tfrac{\partial}{\partial q_i}), \sigma(f)\} &= \{p_i, f\} = \tfrac{\partial f}{\partial q_i}; \\ \sigma\big(\big[\tfrac{\partial}{\partial q_i}, f\big]\big) &= \sigma(\tfrac{\partial}{\partial q_i} f - f \tfrac{\partial}{\partial q_i}) = \sigma(\tfrac{\partial f}{\partial q_i}) = \tfrac{\partial f}{\partial q_i}. \end{aligned}$$

≫ *Questions?* ≪

∼ § Moment Maps § ∼

Hamiltonian action

Let M be a symplectic manifold with a smooth G-action.

- We say it is **symplectic**, if ω is preserved.
  Equivalently, the induced map $\mathfrak{g} \to \mathfrak{X}(M)$ factor through symplectic vector field
  $$\mathfrak{g} \to \mathfrak{X}(M)^{\omega} \to \mathfrak{X}(M).$$
- We say it is **Hamiltonian**, if the induced map $\mathfrak{g} \to \mathfrak{X}(M)$ factors through
  $$\mathfrak{g} \xrightarrow{H} \mathcal{C}(M) \xrightarrow{H \mapsto X_H} \mathfrak{X}(M),$$
  where we assume H to be Lie algebra homomorphism.

Moment map

- For a Hamiltonian action of G on M, we call

$$\mu : M \longrightarrow \mathfrak{g}^* \qquad x \longmapsto [X \mapsto H(X)(x)]$$

the **moment map**.

Theorem

Assume G is connected, then the moment map is G-equivariant. Besides,

$$\mu^* : S\mathfrak{g} = \Bbbk[\mathfrak{g}^*] \to \mathcal{C}(M)$$

preserves the Poisson bracket.

The proof

- Firstly, to show it preserves Poisson bracket, it suffices to check at the generator due to the Leibniz rule.
- For $X \in \mathfrak{g} = \Bbbk[\mathfrak{g}^*]^1$,

$$\mu^*(X)(x) = X(\mu(x)) = X(H(-)(x)) = H(X)(x).$$

So μ^* is induced by H. So it follows from definition.

- To show it is equivariant,

$$\forall g \in G, x \in M, \qquad \mu(gx) = \mathrm{Ad}_g\, \mu(x),$$

it suffices to show the infinitesimal version

$$\forall X \in \mathfrak{g}, x \in M, \qquad \mu_*(X_x^{\#}) = \mathrm{ad}_X\, \mu(x),$$

where $X^{\#} = X_{H(X)} \in \mathfrak{X}(M)$ is the image of $X \in \mathfrak{g}$.
- Let $Y \in \mathfrak{g} \in T^*\mathfrak{g}^* = \mathfrak{g}$,

$$\begin{aligned}
\mu_*(X_x^{\#})(Y) &= (X_x^{\#})(\mu^* Y) = (X_x^{\#})(H(Y)) \\
&= \{H(X), H(Y)\}(x) = H([X, Y])(x), \\
\mathrm{ad}_X\, \mu(x)(Y) &= \left[\,\mathrm{ad}_X(H(-)(x))\right](Y) \\
&= \left[H([X, -])(x)\right](Y) = H([X, Y])(x).
\end{aligned}$$

A Question

- For a vector field $X \in \mathfrak{X}(M)$, it induces a infinite small homoemorphism of M, then induces a infinite small homoemorphism of T^*M, so finally defines a vector field $\tilde{X} \in \mathfrak{X}(T^*M)$. How to compute it?
- Denote $g_t : M \to M$ a family of homoemorphism, denote $g_t^* : T^*M \to T^*M$. Let $f \in \mathcal{C}(T^*M)$.
- Firstly, we pick coordinate, $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ over T^*U. Assume $X = \sum X^i(q)\frac{\partial}{\partial q_i}$.
- To do the computation, it suffices to compute $\lim_{t \to 0} \frac{f(g_t^*\alpha) - f(\alpha)}{t}$ for $\alpha \in T_x^*M$ for $f = p_i$ and q_i.

- For the case $f = q_i$, it is clear, the answer is X^i. More exactly,

$$\tilde{X}^i(p, q) = X^i(p).$$

- To do the rest case, we pick an $\alpha \in \Omega^1(U)$, and $x \in M$, consider

$$f(g_t^*(\alpha_x)) = f((g_{-t}^*\alpha)_{g_t x}).$$

Then using $H(t, s) = f((g_t^*\alpha)_{g_s x})$, and taking in $f = p_i$,

$$\begin{aligned} H_t(0,0) &= p_i(L_X\alpha)_x = \left\langle \frac{\partial}{\partial q_i}, L_X\alpha \right\rangle_x \\ H_s(0,0) &= (X(p_i(\alpha_\bullet)))(x) = X \left\langle \frac{\partial}{\partial q_i}, \alpha \right\rangle_x . \end{aligned}$$

