Lecture 8 — Representation of associative algebra (I)

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Functor category

Auslander–Reiten theory

References

Thanks

Recall

- We still assume the modules mentioned to be finite dimensional.
- What is easy

$$\boxed{\text{projective modules}} \leftrightarrow \boxed{\text{simple modules}} \leftrightarrow \boxed{\text{injective modules}}.$$

- What is difficult

$$\boxed{\text{indecomposable modules}}.$$

- Can we find a category such that indecomposable modules is "easy"?

Functor category

- Let R be a finite dimensional k-algebra. Consider $\mathrm{Fun}(R)$ the full subcategory of additive functors of $(k\text{-vec})^{R\text{-mod}}$ in $\mathcal{A}^{\mathcal{C}}$. Or more exactly

$$\mathrm{Fun}(R) = \begin{cases} \textbf{Object}: & F : R\text{-mod} \to k\text{-vec} \quad \text{additive functor} \\ \textbf{Morphism}: & \Big\downarrow \alpha \quad \text{natural transform} \\ & F' = R\text{-mod} \to k\text{-vec} \end{cases}$$

- It forms a sub-abelian category of $\mathcal{A}^{\mathcal{C}}$

Yoneda embedding

- We can define the **Yoneda embedding** a cotravariant functor

$$\tilde{*}: R\text{-mod} \to \mathrm{Fun}(R) \qquad M \mapsto \mathrm{Hom}_R(M, -).$$

Theorem (Yoneda)

$$\mathrm{Mor}_{\mathrm{Fun}(R)}(\tilde{M}, F) = F(M).$$

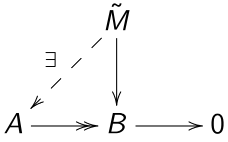
$$\mathrm{Hom}_R(M, N) = \mathrm{Mor}_{R\text{-mod}}(M, N) = \mathrm{Mor}_{\mathrm{Fun}(R)}(\tilde{N}, \tilde{M}).$$

$$M \tilde{\oplus} N = \tilde{M} \oplus \tilde{N} \qquad \widetilde{\mathrm{cok}\, f} = \ker \tilde{f}.$$

- We call an $F \in \mathrm{Fun}(R)$ **representable** if $F \cong \tilde{M}$.

Projective object

- For an $M \in R$-mod, $\tilde{M}$ is projective



$$\iff \mathrm{Mor}(\tilde{M}, A) \twoheadrightarrow \mathrm{Mor}(\tilde{M}, B)$$

$$\iff A(M) \twoheadrightarrow B(M)$$

$$\Longleftarrow A \twoheadrightarrow B$$

- So

$$\boxed{\text{representability}} \Rightarrow \boxed{\text{projectivity}}$$

Projective object

- We call an $F \in \mathrm{Fun}(R)$ is **finitely generated** if it is isomorphic to some quotient of $\tilde{M}$.
- If F is finitely generated and projective, then it is representable.

$$\begin{aligned} & F \text{ projective} \\ \iff & \text{some nontrivial idempotent } \tilde{e} : \tilde{M} \to F \to \tilde{M} \\ \iff & \text{some nontrivial idempotent } e : M \to M, F = e(\tilde{M}) \end{aligned}$$

$$\boxed{\text{projectivity} + \text{f.g.}} \Rightarrow \boxed{\text{representability}}.$$

- Through the Yoneda embedding,

$$\{R\text{-module}\} \stackrel{1:1}{\longleftrightarrow} \{\text{f.g. projective object}\}$$

$$\{\text{indecomposable } R\text{-module}\} \leftrightarrow \{\text{f.g. indecomposable projective object}\}$$

Simple objects

- Define rad the intersection of maximal proper sub-object (may not exist).
- For any simple object $S \in \mathrm{Fun}(R)$, there is a unique indecomposable $M \in R$-mod, such that $F = \tilde{M}/\operatorname{rad}\tilde{M}$.
- Actually, M is the indecomposable module such that

$$S(M) \neq 0 \iff \exists \tilde{M} \overset{\neq 0}{\to} S \overset{S \text{ is simple}}{\Rightarrow} \exists \tilde{M} \twoheadrightarrow S.$$

  It is unique by the general fact of projective covering.
- Then we get

$$\{R\text{-module}\} \overset{1:1}{\longleftrightarrow} \{\text{semisimple object in } \mathrm{Fun}(R)\}$$

$$\{\text{indecomposable } R\text{-module}\} \leftrightarrow \{\text{simple object}\}$$

Auslander–Reiten quiver

- When M, N is indecomposable,

$$\text{rad}\, \tilde{M}(N) = \{M \xrightarrow{f} N : f \text{ is not invertible}\} := \text{rad}(M, N).$$

$$\text{rad}(\text{rad}\, \tilde{M})(N)\{M \xrightarrow{f} L \xrightarrow{g} N : f \in \text{rad}(M, L), g \in \text{rad}(L, N)\} := \text{rad}^2(M,$$

