Quiver Representations I

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Associative Algebras

- Let $\mathbb{k}$ be a field.
- Let A be a (finite dimensional) algebra over $\mathbb{k}$.
- Example 1: $\mathbb{M}\text{at}_{n\times n}(\mathbb{k})$.
- Example 2: $n \times n$ upper triangular matrices.
- Example 3: $\mathbb{k}[t]$ (this is not finite dimensional).
- Example 4: Group algebra or finite groups.
- A *representation* of A is a finite dimensional module of A.
- Main task in representation theory of associative algebra is

$$\boxed{\text{Given an algebra, classify all reps}}.$$

Examples

- For $\mathbb{C}[G]$ of some finite group G, then

$$\boxed{\text{Classification}} \iff \underbrace{\boxed{\text{Classification of Irrep}}}_{\text{by characters}}$$

- For $\mathbb{k}$

$$\boxed{\text{Classification}} \iff \underbrace{\boxed{\text{Classification of vector spaces}}}_{\text{by dimension}}$$

Examples

- For $\mathbb{k}[x]$, then

$$\boxed{\text{Classification}}$$

$$\Updownarrow$$

$$\boxed{\text{Classification of a vector space + an linear endormorphism}}$$

$$\Updownarrow$$

$$\underbrace{\boxed{\text{Classification of conjugacy class of a matrix}}}_{\text{by Jordan form}}$$

- For $\mathbb{k}[x]/(x^2)$,

$$\boxed{\text{Classification}}$$

$$\Updownarrow$$

$$\boxed{\text{Classification of conjugacy class of 2-nilpotent matrices}}$$

Examples

- For $\Bbbk \times \Bbbk$,

$$\boxed{\text{Classification}} \iff \boxed{\text{Classification of two vector spaces}}$$

- For $\begin{pmatrix} \Bbbk & \Bbbk \\ & \Bbbk \end{pmatrix}$, then

$$\boxed{\text{Classification}}$$

$$\Updownarrow$$

$$\boxed{\text{Classification of a pair of vector + a linear map}}$$

$$\Updownarrow$$

$$\underbrace{\boxed{\begin{array}{c}\text{Classification of equivalence class} \\ \text{of a linear transform}\end{array}}}_{\text{by rank}}$$

Examples

- For $\mathbb{k}[x, y]$,

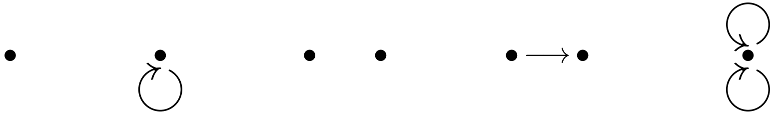
$$\boxed{\text{Classification}}$$
$$\Updownarrow$$
$$\boxed{\text{Classification of conjugacy class of a pair of commuting matrices}}$$

- For $\mathbb{k}\langle x, y\rangle$, then

$$\boxed{\text{Classification}}$$
$$\Updownarrow$$
$$\boxed{\text{Classification of conjugacy class of a pair of matrices}}$$

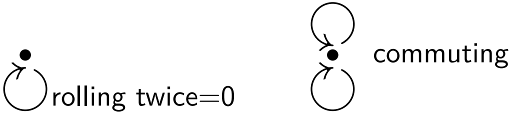
Quivers

- We can summary above examples by quivers (oriented graphs)



$\Bbbk$ $\qquad$ $\Bbbk[t]$ $\qquad$ $\Bbbk \times \Bbbk$ $\qquad$ $\begin{pmatrix} \Bbbk & \Bbbk \\ & \Bbbk \end{pmatrix}$ $\qquad$ $\Bbbk \langle x, y \rangle$

- Not every algebra can be recovered from quivers. But if we know the relations the arrows should satisfy, then it can.



$\Bbbk[t]/t^2$ $\qquad$ $\Bbbk[x, y]$

Undecidedness

Theorem (Baur; Kokorin and Mart'yanov)

The algebra $\mathbb{k}\langle x, y\rangle$ *is undecidable.*

- A is said to be *decidable* if there is a Turing machine algorithm which will decide the truth or falsehood of any sentence in the language of finite dimensional A-modules.
- W. Baur. Decidability and undecidability of theories of abelian groups with predicates for subgroups. Compositio Math. 31 (1975), 23-30.
- A. I. Kokorin and V. I. Mart'yanov. Universal extended theories. Algebra, Irkutsk (1973), 107-114.

Types

- A is said to be *of finite representation type* if there are finite many representations $V_1, V_2, \ldots$, such that any representation M can be written as

$$M \cong V_1 \underset{\mathbb{k}}{\otimes} M_1 \; \oplus \; V_2 \underset{\mathbb{k}}{\otimes} M_2 \; \oplus \; \cdots ,$$

with $M_1, M_2, \ldots$ are finite dimensional $\mathbb{k}$-vector spaces (i.e. the multiplicity spaces).