- So

$$\lim_{t\to 0}\frac{f(g_t^*\alpha)-f(\alpha)}{t} = -\left\langle \frac{\partial}{\partial q_i}, L_X\alpha \right\rangle_x + X\left\langle \frac{\partial}{\partial q_i}, \alpha \right\rangle_x$$
$$= \left\langle [X, \frac{\partial}{\partial q_i}], \alpha \right\rangle_x$$
$$= \left\langle -\sum_j \frac{\partial X^j}{\partial q_i}\frac{\partial}{\partial q_j}, \alpha \right\rangle_x = -\sum_j \frac{\partial X^j}{\partial q_i} p_j(\alpha_x).$$

- As a result,

$$\tilde{X} = \sum X^i(q)\frac{\partial}{\partial q_i} - \sum_i \left(\sum_j \frac{\partial X^j}{\partial q_i} p_j \right) \frac{\partial}{\partial p_i}$$

- On the other hand, under $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$,

$$X_{p_i} = \frac{\partial}{\partial q_i}, \qquad X_{q_i} = -\frac{\partial}{\partial p_i}.$$

- The symbol $\sigma(X) \in \mathcal{C}(T^*M)$ of X, that is by natural pairing, $\alpha_x \mapsto \langle X, \alpha_x \rangle$,

$$\sigma(X) = \sum X^i(q) p_i.$$

So

$$d\sigma(X) = \sum X^i(q) dp_i + \sum_i \left(\sum_j \frac{X^j(q)}{\partial q_i} \right) dq_i.$$

- It corresponds to the Hamiltonian vector field

$$X_{\sigma(X)} = \sum X^i(q) \frac{\partial}{\partial q_i} - \sum_i \left(\sum_j \frac{\partial X^j}{\partial q_i} p_j \right) \frac{\partial}{\partial p_i}$$

exactly $\tilde{X}$.

Example

- Let M be a smooth manifold acted by G. Then the action of G on T^*M is Hamiltonian, with moment map

$$\mu : T^*M \longrightarrow \mathfrak{g}^* \qquad \alpha_x \longmapsto \left[X \mapsto \left\langle X^{\#}, \alpha_x \right\rangle \right].$$

- In particular, let $\mathbb{R}^3$ acts on $T^*\mathbb{R}^3$ by translation, the moment map is

$$\mu : T^*\mathbb{R} \longrightarrow (\mathbb{R}^3)^* \qquad (\mathbf{p}, \mathbf{q}) \longmapsto [\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{p} \rangle].$$

So under the inner product, it is actually given by **p** the linear momentum.

Example

- In particular, let SO_3 act on $T^*\mathbb{R}^3$ by rotation, the moment map is

$$\mu : T^*\mathbb{R} \longrightarrow (\mathfrak{so}_3)^* \qquad (\mathbf{p}, \mathbf{q}) \longmapsto [A \mapsto \langle A\mathbf{q}, \mathbf{p}\rangle].$$

- But for $\mathfrak{so}_3$, it is famous that

$$(\mathfrak{so}_3, [,]) \cong (\mathbb{R}^3, \times) \qquad \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \longleftrightarrow (a_1, a_2, a_3).$$

So under the inner product over $\mathbb{R}^3$, the moment map is $\mathbf{q} \times \mathbf{p}$, the angular momentum.

Example

- Consider two vector space V, W, and $G = \mathrm{GL}(V)$. Note that by the trace pairing, $\mathrm{Hom}(V, W)^* = \mathrm{Hom}(W, V)$. The map induced by left multiplication

$$G \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, W) \qquad (g, f) \mapsto (f \circ g^{-1})$$

induces

$$\mathfrak{gl}(V) \longrightarrow \mathfrak{X}^*(\mathrm{Hom}(V, W)) \qquad X \longmapsto [f \mapsto -f \circ X].$$

So the moment map is

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V) \longrightarrow \mathfrak{gl}^*(V) \qquad (f, g) \longmapsto [X \longmapsto \mathrm{tr}(-f \circ$$

So under the trace pairing, $\mathfrak{gl}^*(V) = \mathfrak{gl}(V)$, it is given by $(f, g) \mapsto -gf$.

Example

- Similarly, consider the map induced by adjoint action

$$G \times \mathrm{End}(V) \longrightarrow \mathrm{End}(V) \qquad (g, f) \mapsto (g \circ f \circ g^{-1})$$

induces

$$\mathfrak{gl} \longrightarrow \mathfrak{X}^*(\mathrm{End}(V)) \qquad X \longmapsto [f \mapsto [X, f]].$$

So the moment map is

$$\begin{array}{rcl} \mathrm{End}(V) \oplus \mathrm{End}(V) & \longrightarrow & \mathfrak{gl}^* \\ (f, g) & \longmapsto & [X \longmapsto \mathrm{tr}([X, f] \circ g)] = \mathrm{tr}(X \circ [f, g]). \end{array}$$

So under the trace pairing, $\mathfrak{gl}^*(V) = \mathfrak{gl}(V)$, it is given by $(f, g) \mapsto [f, g]$.

