Geometry and Representation Seminar

Springer Theory

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§ Introduction §

Introduction

- It is well-known that the irreducible representations of $\mathfrak{S}_n$ are parameterized by Young diagrams of n-boxes.
- It is standard that the conjugation classes of nilpotent matrices are also parameterized by Young diagrams of n-boxes.
- They looks like a combinatorial coincidence, but can they be related without the intermediary of Young diagrams?

Introduction

Flag manifolds

Springer Resolution

Borel–Moore Homology

Convolution

Remarks

Thanks

$\gg$ *Questions?* $\ll$

§ Flag manifolds §

Flag manifolds

- Consider an n-dimensional space V, a **(complete) flag** is a series of subspaces

$$F^0 \subseteq F^1 \subseteq \cdots \subseteq F^{n-1} \subseteq F^n$$

with $\dim F^i = i$. We denote $\mathcal{F}\ell(V)$ the set of all the (complete) flags in V.

As Homogenous Space

- Let $G = \mathrm{GL}_n$, and B the subgroup of upper triangular matrices. Consider the map

$$\mathrm{span} : G \longrightarrow \mathcal{F}\ell(V) \qquad A \longmapsto F_A$$

where $F_A = (F_A^0 \subseteq \cdots \subseteq F_A^n)$ with

$$F_A^i = \text{the space spanned by first } i\text{-column vectors of } A.$$

- One can check that this induces a bijection $G/B \cong \mathcal{F}\ell(\mathbb{C}^n)$.

Bruhat decomposition

- The double cosets (by linear algebra)

$$B\backslash G/B \xleftrightarrow{1:1} \mathfrak{S}_n.$$

- We have the decomposition of G/B, by

$$G/B = \bigsqcup_{w\in\mathfrak{S}_n} BwB/B, \qquad BwB/B \cong \mathbb{C}^{\ell(w)}.$$

- The same can be done for $\mathcal{F}\ell(V)$ after fixing a choice of one flag correspondent $1\cdot B/B$. The cellular can be described inside $\mathcal{F}\ell(V)$, called **relative position**.

Orbits

- We also know that

$$\begin{aligned} * \times_G (G/B \times G/B) &= * \times_G (G \times_B G/B) \\ &\text{by } (xB, yB) \mapsto (x \times_B x^{-1}yB) \\ &= * \times_B G/B \end{aligned}$$

So

$$\{G\text{-orbit of } G/B \times G/B\} \xleftrightarrow{1:1} \{B\text{-orbit of } G/B\}$$

Schubert cells

- We call $\{BwB/B : w \in \mathfrak{S}_n\}$ the **Schubert cells** of G/B.
- Note that $BwB/B \cong \mathbb{C}^{\ell(w)}$.
- The corresponding G-orbit

$$\{(xB, yB) : x^{-1}y \in BwB\}$$

is also called the **Schubert cells** (note: they are not cells).
- Note that it is a fibre bundle over G/B with fibre $BwB/B \cong \mathbb{C}^{\ell(w)}$.

≫ *Questions?* ≪

§ Springer Resolution §

Stable Flags

- For $A \in \mathrm{End}(V)$, and a flag $F = (F^0 \subseteq \cdots \subseteq F^n) \in \mathcal{F}\ell(V)$, we say F is A-**stable** if each F^i is A-invariant.
- Since we interested in nilpotent matrices, denote

$$\mathcal{N}\imath\ell(V) = \{A \in \mathrm{End}(V) : A \text{ is nilpotent}\}.$$

Denote the pair of the relation of "stable"

$$\widetilde{\mathcal{N}\imath\ell}(V) = \{(A, F) \in \mathcal{N}\imath\ell(V) \times \mathcal{F}\ell(V) : F \text{ is } A\text{-stable}\}.$$

Two projections

- On one hand, the projection

$$\widetilde{\mathcal{N}\iota\ell}(V) \to \mathcal{N}\iota\ell(V)$$

is called the **Springer resolution**.

- On the other hand, another projection

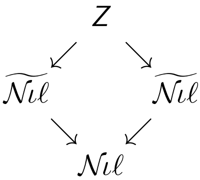
$$\widetilde{\mathcal{N}\iota\ell}(V) \to \mathcal{F}\ell(V)$$

is exactly the cotangent bundle projection of $\mathcal{F}\ell(V)$.

