Quiver Representations II

Xiong Rui

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A linear algebra problem

- For $1 \leq i < j \leq n$, and angles

  $$\theta_{ij} \in \mathbb{R}/2\pi\mathbb{Z}$$

  can we find *linear independent* vectors

  $$v_1, \ldots, v_n \in \mathbb{R}^n \qquad \text{such that} \qquad \angle(v_i, v_j) = \theta_{ij}.$$

- Example 1. when $n = 2$, always possible.
- Example 2. We can find in $\mathbb{R}^3$ three vectors which are pairwise in 119.999°; but cannot find for 120°.

A linear algebra problem

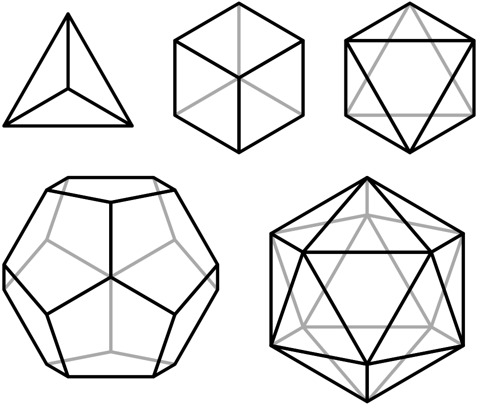
- Let us put

$$\begin{pmatrix} 1 & \cos\theta_{12} & \cdots & \cos\theta_{1n} \\ \cos\theta_{12} & 1 & \cdots & \cos\theta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\theta_{1n} & \cos\theta_{2n} & \cdots & 1 \end{pmatrix}$$

- We can find such vectors if and only if this matrix is positive-definite. (Is this clear?)

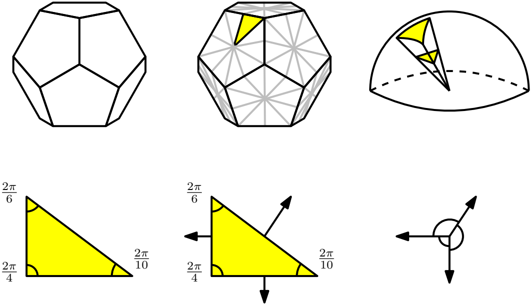
Classification of Regular Polyhedra

- How many regular polyhedra in $\mathbb{R}^3$? (Answer: 5 of them, Platonic solid).



Classification of Regular Polyhedra

- We can do as follows



- It reduce to find vectors v_1, v_2, v_3 with

$$\pi - \angle(v_1, v_2) = \frac{\pi}{a}, \quad \pi - \angle(v_2, v_3) = \frac{\pi}{b}, \quad \pi - \angle(v_1, v_3) = \frac{\pi}{2}.$$

Each face is an *a*-gon; each vertex joints *b* faces.

Conclusion

- Then

$$\text{(above problems)} \iff \begin{pmatrix} 1 & -\cos\frac{\pi}{a} & \\ -\cos\frac{\pi}{a} & 1 & -\cos\frac{\pi}{b} \\ & -\cos\frac{\pi}{b} & 1 \end{pmatrix} \text{ is Pst-Dfnt.}$$

That is,

$$a \geq 3, b \geq 3, \qquad 1 - \cos^2\frac{\pi}{a} - \cos^2\frac{\pi}{b} > 0.$$

Only $(3,3),(3,4),(3,5),(4,3),(5,3)$ serves. Since

$$n = 3 \Longrightarrow \frac{1}{4} = \cos^2\frac{\pi}{n}, \qquad n \geq 4 \Longrightarrow \frac{1}{2} \leq \cos^2\frac{\pi}{n} \leq 1$$

Integral Quadratic Forms

- Consider

$$Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i<j} a_{ij} x_i x_j$$

with $a_{ij} \in \mathbb{Z}_{\geq 0}$. When

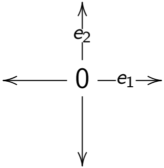
$$x \neq 0 \Rightarrow Q > 0, \quad \text{that is , } Q \text{ is positive-definite?}$$

- Example 1. $x^2 + y^2$ is positive-definite.
- Example 2. $x^2 + y^2 - xy$ is positive-definite.
- Example 3. $x^2 + y^2 - 2xy$ is only semi-positive-definite.
- Example 4. $x^2 + y^2 + z^2 - xy - xz$ is positive-definite.
- Example 5. $x^2 + y^2 + z^2 - xy - xz - yx$ is semi-PD.