Digression

- If G is compact and semisimple, so $\mathfrak{g}$ is semisimple, then the moment map always uniquely exists. Since $H^1(\mathfrak{g};\mathbb{R}) = 0$ implies $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, but

$$\begin{aligned} d\omega(X,Y) &= di_Y i_X\omega = L_Y i_X\omega - i_Y di_X\omega \\ &= L_Y i_X\omega - i_Y L_X\omega + i_Y i_X d\omega = L_Y i_X\omega = L_Y(\omega(X,-)) \\ &= (L_Y\omega)(X,-) + \omega(L_Y X,-) = \omega(-,[X,Y]). \end{aligned}$$

In conclusion $X_{\omega(X,Y)} = \omega(-,[X,Y])$. So $\mathfrak{g}$ takes value in the Hamiltonian vector fields.

Digression

- Since $H^2(\mathfrak{g};\mathbb{R})=0$, so the first row split

$$\begin{array}{ccccccc} 0\to & H^0(M) & \longrightarrow & \text{pull back} & \longrightarrow & \mathfrak{g} & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0\to & H^0(M) & \longrightarrow & \mathcal{C}(M) & \longrightarrow & \mathfrak{X}^{\mathrm{Ham}}(M) & \to 0 \end{array}$$

So we get desired lift.

$\gg$ *Questions?* $\ll$

∼ § Hamiltonian Reduction § ∼

Calculation

- For further use, let us denote $X^{\#} \in \mathfrak{X}(M)^{\omega}$ for $X \in \mathfrak{g}$ the corresponding vector field. Note that $X^{\#} = X_{H(X)}$.
- For $X \in \mathfrak{g}$, as a function over $\mathfrak{g}^*$ then $\mu^*(X) = H(X)$, since

$$\mu^*(X)(x) = \langle X, \mu(x) \rangle = \langle X, H(-)(x) \rangle = H(X)(x).$$

- For $X \in \Omega^1(\mathfrak{g}^*) = \mathfrak{g}$, then $\mu^*(X) = dH(X)$. Since

$$\mu^*(X) = \mu^*(dX) = d\mu^*(X) = dH(X).$$

- For $X \in \mathfrak{g}$, and $v \in TM$, then $\langle X, \mu_*(v) \rangle = dH(X)(v) = \omega(v, X^{\#})$. Since

$$\langle X, \mu_*(v) \rangle = \langle \mu^* X, v \rangle = \langle dH(X), v \rangle = \omega(v, X^{\#}).$$

Moment map again

Theorem

Let M be symplectic manifold, with a Hamiltonian G-action with moment map μ. At $x \in M$,

$$\begin{aligned} \ker d\mu &= T_x(Gx)^{\omega\perp} = \left\{X \in T_x M : \substack{Y \in T(Gx) \\ \Rightarrow \omega(X,Y)=0}\right\}, \\ \operatorname{im} d\mu &= \mathfrak{g}_x^{\perp} = \left\{\lambda \in \mathfrak{g} : \substack{X \in \mathfrak{g}, X_x^{\#}=0 \\ \Rightarrow \lambda(X)=0}\right\}. \end{aligned}$$

- Note that $T_x(Gx)$ is exactly $\{Y_x^{\#} : Y \in \mathfrak{g}\}$, the assertion for kernel is clear.
- For $\lambda \in \mathfrak{g}^*$, $\lambda(X)$ can be presented by $\omega(v, X_x^{\#})$ if $X_x^{\#} = 0 \Rightarrow \lambda(X) = 0$ by nondegenerateness of ω.

Restriction

- Let p be a regular value of μ. The preimage $\mu^{-1}(p)$ is a manifold. Let $\mathbb{O}_p \subseteq \mathfrak{g}^*$ be the orbit of p.
- The restriction of ω on $X = \mu^{-1}(\mathbb{O}_p)$ is generally not nondegenerate. For $x \in M$, we should consider $\operatorname{rad} \omega|_{T_x X}$. Note that for $v \in T_x X$, then
$$\omega(v, X^{\#}) = dH(X)(v) = \langle X, \mu^*(v) \rangle = 0.$$
So $T_x(Gx) \subseteq \operatorname{rad} \omega|_{T_x X}$.
- Note that $\dim \operatorname{rad} \omega|_{T_x X} = \operatorname{codim} X = \operatorname{codim} \mathbb{O}_p = \dim\{g \in G : \operatorname{Ad}_g p = p\}$.
- To ensure $T_x(Gx) = \operatorname{rad} \omega|_{T_x X}$, and X/G a manifold we should assume the action of G is proper, and the stablizer G_p acts on X freely.

Theorem (Marsden–Weinstein)

Let M be symplectic manifold, with a proper Hamiltonian G-action with moment map μ. Let $p \in \mathfrak{g}^*$ be a regular value of μ, and the stablizer G_p acts on $\mu^{-1}(p)$ freely, then

$$\mu^{-1}(p)/G_p = \mu^{-1}(\mathbb{O}_p)/G$$

has a natural symplectic structure.

$$\begin{array}{ccccc} \mu^{-1}(p)/G_p & & & & M \\ & \nwarrow & & \nearrow & \\ & & \mu^{-1}(p) & & \end{array}$$

Theorem (Marsden–Weinstein)

Let M be symplectic manifold, with a proper Hamiltonian G-action with moment map μ. Assume G acts on $\mu^{-1}(0)$ freely, then

$$\mu^{-1}(0)/G$$

has a natural symplectic structure.