Steinberg Varietier

- Now consider the fibre product

$$
\begin{aligned}
Z &= \widetilde{\mathcal{N}\iota\ell} \underset{\mathcal{N}\iota\ell}{\times} \widetilde{\mathcal{N}\iota\ell} \\
&= \left\{(A, F_1, F_2) : \begin{array}{c} F_1 \text{ and } F_2 \\ \text{are both } A\text{-stable} \end{array}\right\}
\end{aligned}
$$



This is called the **Steinberg variety**.

Two Projections

- We denote Z_X or $\mathcal{F}\ell_X$ the preimage of $X \subseteq \mathcal{N}\iota\ell$ in Z or $\widetilde{\mathcal{N}\iota\ell}$.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ \widetilde{\mathcal{N}\iota\ell} & \searrow & \downarrow & \swarrow & \widetilde{\mathcal{N}\iota\ell} \\ & & \mathcal{N}\iota\ell & & \end{array} \qquad \begin{array}{ccccc} & & Z_X & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{F}\ell_X & \searrow & \downarrow & \swarrow & \mathcal{F}\ell_X \\ & & X & & \end{array}$$

- Then Z is disjoint union of $Z_{\mathbb{O}}$ with $\mathbb{O}$ goes through all conjugation classes of $\mathcal{N}\iota\ell$.

Two Projections

- On the other hand

$$
\begin{aligned}
Z &= \widetilde{\mathcal{N}il} \times_{\mathcal{N}il} \widetilde{\mathcal{N}il} \subseteq \widetilde{\mathcal{N}il} \times \widetilde{\mathcal{N}il} \\
&= \text{cotangent bundle of } G/B \times \text{cotangent bundle of } G/B \\
&= \text{cotangent bundle of } G/B \times G/B
\end{aligned}
$$

- Actually, Z is the disjoint union of conormal bundles of Schubert cells in $G/B \times G/B$. We abuse of notation and call them the Schubert cells in Z (note: they are not cells).
- Actually the closure of them are the irreducible components.

Formulation of Springer Theory

- The fibre of $\widetilde{\mathcal{N}\iota\ell}$ at $x \in \mathcal{N}\iota\ell$, is

$$\mathcal{F}\ell_x = \{F \in \mathcal{F}\ell(V) : F \text{ is } x\text{-stable}\}.$$

- Some homology theory (Borel–Moore homology) H of Z can be equipped with a operator called **convolution**

$$H(Z) \otimes H(Z) \xrightarrow{*} H(Z), \qquad H(Z) \otimes H(\mathcal{F}\ell_x) \xrightarrow{*} H(\mathcal{F}\ell_x).$$

- Springer theory: $H(Z)$ is the group algebra of $\mathfrak{S}_n$, and $H(\mathcal{F}\ell_x)$ the full list of irreducible representations with x going through all G-conjugation class of $\mathcal{N}\iota\ell$.

≫ *Questions?* ≪

§ Borel–Moore Homology §

Dimensions

- Let Y_w be the Schubert cells in Z correspondent to $w \in \mathbb{S}_n$. Since it is conormal bundle, then

$$\dim_{\mathbb{C}} Y_w = 2 \dim_{\mathbb{C}} \mathcal{F}\ell \, .$$

- Let $Z_{\mathbb{O}}$ be preimage of one of conjugation classes of $\mathcal{N}\iota\ell$, then

$$\dim_{\mathbb{C}} Z_{\mathbb{O}} = 2 \dim_{\mathbb{C}} \mathcal{F}\ell \, .$$

  This is nontrivial and proved by symplectic geometry.

The Correct Homology Theory

- For a locally compact space X, assume it is embedded in certain m-dimensional smooth manifold M, then we define

$$H_*^{\mathrm{BM}}(X) = H^{m-*}_{\mathrm{singular}}(M, M \setminus X).$$

- This does not depend on the choice of embedding. In particular, if X itself is smooth of dimensional n, then $H_*^{\mathrm{BM}}(M) = H^{n-*}_{\mathrm{singular}}(M)$.
- We denote the topmost degree part of Borel–Moore homology,

$$H(X) = H^{\mathrm{BM}}_{\mathrm{top}}(X; \mathbb{C}).$$

This is what we will use.