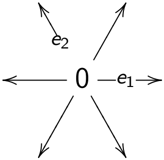
Integral Quadratic Forms

- For a positive-definite Q, a_{ij} never takes 2. Since
$$Q = x^2 + y^2 - 2xy + \cdots$$
takes $(x, y, \cdots) = (1, 1, 0, \ldots)$ as a zero.
- Consider
$$R = \{x \in \mathbb{Z}^n : Q(x) = 1\}.$$
Then $e_i \in R$, and if $x, y \in R$, the reflection
$$s_y x \longmapsto x - \frac{2Q(x, y)}{Q(x)} y \in R.$$
Here $2Q(x, y) = Q(x + y) - Q(x) - Q(y)$.

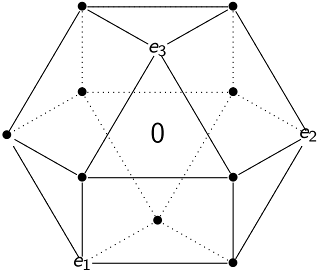
Gallary



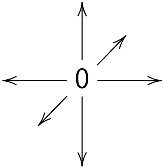
$x^2 + y^2$



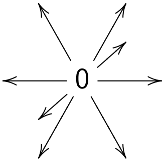
$x^2 - xy + y^2$



$x^2 - xz + y^2 - yz + z^2$



$x^2 + y^2 + z^2$



$x^2 - xy + y^2 + z^2$

Classification

- Above problems reduce to the question, when

$$\begin{pmatrix} 2 & a_{12} & \cdots & a_{1n} \\ a_{12} & 2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 2 \end{pmatrix} \qquad - a_{ij} \in A$$

is positive-definite for a given discrete set A. For example,

$$A = \{2\cos\tfrac{\pi}{2}, 2\cos\tfrac{\pi}{3}, 2\cos\tfrac{\pi}{4}, \ldots\} = \{0, 1, \sqrt{2}, \cdots\}$$

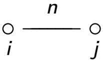
$$A = \{0, 1, 2, \ldots\} \qquad \text{(essentially } \{0,1\} \text{).}$$

Classification

- The answer to $A = \{0, 1, 2, \ldots\}$ (essentially $\{0, 1\}$) is the special case.
- The answer to

$$A = \{2\cos\tfrac{\pi}{2}, 2\cos\tfrac{\pi}{3}, 2\cos\tfrac{\pi}{4}, \ldots\} = \{0, 1, \sqrt{2}, \cdots\}$$

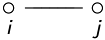
  is given by Coxeter.
- We put



$$\iff \quad -a_{ij} = 2\cos\frac{\pi}{n},$$

  and

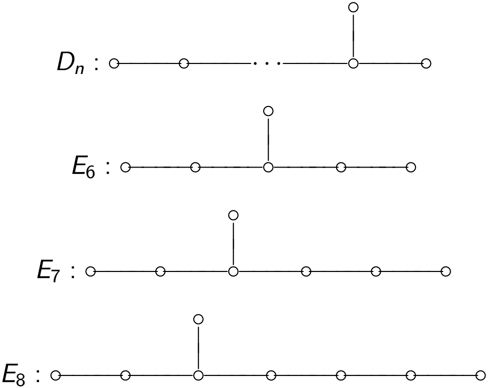
$$n = 3$$



$$n = 2$$

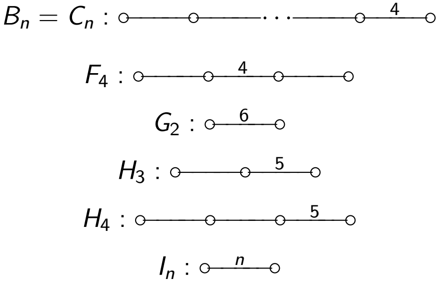
Coxeter Diagrams

With $-a_{ij} \in \{0, 1\}$ (i.e. simply-laced)

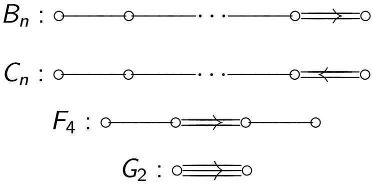


Coxeter Diagrams

With some $a_{ij} \neq \{0, 1\}$



Dynkin Diagram (in this case, the problem is not only for angles, but also for lengths. Precise formulation later.)