Theorem

Let M be a manifold with free proper G-action, then

$$T^*(M/G) = \mu^{-1}(0)/G.$$

≫ *Questions?* ≪

$\sim$ § Algebraic Reduction § $\sim$

Algebraic Reduction

- Now, turn everything into algebraic.

Manifolds	Varieties
Symplectic Structure	Poisson Structure

We say a variety X having a **Poisson strucuture** if it have over $\mathcal{O}_X$.

- In this case, we can also consider moment map, but since variety is of singularities, we should not only consider the regular value.

Algebraic Reduction

- Let X be a nonsingular affine algebraic variety acted by reduction group G. Now assume $\mu : T^*X \to \mathfrak{g}^*$ is the moment map.

Theorem

For χ a character, then $\mu^{-1}(0)//_\chi G$ has a Poisson structure. Besides,

$$\mu^{-1}(0)//_\chi G \to \mu^{-1}(0)//G$$

preserves Poisson structure.

The proof

- Since $\mu^{-1}(0)$ is defined by $\mu = 0$, that is $\left\{x \in X : {}^{\forall X \in \mathfrak{g},}_{\mu(x)(X)=0}\right\}$. But $\mu(x)(X) = H(X)(x)$, so $\mu^{-1}(0)//G$ is defined by

$$\left(\Bbbk[X]/\left\langle H(X) : X \in \mathfrak{g}\right\rangle\right)^G.$$

- We simply define

$$\{f, g\} = \{\hat{f}, \hat{g}\} \mod \langle H(X) : X \in \mathfrak{g}\rangle$$

where $\hat{f}, \hat{g}$ is some lift.

The proof (continued)

- To show this is well-defined, we should show $\langle H(X) : X \in \mathfrak{g} \rangle$ is stable under this bracket. For $f \in \left(\Bbbk[X] / \langle H(X) : X \in \mathfrak{g} \rangle \right)^G$,

$$\begin{aligned} \{f, H(X)g\} &= \{f, H(X)\}g + \{f, g\}H(x); \\ \{f, H(x)\} &= X^{\#} f \in \langle H(X) : X \in \mathfrak{g} \rangle . \end{aligned}$$

  This follows from G-invariance.
- Then, we also need to show $\{f, g\}$ is G-invariant. But the action of G commutes with bracket.
- The proof of $\mu^{-1}(0) /\!/_{\chi} G$ is similar, since it is locally f/g with $f, g \in \mu^{-1}(0)^{G, \chi^n}$.

Theorem

Let $X^s \subseteq X$ be the subvariety of χ-stable points, so that $X^s//_\chi G \subseteq X//_\chi G$ is a smooth subvariety; similarly, let $(\mu^{-1}(0))^s \subseteq \mu^{-1}(0)$ be the subvariety of χ-stable points, so that $\mu^{-1}(0)^s//_\chi G \subseteq \mu^{-1}(0)//_\chi G$ is smooth. Then the restriction of the Poisson bracket to $\mu^{-1}(0)^s//_\chi G$ is nondegenerate, i.e. comes from a symplectic form on it, and $\mu^{-1}(0)^s//_\chi G$ contains $T^*(X^s//G)$ as an open (possibly empty) subset.

- (By our discussion) This is not true in general, but it is true for quiver varieties, see Theorem 2.3.
- I believe the above follows from the proof of Marsden–Weinstein reduction theorem, and Luna slide theorem. In this case, stable points are the point acted freely.

≫ <u>*Questions?*</u> ≪

∼ § Lagrangian Submanifolds § ∼

Lagrangian submanifolds

- For a symplectic vector space V, a subspace $W \subseteq V$ is called $\left\{ \begin{array}{c} \textbf{isotropic} \\ \textbf{coisotropic} \\ \textbf{Lagrangian} \end{array} \right\}$ if $\left\{ \begin{array}{l} W \subseteq W^{\perp\omega} \\ W \supseteq W^{\perp\omega} \\ W = W^{\perp\omega} \end{array} \right.$.
- For a symplectic manifold M, $N \subseteq M$ a submanifold is called $\left\{ \begin{array}{c} \textbf{isotropic} \\ \textbf{coisotropic} \\ \textbf{Lagrangian} \end{array} \right\}$ if T_xN is $\left\{ \begin{array}{c} \text{isotropic} \\ \text{coisotropic} \\ \text{Lagrangian} \end{array} \right\}$ in T_xM for all $x \in N$.
- In particular, if N is Lagrangian in M, then $\dim N = \frac{1}{2}\dim M$.