Two projections

- On one hand,

$$\begin{array}{rl} Z = \bigsqcup_{w\in\mathfrak{S}_n} Y_w \\ \dim Z = \dim Y_w \end{array} \implies \begin{array}{rl} H(Z) &= \bigoplus_{w\in\mathfrak{S}_n} H(Y_w) \\ &= \bigoplus_{w\in\mathfrak{S}_n} \mathbb{C}\cdot[Y_w]. \end{array}$$

- Actually, by a long exact sequence argument, we can show that

$$H_*^{\mathrm{BM}}(Z) = \bigoplus_{w\in\mathfrak{S}_n} H_*^{\mathrm{BM}}(Y_w).$$

Two projections

- On the other hand,

$$\begin{array}{r} Z = \bigsqcup_{\text{conj cls } \mathbb{O}} Z_{\mathbb{O}} \\ \dim Z = \dim Z_{\mathbb{O}} \end{array} \implies H(Z) = \bigoplus_{\text{conj cls } \mathbb{O}} H(Z_{\mathbb{O}})$$

- We will show that

$$H(Z) = \bigoplus_{\text{conj cls } x} H(\mathcal{F}\ell_x) \otimes H(\mathcal{F}\ell_x).$$

- For each conjugation class $\mathbb{O}$ of $\mathcal{N}\iota\ell$, and $x \in \mathbb{O}$. Consider the spectral sequence of $Z_{\mathbb{O}} \to \mathbb{O}$ with fibre Z_x. In the case $G = \mathrm{GL}_n$, $\mathbb{O}$ is simply-connected, we see

$$H(Z_{\mathbb{O}}) = \underbrace{H(\mathbb{O})}_{=\mathbb{C}} \otimes H(Z_x) = H(Z_x) = H(\mathcal{F}\ell_x) \otimes H(\mathcal{F}\ell_x).$$

- In general, the fundamental group of $\mathbb{O}$ is

$$\begin{aligned} G(x) &= G_x/\text{component of } 1 \in G_x \\ G_x &= \{g \in G : \mathrm{ad}_g\, x = x\}. \end{aligned}$$

So $H(Z_{\mathbb{O}}) = (H(\tilde{\mathbb{O}}) \otimes H(Z_x))^{G(x)} = (H(\mathcal{F}\ell_x) \otimes H(\mathcal{F}\ell_x))^{G(x)}$. Note that $G(x)$ acts on the homology of the universal covering $H(\tilde{\mathbb{O}})$ trivially, since $\mathbb{O}$ is oriented.

≫ *Questions?* ≪

§ Convolution §

Convolution

- For smooth manifolds $\left\{\begin{smallmatrix}A\\B\\C\end{smallmatrix}\right.$, and closed subsets $\left\{\begin{smallmatrix}X\subseteq A\times B\\Y\subseteq B\times C\end{smallmatrix}\right.$.

  Denote the projections $A\times B\times C\begin{cases}\xrightarrow{p_1} B\times C\\ \xrightarrow{p_2} A\times C\\ \xrightarrow{p_3} A\times B\end{cases}$.
- Denote

$$\begin{aligned} X\circ Y &= p_2(p_3^{-1}(X)\cap p_1^{-1}(Y)) \\ &= \left\{(a,c)\in A\times C : \exists b\in B, \left\{\begin{smallmatrix}(a,b)\in A\\(b,c)\in B\end{smallmatrix}\right.\right\}. \end{aligned}$$

  To be well-defined, we need to assume $p_3^{-1}(X)\cap p_1^{-1}(Y)\to X\circ Y$ is proper.