Relation to Quiver Representations

- Let $Q = (I, H)$ be a quiver, with $I = \{1, \dots, n\}$. For a representation V, we denote

$$\underline{\dim}\, V = (\dim V_1, \dim V_2, \dots, \dim V_n).$$

$$\begin{tikzcd}
 & \circ \arrow[d] & \\
\circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[r] & \circ
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V_1 \arrow[d, "f"] & \\
V_2 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & V_3 \arrow[r, "k"] & V_4
\end{tikzcd}$$

$$\underline{\dim} = (\dim V_1, \dim V_2, \dim V_3, \dim V_4).$$

Quiver Varieties

- Then, given dimension vector $x = (x_1, x_2, \ldots, x_n)$, how many isomorphic classes of dimension x?
- It is

$$\prod_{i \to j} \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}^{x_i}, \mathbb{k}^{x_j}) \Big/ \text{isomorphism}$$

$$\underline{\dim} = (1, 2, 3, 4)$$

$$\begin{tikzcd}
 & \mathbb{k}^1 \arrow[d, "f"] & \\
\mathbb{k}^2 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \mathbb{k}^3 \arrow[r, "k"] & \mathbb{k}^4
\end{tikzcd}$$

$$(f, g, h, k) \in \mathrm{Hom}(\mathbb{k}^1, \mathbb{k}^3) \times \mathrm{Hom}(\mathbb{k}^2, \mathbb{k}^3) \times \mathrm{Hom}(\mathbb{k}^2, \mathbb{k}^3) \times \mathrm{Hom}(\mathbb{k}^3, \mathbb{k}^4)$$

Quiver Varieties

- Actually,

$$\prod_{i\to j}\mathrm{Hom}_{\Bbbk}(\Bbbk^{x_i},\Bbbk^{x_j})\Big/\text{isomorphism}$$

$$\|$$

$$\prod_{i\to j}\mathrm{Hom}_{\Bbbk}(\Bbbk^{x_i},\Bbbk^{x_j})\Big/\text{conjugation action of }\prod_{i\in I}\mathrm{GL}(\Bbbk^{x_i})$$

$$\begin{tikzcd}
& & \Bbbk^1 \arrow[dd] \arrow[dr, "g_1"] & & & \\
& & & \Bbbk^1 \arrow[dd] & & \\
\Bbbk^2 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "g_2"] & & \Bbbk^3 \arrow[rr] \arrow[dr, "g_3"] & & \Bbbk^4 \arrow[dr, "g_4"] & \\
& \Bbbk^2 \arrow[rr, shift left] \arrow[rr, shift right] & & \Bbbk^3 \arrow[rr] & & \Bbbk^4
\end{tikzcd}$$

$$(g_1,g_2,g_3,g_4)\in\mathrm{GL}(\Bbbk^1)\times\mathrm{GL}(\Bbbk^2)\times\mathrm{GL}(\Bbbk^3)\times\mathrm{GL}(\Bbbk^4)$$

Then?

- In general, for a manifold

$$\dim(X/G) \geq \dim X - \dim G.$$

- So

$$\dim\left(\prod_{i\to j}\mathrm{Hom}_{\mathbb{k}}(\mathbb{k}^{x_i},\mathbb{k}^{x_j})\Big/\prod_{i\in I}\mathrm{GL}(\mathbb{k}^{x_i})\right)$$
$$\geq \dim\prod_{i\to j}\mathrm{Hom}_{\mathbb{k}}(\mathbb{k}^{x_i},\mathbb{k}^{x_j}) - \dim\prod_{i\in I}\mathrm{GL}(\mathbb{k}^{x_i})$$

- Note that the constant matrix $\in \prod \mathrm{GL}(\mathbb{k}^{x_i})$ acts trivially. So the above inequality is strict for $x \neq 0$.

- So, if Q has finite representation type, then at least

$$\dim\left(\prod_{i\to j}\mathrm{Hom}_{\Bbbk}(\Bbbk^{x_i},\Bbbk^{x_j})\Big/\prod_{i\in I}\mathrm{GL}(\Bbbk^{x_i})\right)=0$$

  (i.e. discrete).

- By above, for any dimension vector $x\neq 0$,

$$\begin{aligned}0 &< \dim\textstyle\prod_{i\in I}\mathrm{GL}(\Bbbk^{x_i})-\dim\prod_{i\to j}\mathrm{Hom}_{\Bbbk}(\Bbbk^{x_i},\Bbbk^{x_j})\\ &=\textstyle\sum_{i=1}^n x_i^2-\sum_{i<j}h_{ij}x_ix_j.\end{aligned}$$

  here $h_{ij}=\#\{i\to j\}+\#\{j\to i\}$.