Examples

- Let X, Y be symplectic manifolds, equip $X \times Y$ the symplectic form by
$$\omega_X \oplus (-\omega_Y) = \mathrm{pr}_X^* \omega_X - \mathrm{pr}_Y^* \omega_Y.$$
- Then $f : X \to Y$ is symplectic (that is, $f^* \omega_Y = \omega_X$) if and only if the graph
$$\mathrm{graph}(f) = \{(x, f(x)) \in X \times Y : x \in X\}$$
is Lagrangian in $X \times Y$.
- Since f factors through $X \cong \mathrm{graph}(f) \xrightarrow{\iota} X \times Y \xrightarrow{\mathrm{pr}_Y} Y$.
$$\iota^* \omega_{X \times Y} = \iota^* (\mathrm{pr}_X^* \omega_X - \mathrm{pr}_Y^* \omega_Y) = \omega_X - f^* \omega_Y.$$

Examples

- Let M be a manifold, and $\alpha \in \Omega(M)$ a 1-form. Then it is closed if and only if the image of $\alpha : M \to T^*M$,

$$\text{image}(\alpha) = \{(\alpha_x) \in T^*M : x \in M\}$$

is Lagrangian in T^*M.

- Since α factors through $M \to \text{image}\,\alpha \to T^*M$.

$$\alpha^*\omega = \alpha^*(d\lambda) = d(\alpha^*\lambda) = d\alpha,$$

where λ is the tautological form over T^*M.

Examples

- Let M be a manifold, N be a submanifold. Then the conormal bundle

$$N^*(N, M) = \{\alpha \in T_x^*(M) : \alpha(T_x N) = 0\} = (T_x M / T_x N)^*$$

  is Lagrangian in T^*M.
- This follows from local computation.

Example

- We keep the convention of product of symplectic form (with a minus).
- Let $f : M \to N$ be a smooth function, denote its infinitesimal graph

  $$\text{graph}(f^*) = \left\{(x, f^*\alpha, f(x), \alpha) \in T^*M \times T^*N : x \in M, \alpha \in T^*_{f(x)}N\right\}$$

  is Lagrangian.
- Actually, $f^*(T^*N) \cong \text{graph}(f^*) \subseteq T^*M \times T^*N$.

Tips

- Let $f : M \to N$, and $f^*(T^*N)$ the pull back of cotangent bundle of N. Then the pull back of tautological form of T^*N and T^*M coincide by direct computation

$$\begin{array}{ccc} f^*(T^*N) & \xrightarrow{df} & T^*M \\ \downarrow & & \\ T^*N & & \end{array}$$

- Consider $\begin{bmatrix} f^*(T^*N) & \to & T^*M \\ \downarrow & \searrow & \uparrow \\ T^*N & \leftarrow & T^*M \times T^*N \end{bmatrix}$ So $\mathrm{pr}_1^* \lambda_M - \mathrm{pr}_2^* \lambda_N$ pull back to $f^*(T^*M)$ to be zero.

Tips

- Pick local coordinates $(q_i) = (q_1, \dots, q_m)$ for M, and $(Q_j) = (Q_1, \dots, Q_n)$ for N. Assume $Q_j = f_j(q)$, and denote $f_{ji} = \frac{\partial f_j}{\partial q_i}$. Then

$$f^* : T^*_{f(q)}N \longrightarrow T^*_q M : \qquad \sum_j P_j dQ_j \longmapsto \sum_{ij} P_j(y) f_{ji}(q) dq_i$$

- So the map $f^*(T^*N) \to T^*M$ is locally given by $\left\{ \begin{smallmatrix} p_i = \sum_j P_j(y) f_{ji}(y) \\ q_i = q_i \end{smallmatrix} \right.$, so

$$\text{pull back of } p_i dq_i = \sum_j P_j(y) f_{ji}(q) dq_i.$$

- So the map $f^*(T^*N) \to T^*N$ is locally given by $\left\{ \begin{smallmatrix} P_i = P_i \\ Q_i = f_j(q) \end{smallmatrix} \right.$, so

$$\text{pull back of } P_i dQ_i = \sum_j P_j(y) f_{ji}(q) dq_i.$$

Digression

- We keep the convention of product of symplectic form (with a minus). By a twist

  $$\sigma : T^*M \times T^*N \longrightarrow T^*(M \times N) \quad (x, \alpha) \times (y, \beta) \longmapsto (x, y, \alpha, -\beta)$$

  They are identified.
- On one hand, given a symplectic map $f : T^*M \to T^*N$, we get $\mathrm{graph}(f) \subseteq T^*M \times T^*N$. On the other hand, for $F \in \mathcal{C}(M \times N)$ such that $\mathrm{graph}(dF) \subseteq T^*(M \times N)$. If they coincide, we say F is the generating function of f. Equivalently,

  $$f : (x, (d_x F)_{(x,y)}) \longmapsto (y, -(d_y F)_{(x,y)})$$

  is well-defined.

Digression

- For example, if $F \in \mathcal{C}(M \times M)$ is the generating function of $f : T^*M \to T^*M$, then the fixed point of f is one-to-one correspondent to the critical point of $\varphi(x) = F(x, x) : \mathcal{C}(M)$. Since

$$\boxed{\text{point of } T^*M \text{ with } (x, d_xF) = (y, -d_yF)} = \boxed{\text{point } x \text{ of } M \text{ such that } (d_xF)_{(x,x)} = (-d_xF)_{(x,x)}} = \boxed{\text{point } x \text{ of } M \text{ such that } (d\varphi)|_x = 0}$$

$\gg$ *Questions?* $\ll$

∼ § Symplectic Resolutions § ∼

Resolution of Singularities

- For variety $\mathcal{M}$ and nonsingular variety $\tilde{\mathcal{M}}$, a morphism

$$\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$$

  is called a **resolution of singularities** if it is proper and birational.