Convolution

$$
\begin{array}{ccc}
H_*^{\mathrm{BM}}(X) \otimes H_*^{\mathrm{BM}}(Y) & = & H^*(A\times B, A\times B \setminus X) \otimes H^*(B\times C, B\times C\setminus Y) \\
\downarrow p_3^* \otimes p_1^* & & p_3^* \otimes p_1^* \downarrow \\
H_*^{\mathrm{BM}}(p_3^{-1}(X)) \otimes H_*^{\mathrm{BM}}(p_1^{-1}(Y)) & = & H^*(A\times B\times C, A\times B\times C\setminus p_3^{-1}(X)) \otimes H^*(A\times B\times C, A\times B\times C\setminus p_1^{-1}(Y)) \\
\downarrow \bullet & & \smile \downarrow \\
H_*^{\mathrm{BM}}(p_3^{-1}(X) \cap p_1^{-1}(Y)) & \longrightarrow & H^*(A\times B\times C, A\times B\times C\setminus p_3^{-1}(X) \cap p_1^{-1}(Y)) \\
\downarrow (p_2)_* & & \\
H_*^{\mathrm{BM}}(X \circ Y) & &
\end{array}
$$

The diagram is not dimension sensiti

Convolution

- As a result,

$$H_x^{\text{BM}}(X) \otimes H_y^{\text{BM}}(Y) \to H_z^{\text{BM}}(X \circ Y)$$

where $x + y = z + \dim B$.

- In our case, $A = B = C = \mathcal{N}\imath\ell$, or $\{ {A=B=\mathcal{N}\imath\ell \atop C=\text{pt}}$, we get

$$\begin{aligned} H(Z) \otimes H(Z) &\to H(Z) \\ H(Z) \otimes H(\mathcal{N}\imath\ell) &\to H(\mathcal{N}\imath\ell) \\ H(Z) \otimes H(Z_{\overline{\mathbb{O}}}) &\to H(Z_{\overline{\mathbb{O}}}) \\ H(Z) \otimes H(\mathcal{F}\ell_x) &\to H(\mathcal{F}\ell_x) \end{aligned}$$

Group Algebra

- Now we can show $H(Z) \cong \mathbb{C}[\mathfrak{S}_n]$. We want to consider the graph of $w \in \mathfrak{S}_n$ in $\mathcal{N}\iota\ell \times \mathcal{N}\iota\ell$. Then it will be clear $\mathrm{graph}(u) \circ \mathrm{graph}(v) = \mathrm{graph}(uv)$.
- But it is a little tricky, since w does not act on $\mathcal{F}\ell$ algebraically, and the graph of this action needs not to lie in Steinberg variety Z.
- But anyway, whenever we defined a suitable choice, it would suffice to check its action over $H_*^{\mathrm{BM}}(\mathcal{N}\iota\ell)$.

Group Algebra

- One way is, only to describe the simple reflection, which is not hard to work out a proper approximation (need to be proper, since Borel–Moore homology is only properly homotopy invariant). See Lusztig and Kazhdan.
- Another way is more algebraic. The key point is the following. Consider the regular semisimple elements in $\mathfrak{g} = \mathbb{M}_{n\times n}(\mathbb{C})$,

$$\mathfrak{g}^{\mathrm{rs}} = \{\text{diagonalizable matrix with distinct eigenvalues}\}.$$

Now $\tilde{\mathfrak{g}}^{\mathrm{rs}} = \{(x, F) \in \mathfrak{g}^{\mathrm{rs}} : F \text{ is } x \text{ stable}\}$ is a Galois (normal) covering of $\mathfrak{g}^{\mathrm{rs}}$.

Group Algebra

- Then we can somehow taking limits

$$
\begin{array}{ccccccccc}
 & & \tilde{\mathfrak{g}}^{\mathrm{rs}} \times_{\mathfrak{g}^{\mathrm{rs}}} \tilde{\mathfrak{g}}^{\mathrm{rs}} & & & & & Z & \\
 & \swarrow & & \searrow & & & \swarrow & & \searrow \\
\tilde{\mathfrak{g}}^{\mathrm{rs}} & & & & \tilde{\mathfrak{g}}^{\mathrm{rs}} \quad \overset{\text{Specialization}}{\Longrightarrow} & \widetilde{\mathcal{N}\iota\ell} & & & \widetilde{\mathcal{N}\iota\ell} \\
 & \searrow & & \swarrow & & & \searrow & & \swarrow \\
 & & \mathfrak{g}^{\mathrm{rs}} & & & & & \mathcal{N}\iota\ell &
\end{array}
$$

- Note that in general we can only take limit along curves, in this case, it does not depend on the choice because we can check the action over $H_*^{\mathrm{BM}}(\mathcal{N}\iota\ell)$. However, see Chriss and Ginzburg 3.4.11 and 7.3.20.

