Lecture 9 — Representation of associative algebra (II)

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Hereditary algebra

- We consider only the algebraically closed case.
- We call an algebra is **hereditary** if its projective dimension ≤ 1, i.e. $\mathrm{Ext}^{>1} = 0$. Equivalently, any submodule of projective module is again projective.
- As a result, any homomorphism between indecomposable projective modules is zero of injective. So we can introduce a partial order of "submodule".
- In particular, for an indecomposable projective modules, $\mathrm{End}_R(P) = k$.

Some reduction

- Since projective dimension is a property of category, so it suffices to consider basic algebra R.
- The corresponding surjective algebra homomorphism from path algebra $k\Gamma \to R$ is an isomorphism.
  Hint: we have a partial order on the vertices, so one can uniquely write an element of R into sum of paths by induction on this order.
- If a quiver Γ is acyclic (no loop and no self-circle), then its path algebra $k\Gamma$ is hereditary (we will see next page).
- In other words,

$$\boxed{\text{Hereditary algebra}} \approx \boxed{\text{Quiver algebra}}.$$

Remind of Path algebra

- Let Γ be an acyclic (no loop and no self–circle) quiver, and $R = k\Gamma$. For each vertex v, denote e_v the path of length 0 staying at v.
- It is efficient to consider right modules, as we mentioned in lecture 7, since our conversion of product of path.
- Let V be a right $k\Gamma$-module, we consider $V_v = Ve_v$ for any vertex v, and $[V_w \xrightarrow{\gamma} V_v]$ for any edge $[w \xrightarrow{\gamma} v]$ in Γ. These data form a "diagram of vector spaces in the shape of Γ". Conversely, any such diagram forms a right $k\Gamma$-module.
- Notice: no commutative condition is assumed on diagram.

Review of Path algebra

- $e_v R$ forms the set of indecomposable projective modules.

	is spanned by
R	paths
$\operatorname{rad} R$	paths of length ≥ 1
$e_v R$	paths which begins with v
$\operatorname{rad} e_v R$	paths which begins with v of length ≥ 1

So

$$\operatorname{rad} e_v R = \bigoplus_{v \to w} e_w R.$$

In particular, $\operatorname{Ext}^{>1}(e_v R / \operatorname{rad} e_v R, -) = 0$, then so all the modules.

When the path algebra is of finite representation type

- If so, then for any given dimension of Ve_v at each vertex, say $\dim Ve_v = m_v$, the right module V has only finite many isomorphic classes.

$$\boxed{\begin{array}{l} k\Gamma \text{ structure over } V \text{ such} \\ \text{that } Ve_v = k^{m_v}. \end{array}} = \prod_{\text{edge } \gamma: v \to w} \mathrm{Hom}(Ve_v, Ve_w).$$

- Two structures are isomorphic if and only if they are in the same orbit under the action of

$$\prod_{\text{vertex } v} \mathrm{GL}(Ve_v).$$

When the path algebra is of finite representation type

- For a quiver Γ, assume the vertices is $\{1, \dots, n\}$, and there is a_{ij} many edges $i \to j$. Denote the **Cartan matrix** $C = (c_{ij}) = (\delta_{ij} - a_{ij}) + (\delta_{ji} - a_{ji})$. Consider the bilinear form

$$B(x) = \sum_i x_i^2 - \sum_{i<j} a_{ij} x_i x_j = \frac{1}{2} x^\top C x.$$

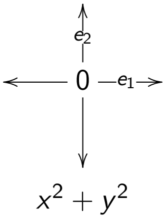
- Then consider the dimension of

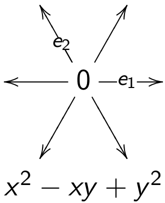
$$\prod_{\text{edge } \gamma : v \to w} \mathrm{Hom}(Ve_v, Ve_w) \Big/ \prod_{\text{vertex } v} \mathrm{GL}(Ve_v) \text{ is finite.}$$

To ensure this is finite we need $B(m) \geq 1$ for all $m = (m_i)$. Since the action of a scalar in the group is trivial.