- One can define the **Auslander–Reiten quiver**

$$\text{vertices} = \{\text{indecomposable modules}\},$$
$$\dim \frac{\text{rad}(M,N)}{\text{rad}^2(M,N)} \text{ many edges } M \to N.$$

Remarks

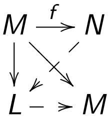
- Actually, $\mathrm{Fun}(R)$ is equivalent to the category of modules of its **Auslannder algebra (may be infinite dimensional)**. Then the radical is literally radical.
- Actually, denote $S_M = \tilde{M}/\operatorname{rad}\tilde{M}$,

$$\begin{aligned}\mathrm{Ext}_{\mathrm{fun}(R)}(S_M, S_N) &= \mathrm{Hom}_{\mathrm{fun}(R)}(\operatorname{rad}\tilde{M}/\operatorname{rad}^2\tilde{M}, S_N) \\ &\stackrel{?}{=} \mathrm{Hom}_{\mathrm{fun}(R)}(S_N, \operatorname{rad}\tilde{M}/\operatorname{rad}^2\tilde{M}) \\ &= \mathrm{Hom}_{\mathrm{fun}(R)}(\tilde{N}, \operatorname{rad}\tilde{M}/\operatorname{rad}^2\tilde{M}) \\ &= \frac{\operatorname{rad}(M,N)}{\operatorname{rad}^2(M,N)}\end{aligned}$$

Minimal resolution

- A map $M \xrightarrow{f} N$ is said to be **left almost split** if f does not admit a retraction and

  For any module L, any map $M \to L$, either factors through N, either admits a retraction.



$$\text{is called } \textbf{left minimal} \text{ if } \begin{array}{ccc} M & \xrightarrow{f} & N \\ \| & & \downarrow \forall h \\ M & \xrightarrow[f]{} & N \end{array} \text{ is invertible.}$$

- Assume M is indecomposable,

  $\tilde{N} \xrightarrow{\tilde{f}} \tilde{M} \to \tilde{M}/\operatorname{rad}\tilde{M} \to 0$ is a minimal resolution
  $\iff$ f is left almost split and left minimal.

Almost split sequence

- If a short exact sequence

$$0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$$

satisfies the following equivalent conditions, we will call it **almost split sequence**

  - f is left almost split and left minimal.
  - f is left almost split and g is right almost split.
  - g is right almost split and right minimal.
  - M is indecomposable, and g is right almost split.
  - L is indecomposable, and f is left almost split.

Auslander–Reiten transpose

- Let $M \in R$-mod,

$$\begin{array}{cl} P_1 \xrightarrow{f} P_0 \to M \to 0 & \text{minimal projective resolution} \\ \Downarrow & \text{apply } *^t = \operatorname{Hom}_R(*, R) \\ 0 \to M^t \to P_0^t \underset{f^t}{\to} P_1^t \to \operatorname{Tr} M \to 0 & \operatorname{Tr} M = \operatorname{cok} f^t \end{array}$$

- Note that $\operatorname{Tr} : R\text{-mod} \to R^{\mathrm{op}}\text{-mod}$,

$$\boxed{\text{nonprojective left modules}} \underset{\operatorname{Tr}}{\overset{\operatorname{Tr}}{\rightleftarrows}} \boxed{\text{nonprojective right modules}}$$

Stable homomorphism

- For M, N, denote the **stable homomorphism**

$$\underline{\mathrm{Hom}}_R(M,N) = \frac{\mathrm{Hom}_R(M,N)}{\{M \to P \to N : \text{ for some projective } P\}}$$

Theorem (Auslander–Reiten Formula)

For $M, N \in R$-mod,

$$\begin{aligned}\underline{\mathrm{Hom}}_R(M,N) &= \mathrm{Tor}_R(\mathrm{Tr}\, M, N) \\ &= \mathrm{D}\,\mathrm{Ext}^R(\mathrm{Tr}\, M, \mathrm{D}\, N) = \mathrm{D}\,\mathrm{Ext}_R(N, \mathrm{D}\,\mathrm{Tr}\, M),\end{aligned}$$

where $DM = M^{\vee}$ the dual space.

Auslannder–Reiten Theorem

Theorem (Auslannder–Reiten)

If M is indecomposable and non-projective, then there exists an almost split sequence

$$0 \to \mathrm{D}\,\mathrm{Tr}\, M \to E \to M \to 0.$$

$$0 \neq * \in \quad \mathrm{D}(S_M) \hookrightarrow \mathrm{D}\,\underline{\mathrm{Hom}}(M, M) \cong \mathrm{Ext}(M, \mathrm{D}\,\mathrm{Tr}\, M) \quad \ni [E].$$

When M is indecomposable and projective, then $\mathrm{rad}\, M \to M$ *is right almost split and right minimal.*

Auslannder–Reiten Theorem

Theorem (Auslannder–Reiten)

If M is indecomposable and non-injective, then there exists an almost split sequence

$$0 \to M \to E \to \operatorname{Tr} \operatorname{D} M \to 0.$$

When M is indecomposable and injective, then $M \to M/\operatorname{soc} M$ is left almost split and left minimal.