- For variety $\mathcal{M}$ with Poisson structure and nonsingular variety $\tilde{\mathcal{M}}$ with symplectic structure, a Poisson morphism

$$\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$$

  is called a **symplectic resolution of singularities** if it is proper and birational.

- Assume a nonsingular affine variety X is acted by a reductive group G. Denote

$$\mathcal{M}_\chi = \mu^{-1}(0)//_\chi G$$

the algebraic reduction.

Theorem

Assume $\mathcal{M}_\chi$ is connected and nonsingular and the subset of regular points $\mathcal{M}_0^{reg}$ is nonempty, then

$$\mathcal{M}_\chi \longrightarrow \mathcal{M}_0$$

is a symplectic resolution of singularities.

Example

- Let $X = \Bbbk^2$, and $G = \Bbbk^\times$. Then
$$T^*X = X \oplus X^* = \left\{ (i,j) : \Bbbk \overset{j}{\underset{i}{\rightleftarrows}} \Bbbk^2 \right\}.$$
- The action $G \times X \to X$ is taken to be $\lambda \cdot j = \lambda^{-1} j$, so $G \times T^*X \to T^*X$ is given by $\lambda \cdot (i,j) = (\lambda i, \lambda^{-1} j)$.
- The moment map,
$$T^*X \longrightarrow \underbrace{(\mathrm{Lie}\, G)^*}_{=\Bbbk} \qquad (i,j) \longmapsto -ij.$$

So $\mu^{-1}(0) = \{(i,j) : ij = 0\}$.

- Consider

$$\{(i,j) : ij = 0\}//\mathbb{k}^\times \longrightarrow \mathbb{M}_{2\times 2}(\mathbb{k}) \qquad (i,j) \longmapsto j \circ i.$$

- Claim: the image is $\{A \in \mathbb{M}_{2\times 2}(\mathbb{k}) : \det A = \operatorname{tr} A = 0\}$, and the above morphism is isomorphism. (Linear algebra)
- But

$$\{A \in \mathbb{M}_{2\times 2}(\mathbb{k}) : \det A = \operatorname{tr} A = 0\} = \left\{ \left(\begin{smallmatrix} a & c \\ b & -a \end{smallmatrix}\right) : a^2 + bc = 0 \right\}$$

  is a quadric in $\mathbb{k}^3$. It obviously has a singularity at the origin.
- Points with trivial stablizer is $\{(i,j) : i \neq 0 \text{ or } j \neq 0\}$.
- Such points lying in a closed orbit is $\{(i,j) : i \neq 0 \text{ and } j \neq 0\}$. Equivalently, $\{(i,j) : ji \neq 0\}$. So $(\mu^{-1}//\mathbb{k}^\times)^{\mathrm{reg}} = \{a^2 + bc = 0\} \setminus 0$.

- Now consider the character $\chi = \mathrm{id}$.

  $$\mathbb{k}^\times \times T^*X \times \mathbb{k} \longrightarrow T^*X \times \mathbb{k} \qquad (\lambda, i, j, z) \longmapsto (\lambda i, \lambda^{-1} j, \lambda z).$$

  So χ-semistable points $\{(i,j) : j \neq 0\}$.

- As a result,

  $$\mu^{-1}(0)/\!/_\chi \mathbb{k}^\times = \{(i,j) : j \neq 0\}/\mathbb{k}^\times.$$

  Consider the fibre

  $$\begin{array}{ccc} \{(i,j) : j \neq 0\}/\mathbb{k}^\times & \longrightarrow & \{j : j \neq 0\}/\mathbb{k}^\times \\ \| & & \| \\ \left\{(V,i) : {\scriptstyle \dim V = 1, V \subseteq \mathbb{k}^2 \atop i : \mathbb{k}^2 \to V : i|_V = 0}\right\} & \longrightarrow & \mathbb{P}^1_{\mathbb{k}}. \end{array}$$

  We see $\mu^{-1}(0)/\!/_\chi \mathbb{k}^\times$ is a line bundle over $\mathbb{P}^1_{\mathbb{k}}$.

- Actually, this is the cotangent bundle of $\mathbb{P}^1_{\Bbbk}$. (Directly or by reduction)
- Direct way: a cotangent vector at $V \in \mathbb{P}^1_{\Bbbk}$ is a functional $\Bbbk^2 \to \Bbbk$ restrict V trivially.
- Reduction way: $\mu^{-1}(0)$ contains $T^*(X^s//G)$ as open subset (from construction of quotient).
- Now, $\mu^{-1}(0)//_{\chi}\Bbbk^\times$ is nonsingular, so

$$\mu^{-1}(0)//_{\chi}\Bbbk^\times \longrightarrow \mu^{-1}(0)//\Bbbk^\times$$

is a symplectic resolution of singularities.

