Overview of Representation theory

Lecture 7 — The structures of algebras and groups (III)

Xiong Rui

July 20, 2020

Indecomposable modules

- **From now on, all the modules mentioned are assumed to be finite dimensional**.
- For a nonezero module M, if the only direct summand is 0 and itself, then it is called **indecomposable**.

Theorem (Fitting)

M is indecomposable iff $\mathrm{End}(M)$ is a local ring, i.e. a ring (not necessarily commutative) whose element is nilpotent or invertible.

- Clearly, for a local ring R, $R/\{\text{nilpotent elements}\}$ is a division ring.

Simple modules makes up modules

- If a module M is Noetherian and artinian (finite-dimensional linear space), then there is a decomposition

$$M = M_1 \oplus \cdots \oplus M_n$$

such that each M_i is indecomposable.

Theorem (Krull–Schmidt)

The indecomposable modules are unique with multiplicity.

$$\underbrace{\boxed{\text{indecomposable modules}}}_{\text{Fitting}} \xrightarrow[*]{\text{K–S}} \boxed{\text{all ``small'' modules}}.$$

We lose nearly nothing!

- Naïvely, do we have

$$\text{small modules} \xleftrightarrow{1:1} \text{multiple-sets of indecomposable modules?}$$

- Yes! If we forget the morphisms.
- But this is the main difficulty,

simple	a few of things
indecomposable	everything

Radical

- For an associative algebra R, define its **radical**

$$\operatorname{rad} R = \sum \text{nilpotent ideals.}$$

  It turns out to be the intersection of maximal ideals.
- Then $R/\operatorname{rad} R$ is a semisimple algebra.

Theorem (Wedderburn-Malcev)

If k is perfect, for any k-algebra R, the following short exact sequence

$$0 \to \operatorname{rad} R \to R \to R/\operatorname{rad} R \to 0$$

splits by an algebra homomorphism $R/\operatorname{rad} R \to R$.

Remarks

- For a Lie algebra $\mathfrak{g}$, define

$$\mathrm{rad}_{\mathrm{nil}}\, \mathfrak{g} = \sum \text{nilpotent ideals.}$$

- Then $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is reductive.

Theorem (Levi)

For complex Lie algebra $\mathfrak{g}$, the following short exact sequence

$$0 \to \operatorname{rad}\mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g} \to 0$$

splits by a Lie algebra homomorphism $\mathfrak{g}/\operatorname{rad}\mathfrak{g} \to \mathfrak{g}$.

Radical

- For a (finite dimensional) module M, we can define its **radical**

$$\operatorname{rad} M = \operatorname{rad} R \cdot M.$$

  It turns out to be the intersection of maximal submodules.

- Then $M/\operatorname{rad} M$ is a semisimple module of R and also $R/\operatorname{rad} R$.

Theorem (Nakayama)

Let M, N be two (finite dimensional) modules, and a morphism $N \to M$, then

$$\begin{array}{rrcl} & N & \twoheadrightarrow & M \\ \Leftrightarrow & N & \twoheadrightarrow & M/\operatorname{rad} M \\ \Leftrightarrow & N/\operatorname{rad} N & \twoheadrightarrow & M/\operatorname{rad} M \end{array}$$

$$\begin{array}{ccccccccc} 0 & \rightarrowtail & \operatorname{rad} N & \rightarrowtail & N & \rightarrowtail & N/\operatorname{rad} N & \rightarrowtail & 0 \\ & & \downarrow & & \downarrow & \searrow & \downarrow & & \\ 0 & \rightarrowtail & \operatorname{rad} M & \rightarrowtail & M & \rightarrowtail & M/\operatorname{rad} M & \rightarrowtail & 0 \end{array}$$

Projective cover

- For a module M, its projective cover $P \twoheadrightarrow M$ is defined to be

For any projective $\varphi : P' \twoheadrightarrow M$, it factors throught $P \twoheadrightarrow M$ by a surjection $P' \twoheadrightarrow P$.

$$\begin{array}{ccc} P' & \twoheadrightarrow & M \\ \downarrow & & \| \\ P & \twoheadrightarrow & M \end{array}$$

Theorem ($\iff$ Lift of idempotents)

For k-algebra, the projective cover exists for any (finite dimensional) module. Moreover, there is a bijection

$$\{\textit{projective modules}\} \underset{\textit{projective cover}}{\overset{P \mapsto P/\operatorname{rad} R}{\rightleftarrows}} \{\textit{semisimple modules}\}$$

$$\{\textit{projective indecomposable modules}\} \rightleftarrows \{\textit{simple modules}\}$$