Group Algebra

- On the other hand,

$$H(Z) = \bigoplus_{\text{conj cls } \mathbb{O}} H(Z_{\mathbb{O}})$$
$$H(Z_{\mathbb{O}}) = H(\mathcal{F}\ell_x) \otimes H(\mathcal{F}\ell_x)$$

  are all $H(Z)$-module isomorphisms.

- So we get an isomorphism

$$\mathbb{C}[\mathfrak{S}_n] = H(Z) = \bigoplus_{\text{conj cls } x} \mathrm{End}(H(F_x)).$$

  So this establishes Springer theory.

$\gg$ *Questions?* $\ll$

§ Remarks §

On the Representations

- Note that the inclusion $\begin{pmatrix} \mathrm{GL}_n & \\ & \mathrm{GL}_m \end{pmatrix} \subseteq \mathrm{GL}_{n+m}$ will repeat the induction process of $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{m+n}$. From this, we can see the exact correspondence between $H(F_x)$ and the classic classification of irreducible representation of $\mathfrak{S}_n$.
- Actually, we also proved all the representation of $\mathfrak{S}_n$ is over $\mathbb{Q}$ (splitting field is $\mathbb{Q}$).
- Even for SL_n, $G(x)$ is in general not trivial.
- The dimension of $H(F_x)$ is the number of components of F_x. (Since the irreducible components of F_x share the same dimension, see Chriss and Ginzburg 3.3) One can check this is one-to-one correspondent to the number of standard Young tableaux (so it is the Hook-length).

Sheaves

- It would be not satisfactory if we use too much vague topological method. One way to settle them is to consider them over sheaf-level.
- Let $\pi : X \to Y$ be a proper map with X smooth (mostly resolution of singularities). Denote $\mathcal{C}_X = \mathbb{C}[\dim X]$, it is the dualizing sheaf (it is in general a complex) since X is smooth. Then

$$H_*^{\mathrm{BM}}(X; \mathbb{C}) = H^*(X; \mathcal{C}_X) = H^*(Y, \pi_* \mathcal{C}_X)$$

Sheaves

- Let $Z = X \times_Y X$. Then

$$H_*^{\mathrm{BM}}(Z;\mathbb{C}) = \mathrm{Ext}^*_{D(Y)}(\pi_*\mathcal{C}_X, \pi_*\mathcal{C}_X)$$

- The convolution algebra structure is given by Yoneda product.
- The same for $H_*^{\mathrm{BM}}(X;\mathbb{C})$, the action is given by Yoneda product

$$H_*^{\mathrm{BM}}(X;\mathbb{C}) = H^*(Y, \pi_*\mathcal{C}_X) = \mathrm{Ext}^*_{D(Y)}(\mathcal{O}_Y, \pi_*\mathcal{C}_X).$$

Perverse Sheaves

- It is famous that $\pi_* \mathcal{C}_X$ can be decomposed into direct sum of shifting of perverse sheaves over Y.
- And over Y, the equivariant perverse sheaves are in bijection with local coordinates of orbits.
- But to get $H_*^{\mathrm{BM}}(Z; \mathbb{C})$, it needs to apply Fourier transform.
- Details can be found in Ginzburg, Geometric Methods in Representation Theory.

Springer theory for $\mathcal{U}(\mathfrak{sl}_n)$

- The same idea can be moved to partial flags of vector space V, that is, a series of subspaces

$$0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^{n-1} \subseteq F^n = V$$

  without dimension assumption. Note that its connected component is parameterized by Young diagrams of $\dim V$-boxes of length n.

- We can construct from $\mathcal{N}\imath\ell$ to get $\widetilde{\mathcal{N}\imath\ell}$, and Z. Then there will be a surjection

$$\mathcal{U}(\mathfrak{sl}_n) \longrightarrow H(Z),$$

  and $H(\mathcal{F}\ell_x)$ will be all the representations.

≫ *Questions?* ≪

§ THANKS §

References

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