Springer Resolutions

- The above example is a good example of Springer resolution. Let G be a semisimple Lie group, and $\mathfrak{g}$ its Lie algebra.
- Consider the nilpotent cone

$$\mathcal{N} = \{x \in \mathfrak{g} : x \text{ is nilpotent}\}$$

- Let
$\mathcal{B} = G/B = \{\text{all Borel subgroups}\} = \{\text{all Borel subalgebras}\}$.
- Denote
$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} : x \in \mathfrak{b}\}.$$

Springer Resolutions

- Let $G = \mathrm{SL}_n$ be a semisimple Lie group, and $\mathfrak{sl}_n$ its Lie algebra.
$$\mathcal{N} = \{x \in \mathfrak{sl}_n : x \text{ is nilpotent}\}.$$
and $\mathcal{B} = \{\text{all flags in } \mathbb{k}^n\}$.
$$\tilde{\mathcal{N}} = \{(x, \mathcal{F}) \in \mathcal{N} \times \mathcal{B} : x\mathcal{F}_i \subseteq \mathcal{F}_{i-1}\}.$$
- We can change to GL_n.

Theorem

As the notations above,
(1) $\tilde{\mathcal{N}} = T^*\mathcal{B}$;
(2) The moment map is given by $\begin{bmatrix} \tilde{\mathcal{N}} \longrightarrow \mathfrak{g} \\ (x, \mathfrak{b}) \longmapsto x \end{bmatrix}$;
(3) $\tilde{\mathcal{N}} \to \mathcal{N}$ *is a symplectic resolution of singularities.*

- The tangent space at $\mathfrak{b} \in \mathcal{B}$ is $\mathfrak{g}/\mathfrak{b}$. So cotangent space at $\mathfrak{b} \in \mathcal{B}$ is, by trace pairing κ (Killing form), $\{x \in \mathfrak{g} : \kappa(x, \mathfrak{b}) = 0\} = \{x \in \mathcal{N} : x \in \mathfrak{b}\}$.
- The moment map, still by Killing form κ,

$$\mu : \tilde{\mathcal{N}} \longrightarrow \mathfrak{g}^* = \mathfrak{g} \qquad (x, \mathfrak{b}) \longmapsto [X \mapsto \kappa(x, X)] = x.$$

- Consider regular nilpotent element $x \in \mathfrak{g}$, there is only one Borel subalgebra $\mathfrak{b} \ni x$.
  In the $\mathfrak{sl}_n$ case, it is $x^{n-1} \neq 0$, so the corresponding flag is by $\ker x^i = \operatorname{im} x^{n-i}$.
- But the regular nilpotent element is open and dense. Besides, μ is proper, since it is a closed subvariety of $\mathcal{N} \times \mathcal{B}$.
- It is symplectic since moment map is Poisson.

≫ *Questions?* ≪

∼ § Kähler Manifolds § ∼

Kähler Manifolds

- A **Kälher manifold** is a manifold M with the

$$\left.\begin{array}{r}\text{complex structure } J;\\ \text{symplectic structure } \omega;\\ \text{Riemannian metric } g.\end{array}\right\}\begin{array}{c}\text{compatible}\\ \omega(\cdot,\cdot)=g(J\cdot,\cdot)\end{array},$$

where J is the product with $\mathbf{i}$. We can define a Hermitian (unitary) metric over $T^*_{\mathbb{C}}M$,

$$(x,y)=g(x,y)+\mathbf{i}\omega(x,y).$$

Projective Spaces

- Projective space $\mathbb{C}P^n$ is Kähler, by the **Fubini-Study form**, induced from the metric from the unitary metric S^{2n+1}.
- Since every complex submanifold of a Kähler manifold is Kähler, this implies that every smooth projective variety is Kähler.

First Example

- Assume there the coordinate is $(p_1, \cdots, p_n, q_1, \cdots, q_n)^\top$. There are three kinds of structures over $V = \mathbb{R}^{2n}$, they are the **inner product**

$$g = \sum q_i^2 + \sum p_i^2 \in S^2 V^*, \quad \text{that is,} \quad g(x, y) = y^{\mathbf{t}} \begin{pmatrix} 1_n & \\ & 1_n \end{pmatrix} x,$$

the **symplectic structure**

$$\omega = p_i \wedge q_i \in \Lambda^2 V^*, \quad \text{that is,} \quad \omega(x, y) = y^{\mathbf{t}} \begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix} x,$$

and the **complex structure** with identification $z_i = p_i + \mathbf{i}\, q_i$, or formally, it is given by the multiplication by $\mathbf{i}$, usually denoted by J,

$$J \cdot x = \begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix} x.$$

Computation of Moment map

- Note that $\mathrm{Sp}(V)$ acts on V. It is actually Hamiltonian.

$$\mathfrak{sp}(V) \longrightarrow \mathfrak{X}(V) \qquad A \mapsto [x \mapsto Ax].$$

By $H_A = \frac{1}{2}\omega(x, Ax)$, for $x \in V$, and $y \in T_x^* V = V$,

$$\begin{aligned} dH_A(y) &= \tfrac{1}{2}\omega(dx, Ax)(y) + \tfrac{1}{2}\omega(x, Adx)(y) \\ &= \tfrac{1}{2}\omega(y, Ax) + \tfrac{1}{2}\omega(x, Ay) \\ &= \omega(y, Ax). \end{aligned}$$

Since $\omega(x, Ay) = -\omega(Ax, y)$.