When the path algebra is of finite representation type

- Now $B(x)$ is positive-definite. The integer solution of $B(x) = 1$ forms a root system. Since the roots are all of the same length, so it is classified by the simply-laced Dynkin diagrams.





Gabriel theorem, necessity

Theorem

For a path algebra $k\Gamma$,

of finite representation type	$\Rightarrow$	Γ is a disjoint union of simply-laced Dynkin diagrams after forgetting the directions of edges.

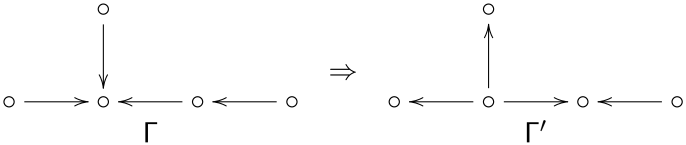
- The rest of efforts will be devoted to show the sufficiency.

Reflection functor

- We are going to construct a pair of functors between

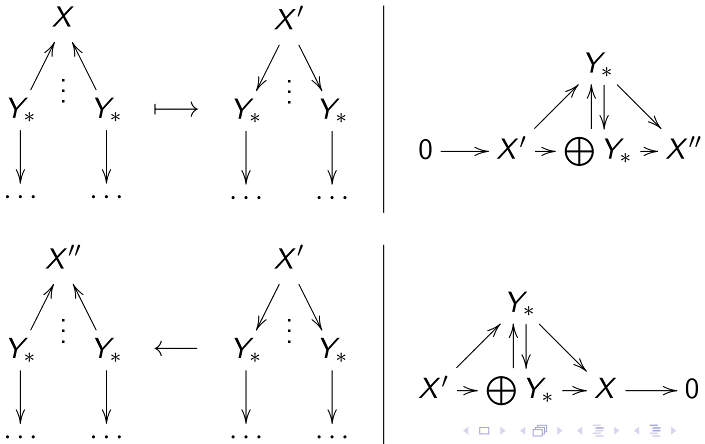
$$\text{mod-}k\Gamma \rightleftarrows \text{mod-}k\Gamma'$$

with Γ' reverse the direction of edges of Γ from one of the sink (only arrows in).



Reflection functor

- We can define the functors to be as the following



Reflection functor

- Then the reflection functor set a bijection between

$$\boxed{\text{ind } k\Gamma\text{-module}} \setminus \begin{smallmatrix} & & k & & \\ & \nearrow & & \nwarrow & \\ 0 & & \cdots & & 0 \\ \cdots & & & & \cdots \end{smallmatrix} \longleftrightarrow \boxed{\text{ind } k\Gamma'\text{-module}} \setminus \begin{smallmatrix} & & k & & \\ & \swarrow & & \searrow & \\ 0 & & \cdots & & 0 \\ \cdots & & & & \cdots \end{smallmatrix}$$

- Moreover, we have a short exact sequence, for indecomposable $\begin{smallmatrix} & & X & & \\ & \nearrow & & \nwarrow & \\ Y_* & & \cdots & & Y_* \\ \cdots & & & & \cdots \end{smallmatrix} \leftrightarrow \begin{smallmatrix} & & X' & & \\ & \swarrow & & \searrow & \\ Y_* & & \cdots & & Y_* \\ \cdots & & & & \cdots \end{smallmatrix}$,

$$0 \to X' \to \bigoplus Y_* \to X \to 0.$$

Reflection functor

- If we denote $\operatorname{Dim}(M) = (\dim M_v)_{v \text{ vertex}}$, then we can compute

$$\operatorname{Dim}\left(\begin{array}{ccccc} & \swarrow & X' & \searrow & \\ Y_* & & \cdots & & Y_* \\ \cdots & & & & \cdots \end{array}\right) = s_i \operatorname{Dim}\left(\begin{array}{ccccc} & \nearrow & X & \nwarrow & \\ Y_* & & \cdots & & Y_* \\ \cdots & & & & \cdots \end{array}\right),$$

  with $s_i x = x - 2B(x, e_i)e_i$, where $e_i = (\delta_{iv})$, the reflection with respect to the vertex i whose direction changed.