Types

- A is said to be *of tame representation type* if for any n, there are finite many $A[t]$-representations $V_1(t), V_2(t), \dots$ parametrized by an indeterminant t such that any A-representation M of dim n can be written as

$$M \cong V_1(t) \underset{\mathbb{k}[t]}{\otimes} M_1 \oplus V_2(t) \underset{\mathbb{k}[t]}{\otimes} M_2 \oplus \cdots ,$$

with $M_1, M_2, \dots$ are finite dimensional $\mathbb{k}[t]$-modules.

Types

- A is said to be *of wild representation type* if the classification of A "includes" the classification of finite dimensional indecomposable representations of $\mathbb{k}\langle x, y \rangle$.

Theorem (Drozd, Crawley–Boevey)

Over algebraic closed field, an algebra is either of finite, tame, or wild representation type.

Quiver Representations

- Now, let us go back to quivers. For a quiver Q, we can assign an algebra called the *path algebra* $\Bbbk[Q]$ of it. Then the classification of "representation of the shape of Q"

$$(\text{vertex}, \text{arrow}) \longrightarrow (\text{vector space}, \text{linear map})$$

  is equivalent to the classification of $\Bbbk[Q]$.
- As we see before, $\Bbbk[t], \Bbbk\langle x, y\rangle, \begin{pmatrix} \Bbbk & \Bbbk \\ & \Bbbk \end{pmatrix}$ are all path algebra of their quivers. But $\Bbbk[t]$ and $\Bbbk[x, y]$ are not.

Gabriel Theorem

Theorem (Gabriel)

Over an algebraic closed field $\mathbb{k}$. Then

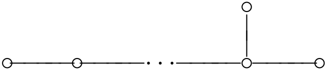
- *a quiver Q is of finite representation type iff Q is a Dynkin diagram without multiple edges (orientation does not matter);*
- *a quiver Q is of tame representation type iff Q is an affine Dynkin diagram without multiple edges (orientation does not matter).*

See next two pages.

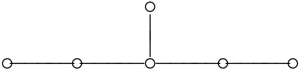
Dynkin Diagrams

A_n :

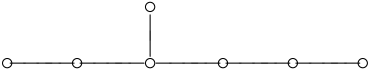
D_n :



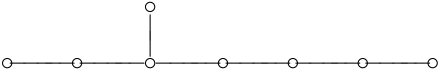
E_6 :



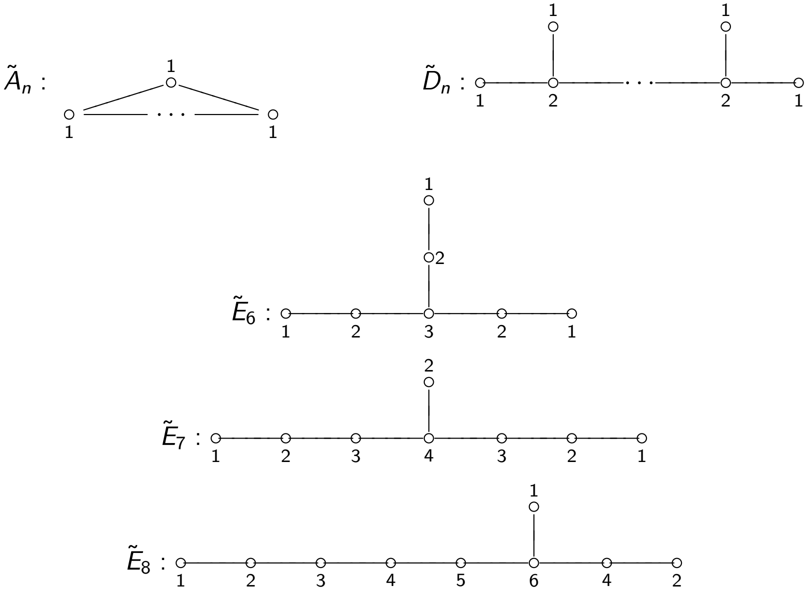
E_7 :



E_8 :



Affine Dynkin Diagrams



Auslander–Reiten Theory

- The idea of AR theory is, not only try to classify modules, but consider maps between them.
- For any subcategory, we define the *AR quiver*

$$\begin{cases} \text{Vertex} : [M] & \begin{array}{l}\text{iso-classes of reps which} \\ \text{cannot be written into smaller } V_1 \oplus V_2\end{array} \\ \text{Arrow} : \begin{array}{c} \downarrow \\ [N] \end{array} & \text{A choice of basis of } \operatorname{rad}(M, N)/\operatorname{rad}^2(M, N). \end{cases}$$

where $\operatorname{rad}(M, N)$ is the space of non-invertible maps from $M \to N$ (it is a linear space), and $\operatorname{rad}(M, N)^2$ the space spanned by $g \circ f$ with $f \in \operatorname{rad}(M, L)$ and $g \in \operatorname{rad}(L, N)$.

Auslander–Reiten Theory

- Roughly, speaking,

$$\begin{cases} \text{Vertex}: & \text{minimal information to recover all reps} \\ \text{Arrow}: & \text{minimal information to recover all morphisms} \end{cases}$$

- For the subcategory of projective modules of a path algebra $\mathbb{k}[Q]$, we will get back the quiver Q.
- Description of Dynkin type quiver — known, combinatorially.
- General algebras — the central topic in representation theory of associative algebras.
- More theory of quivers — another long story.