- So moment map is given by

$$\mu : V \longmapsto \mathfrak{sp}(V)^* \qquad x \mapsto \left[A \longmapsto \tfrac{1}{2}\omega(x, Ax)\right].$$

- It is better to rewrite symplectic form and Riemannian metric in term of complex form. Now we use complex coordinate.

$$\begin{aligned} g(x,y) &= \Re(y^{\mathbf{h}}x) = \tfrac{1}{2}(y^{\mathbf{h}}x + x^{\mathbf{h}}y) \\ \omega(x,y) &= \Im(y^{\mathbf{h}}x) = \tfrac{1}{2\mathbf{i}}(y^{\mathbf{h}}x - x^{\mathbf{h}}y). \end{aligned}$$

- So the action of unitary group $\mathrm{U}(V) \subseteq \mathrm{Sp}(V)$, the moment map is given by

$$\mathbb{C}^n \longrightarrow \mathfrak{u}(\mathbb{C}^n)^* \qquad x \longmapsto \left[A \longmapsto \tfrac{1}{2\mathbf{i}}\big((Ax)^{\mathbf{h}}x\big)\right].$$

Note that $\mathfrak{u} = \{A : A + A^{\mathbf{h}} = 0\}$, and

$$\mathrm{U}(V) = \mathrm{GL}(V) \cap \mathrm{O}(V) = \mathrm{GL}(V) \cap \mathrm{Sp}(V) = \mathrm{Sp}(V) \cap \mathrm{O}(V)$$

$\gg$ *Questions?* $\ll$

∼ § Kähler Reduction § ∼

Theorem

Let V be a Kähler vector space, and $K \subseteq \mathrm{U}(V)$ a compact subgroup, and let $G \subseteq \mathrm{GL}(V)$ be the complexification of K, then for any $x \in \mu^{-1}(0)$, the orbit Gx is closed. And

$$\mu^{-1}(0)/K \to V//G$$

is a bijection.

Example

- Let V be a complex space, over $\mathrm{End}(V)$, by trace pairing, then $\omega(f,g) = \Im \operatorname{tr}(g^{\mathbf{h}} f)$.
- Now $\mathrm{U}(V)$ acts on $\mathrm{End}(V)$ by conjugation, then it factors through $\mathrm{U}(\mathrm{End}(V))$. The moment map is given by

$$\begin{array}{rcl} \mathrm{End}(V) & \longrightarrow & \mathfrak{u}(V)^* \\ x & \longmapsto & \left[A \longmapsto \frac{1}{2\mathbf{i}} \operatorname{tr}\left([A,x]^{\mathbf{h}} x\right) = \frac{\mathbf{i}}{2} \operatorname{tr}(A[x,x^{\mathbf{h}}])\right]. \end{array}$$

- So

$$\mu^{-1}(0) = \{x \in \mathrm{End}(V) : [x,x^{\mathbf{h}}] = 0\} = \{\text{normal operators}\}.$$

As a result,

$$\mathrm{End}(V) /\!/ \mathrm{GL}(V) = \mu^{-1}(0) / \mathrm{U}(V) = \mathbb{C}^n / \mathfrak{S}_n.$$

- Under the assumption above, a point $x \in \mu^{-1}(0)$ is called **regular**, if the stablizer K_x of x in K is trivial.

Theorem

The quotient $\mu^{-1}(0)^{reg}/K$ has a natural structure of a Kähler manifold.
The restriction of the map $\mu^{-1}(0)^{reg}/K \to V^{reg}//G$ is an isomorphism of complex manifolds.
In particular, if K acts on $\mu^{-1}(0)$ freely, then $\mu^{-1}(0)/K \to V//G$ is nonsingular and Kähler.

- For a character $G \to \mathbb{C}^\times$, then the restriction $K \to S^1$, so $\chi_* : \mathfrak{k} \to i\mathbb{R}$. We think $i\chi_*$ as a point of $\mathfrak{k}$.

Theorem

Let V be a Kähler vector space, and $K \subseteq \mathrm{U}(V)$ a compact subgroup, and let $G \subseteq \mathrm{GL}(V)$ be the complexification of K, then for any $x \in \mu^{-1}(i\chi_*)$, the orbit Gx is closed and semistable. And

$$\mu^{-1}(i\chi_*)/K \to V//_\chi G$$

is a bijection.

$\gg$ *Questions?* $\ll$

∼ § HyperKähler Manifolds § ∼

HyperKähler Manifolds

- Let us see the text book.

$\gg$ *Questions?* $\ll$

$\sim$ § THANKS § $\sim$

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