- Note that

$$s_i \operatorname{Dim}\left(\begin{array}{ccccc} & \nearrow & k & \nwarrow & \\ 0 & & \cdots & & 0 \\ \cdots & & & & \cdots \end{array}\right) = -\operatorname{Dim}\left(\begin{array}{ccccc} & \nearrow & k & \nwarrow & \\ 0 & & \cdots & & 0 \\ \cdots & & & & \cdots \end{array}\right).$$

Gabriel theorem, sufficiency

- Then, a combinatorial argument shows, for a quiver Γ whose underlying undirected diagram is loopfree, we can do the arrow-reverse operator to some sink several times and go back to itself. What's more, each vertices are operated once.
- For any composition of reflection with respect to $e_1, \dots, e_n$ in any order, say g, for any v, there is some i such that $g^i v$ has some index negative for some $i \leq \operatorname{ord} g$.
  Hint: consider $v + gv + g^2 v + \cdots$.
- This means, there is a finite series of reflection functors $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N$ such that for any indecomposable M, $\mathcal{S}_i \cdots \mathcal{S}_1(M) = 0$, for some i.
- Thus $\mathcal{S}_{i-1} \cdots \mathcal{S}_1(M)$ is a simple module for the smallest i. So the number of M is finite.

What's more can we say?

- We see, that the reflection functor is reversible on an indecomposable M if and only if the reflection on $\mathrm{Dim}\, M$ is nonnegative. So

$$\boxed{\text{ind } k\Gamma\text{-module}} \leftrightarrow \boxed{\text{positive roots}}.$$

- Actually, we can do the arrow-reverse operator to get any orientation for a quiver of underlying graph loopfree.
  Hint: a such operator is just put a lowest vertex to the topmost.

Remarks

- The ring of projective dimension ≤ 1 is well-understood, for example, a hereditary commutative Noetherian ring is product of Dedekind domain. The Artinian case is nearly what we discussed today. The general noncommutative Dedekind domain is also considered in ring theory.
- The classification of the algebra with radical 2-nilpotent is proven by a stable equivalence to path algebra. More exactly, for an algebra R with $\operatorname{rad}^2 R = 0$, then it is of finite representation type if and only if the quiver associated to the algebra $\begin{pmatrix} R/\operatorname{rad} R & \operatorname{rad} R \\ & R/\operatorname{rad} R \end{pmatrix}$ is a disjoint of simply-laced Dynkin diagrams.

Remarks

- The above theorem only works for algebraically closed field. For general case, it appears non-simply-laced Dynkin diagrams.
- The tame type is also classified by affine Dynkin diagrams (i.e. Euclidean diagrams).
- Actually, we do not get the whole "representation theory" for ADE type quiver. To get the Auslander–Reiten quiver is still a not short way to go.
- To see an isomorphic class to be a orbit under the action of $\prod \mathrm{GL}$ leads to a more geometric consideration, i.e. quiver variety.

Tilting theory

- A question is what is the transpose of a simple module?

$$\bigoplus_{v\to w} e_w R \to e_v R \to S_v \to 0$$

$$Re_v \to \bigoplus_{v\to w} Re_w \to \operatorname{Tr} S_v \to 0$$

Add $P = \bigoplus_{a\neq v} Re_a$, and $T = \operatorname{Tr} S_v \oplus P$.

- When v is a peak (only arrows out), then the (inverse) reflection functor can be expressed as $V \mapsto V'$ where

$$V \otimes_R \begin{pmatrix} Re_v \\ P \end{pmatrix} \to V \otimes_R \begin{pmatrix} \bigoplus_{v\to w} Re_w \\ P \end{pmatrix} \to V' \to 0$$

So $V' = V \otimes_R T$.

Tilting theory

- Conversely, when v be a sink the reflection functor can be expressed as $V' \mapsto V''$,

$$0 \to V'' \to \mathrm{Hom}^R(T, V') \to \mathrm{Hom}^R(T, V'),$$

  for some T, so $V'' = \mathrm{Hom}^R(T, V')$.
- Actually, in above two cases, $\mathrm{End}(T)$ gives the quiver algebra with desired arrow reversed.
- In term of the modern language, the module T is called a **tilting module**.

References for associative algebras

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Thanks