- So

$$\boxed{Q\text{ is of fnt-rep-type}}\Longrightarrow\boxed{Q\text{ is a disjoint union of ADE quiver}}.$$

The converse

- The converse is also true (using reflection functors).
- Amazingly, the following maps are bijection

$$\#\{\text{irreducible reps}\} \xrightarrow{\dim} \#\{\text{simple roots}\}.$$

$$\#\{\text{indecomposable reps}\} \xrightarrow{\dim} \#\{\text{positive roots}\}.$$

  Here the simple roots are the standard basis e_i.
- irre = no proper subrep
- ind = no proper summand.

A type

- For A_n-type.

A_n :


there should be $n(n+1)/2$ many ind-reps.

- For any $1 \leq i \leq n$, consider

$$\cdots \text{——} \overset{i-1}{0} \overset{0}{\text{——}} \overset{i}{\Bbbk} \overset{0}{\text{——}} \overset{i+1}{0} \text{——} \cdots$$

they give all irre reps.

- For $1 \leq i < j \leq n$, consider

$$\cdots \text{——} \overset{i-1}{0} \overset{0}{\text{——}} \overset{i}{\Bbbk} \overset{\text{id}}{\text{——}} \cdots \overset{\text{id}}{\text{——}} \overset{j}{\Bbbk} \overset{0}{\text{——}} \overset{j+1}{0} \text{——} \cdots$$

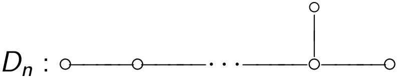
They are indecomposable.

Rep	dim	in usual notation	how many
$\begin{array}{ccccc} & i{-}1 & i & i{+}1 & \\ \cdots & 0 & 1 & 0 & \cdots \end{array}$	e_i	$x_i - x_{i+1}$	n
$\begin{array}{cccccc} & i{-}1 & i & & j & j{+}1 & \\ \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots \end{array}$	$e_i + \cdots + e_j$	$x_i - x_{j+1}$	$\frac{n(n-1)}{2}$

So they give all ind-reps.

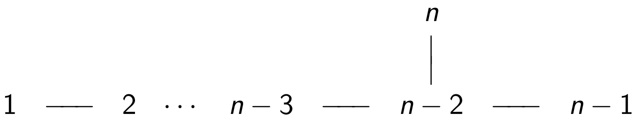
D type

- For D_n-type.



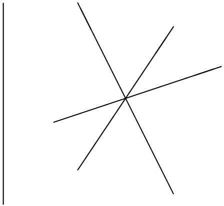
there should be $n(n-1)$ many ind-reps.

- Let us label them by



- There is a phenomenon that

$$\begin{tikzcd}
\Bbbk \times 0 \arrow[dr] & & \\
& \Bbbk \times \Bbbk & \text{diagonal } \Bbbk \arrow[l] \\
0 \times \Bbbk \arrow[ur] & &
\end{tikzcd}$$



  is indcomposable.

- Actually, we need not to be careful about the orientation.

$$\begin{tikzcd}
\Bbbk \times 0 & & \\
& \Bbbk \times \Bbbk \arrow[ul] \arrow[dl] \arrow[r, "\text{addition}"] & \Bbbk \\
0 \times \Bbbk & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \times 0 \arrow[dr] & & \\
& \Bbbk \times \Bbbk \arrow[dl] \arrow[r, "\text{addition}"] & \Bbbk \\
0 \times \Bbbk & &
\end{tikzcd}$$

  are also indcomposable.

0 $\cdots 010 \cdots 0\,0$	e_i	0 $\cdots 01 \cdots 10 \cdots 0\,0$	$e_i + \cdots + e_j$	$\frac{(n-3)(n-2)}{2}$
0 $\cdots\cdots\cdot 1\,0$	e_{n-2}	0 $\cdots 01 \cdots 1\,0$	$e_i + \cdots + e_{n-2}$	$n-2$
0 $\cdots\cdots\cdot 0\,1$	e_{n-1}	0 $\cdots 01 \cdots 1\,1$	$e_i + \cdots + e_{n-1}$	$n-1$
1 $\cdots\cdots\cdot 0\,0$	e_n	1 $\cdots 01 \cdots 1\,0$	$e_i + \cdots + e_{n-2} + e_n$	$n-1$
$-$	$-$	1 $\cdots 01 \cdots 1\,1$	$e_i + \cdots + e_n$	$n-2$
$-$	$-$	1 $\cdots 01 \cdots 12 \cdots 2\,1$	$e_i + \cdots + e_n$ $+ e_j + \cdots + e_{n-2}$	$\frac{(n-3)(n-2)}{2}$

Here 1's and 2's are connected with identity. The map of $1 \to 2$ or $2 \to 1$ is explained above.

For A_n,

$$\#\{\text{ind-reps}\} = \#\{\text{diagonals in } (n+3)\text{-gon}\} - n$$