Auslannder–Reiten Theorem

- In conclusion, if M is indecomposable, the finitely generated minimal resolution exists and is

$$\begin{cases} 0 \to \widetilde{\operatorname{Tr} \operatorname{D} M} \to \tilde{E} \to \tilde{M} \to \tilde{M}/\operatorname{rad} \tilde{M} \to 0 & M \text{ is non-injective} \\ 0 \to \widetilde{M/\operatorname{soc} M} \to \tilde{M} \to \tilde{M}/\operatorname{rad} \tilde{M} \to 0 & M \text{ is injective} \end{cases}$$

- In particular, for M, N indecomposable,

$$\tilde{E}/\operatorname{rad} \tilde{E} \cong \operatorname{rad} \tilde{M}/\operatorname{rad}^2 \tilde{M}$$

  So any $f \in \operatorname{rad}(M, N)/\operatorname{rad}^2(M, N)$ is from $M \to E \overset{\oplus}{\twoheadrightarrow} N$.

- When we take a decomposition of E into indecomposable modules $E = \bigoplus E_i$. Then $\{M \to E \twoheadrightarrow E_i \cong N\}$ forms a basis for $\operatorname{rad}(M, N)/\operatorname{rad}^2(M, N)$.

Theorem

If R is of finite representation type, then its Auslander–Reiten quiver has no multiple edges.

- Otherwise, assume that $M \to N$ has more than 2 edges.
- If $f \in \operatorname{rad}(M, N) \setminus \operatorname{rad}^2(M, N)$, by consider the $\operatorname{im} f$, we know that f is surjective or injective but not isomorphic.
- Without loss of generality that $\dim M > \dim N$.

$$0 \to \operatorname{Tr} \mathrm{D}\, N \to M^2 \oplus E \to N \to 0$$

  So $\dim \operatorname{Tr} \mathrm{D}\, N \geq \dim N$, and $\operatorname{Tr} \mathrm{D}\, N \to M$ has more than 2 edges.
- Continue this process, we get infinite indecomposable modules of different dimensions.

Brauer-Thrall conjecture

Theorem (Brauer-Thrall conjecture)

A finite dimensional algebra is either of finite representation type or there exist indecomposable modules with arbitrarily large dimension.

Theorem (Harada and Sai)

If M_i's are indecomposable and $\dim M_i \leq b$, *Any chain of maps*

$$M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{2^b-1}} M_{2^b}$$

has its composition vanishing.

Brauer-Thrall conjecture

- If $\mathrm{Hom}(M, N) \neq 0$, then M and N can be connected by directed path $M \to \cdots \to N$ in Auslander–Reiten quiver since

$$0 = \mathrm{rad}^{2^b}(M, N) \subseteq \cdots \subseteq \mathrm{rad}(M, N)^2 \subseteq \mathrm{rad}(M, N) \subseteq \mathrm{Hom}(M, N).$$

- For any indecomposable M, there exists an indecomposable P with $\mathrm{Hom}(P, M) \neq 0$.
- But indecomposable projective modules are finite, and by Auslander–Reiten theorem (the existence of minimal resolution), Auslander–Reiten quiver are "locally finite", so there is only finite many vertices.

More functor category

- We can consider $\underline{\mathrm{Fun}}^{\vee}(R) \subseteq \mathrm{Fun}^{\vee}(R)$ the category of contravariant functors F vanishing over all projective modules; Then

$$\boxed{\text{f.g. proj. object}} = \underline{\mathrm{Hom}}(-, M) \qquad \boxed{\text{f.g. inj. object}} = \mathrm{Ext}(-, M).$$

- One can also consider $\underline{\mathrm{Fun}}(R) \subseteq \mathrm{Fun}(R)$ the category of functors F vanishing over all projective modules.Then

$$\boxed{\text{f.g. proj. object}} = \underline{\mathrm{Hom}}(M, -).$$

- The Auslander–Reiten formula describes how this reflects duality functor

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{M \mapsto \underline{\mathrm{Hom}}(M,-)} & \underline{\mathrm{Fun}}(R) \\ {\scriptstyle \mathrm{D\,Tr}} \downarrow & & \downarrow {\scriptstyle \mathrm{D}} \\ R\text{-mod} & \xrightarrow[M' \mapsto \mathrm{Ext}(-,M')]{} & \underline{\mathrm{Fun}}^{\vee}(R) \end{array}$$

References for associative algebras

- Auslander, Reiten, Smalø. Representation theory of Artin algebras.
- Benson. Representations and cohomology.
- Assem, Simson, Skowroński. Elements of the Representation Theory of Associative Algebras Volume 1 Techniques of Representation Theory.

Thanks