Dual

- For a module M, its dual space $M^\vee = \mathrm{Hom}_k(M, k)$ is naturally an R-right module, i.e. an R^{op}-module. Now R^{op} is again a k-algebra, so all the results holds.

$$\boxed{\text{projective cover}} \leftrightarrow \boxed{\text{injective hull}}$$

$$\boxed{\text{radical}} \leftrightarrow \boxed{\text{socle}}$$

where socle is the sum of all simple submodules. There is a bijection

$$\{\text{injective modules}\} \underset{\text{injective hull}}{\overset{I \mapsto \mathrm{soc}\, I}{\rightleftarrows}} \{\text{semisimple modules}\}$$

$$\{\text{injective indecomposable modules}\} \rightleftarrows \{\text{simple modules}\}$$

Morita equivalence

- When k is algebraically closed, if R is semisimple, then it is just product of matrix algebras over k which is Morita equivalent to product of copies of k.

Theorem

When k is algebraically closed, any k-algebra R is Morita equivalent to R' with $R/\operatorname{rad} R$ a direct product of copies of k. More exactly,

$$\{R\text{-modules}\} \cong \{R'\text{-modules}\}.$$

Such algebra R' is called a **basic** *algebra.*

Pierce's trick

- Let R be a basic algebra. Assume

$$R/\operatorname{rad} R = S_1 \oplus \cdots \oplus S_n$$

with each S_i a one-dimension space. The projective cover of above gives rise to

$$R = P_1 \oplus \cdots \oplus P_n.$$

- Use the same trick,

$$\begin{aligned} R &\cong \operatorname{End}_R(R)^{op} & \qquad r \mapsto [s \mapsto sr] \\ &= \operatorname{End}_R(\textstyle\bigoplus_{i=1}^n P_i)^{op} \end{aligned}$$

We cannot simplify, because in which case, there is no general conclusion for $\operatorname{Hom}_R(P_i, P_j)$.

Quiver algebra

- For a quiver (directed diagram) $\Gamma = (V, E)$, define **the path algebra**

$$k\Gamma = \bigoplus_{\text{all paths } \gamma \text{ over } \Gamma} k\gamma$$

with composition

$$[i \to j] \cdot [k \to h] = \begin{cases} [i \to j = k \to h] & j = k \\ 0 & j \neq k. \end{cases}$$

- What we want is to describe R to be a quotient algebra of the path algebra of some quiver.

Quiver algebra

Theorem

For a k-algebra homomorphism $R' \to R$, if $R' \to R/\operatorname{rad} R^2$ is surjective, then so is $R' \to R$.

- So denote $\overline{R} = R/\operatorname{rad} R^2$, and $\overline{P}_i = P/\operatorname{rad} R^2 \cdot P$, then

$$\begin{aligned} \overline{R} &\cong \operatorname{End}_{\overline{R}}(\overline{R})^{op} & r \mapsto [s \mapsto sr] \\ &= \operatorname{End}_{\overline{R}}(\textstyle\bigoplus_{i=1}^{n} \overline{P}_i)^{op} \end{aligned}$$

- Denote

$$\begin{aligned} \operatorname{rad}(P_i, P_j) &= \{P_i \xrightarrow{f} P_j \text{ not invertible}\} \\ &= \begin{cases} \operatorname{Hom}_R(P_i, P_j) & i \neq j \\ \operatorname{rad}\operatorname{End}_R(P_i) & i = j \end{cases} \\ &= \operatorname{Hom}_R(P_i, \operatorname{rad} P_j). \end{aligned}$$

Quiver algebra

- Consider the quiver Γ associated to R,

$$\text{vertices} = \{1, \ldots, n\} \quad \dim \operatorname{rad}(\overline{P}_i, \overline{P}_j) \text{ many edges } i \to j$$

  Note that $\operatorname{rad}(\overline{P}_i, \overline{P}_i)$ is codimension 1 in $\operatorname{End}(\overline{P}_i)$.
- By picking a basis for each $\operatorname{Hom}_{\overline{R}}(\overline{P}_i, \overline{P}_j)$ and lifting to $\operatorname{Hom}_R(P_i, P_j)$, we get a desired surjective algebra homomorphism

$$k\Gamma \to R.$$

- The quiver associated to the path algebra of a quiver is itself.

Remarks

- If we write $P_i = Re_i$, then

$$
\begin{aligned}
\mathrm{Hom}_R(P_i, \operatorname{rad} P_j) &= \mathrm{Hom}_R(Re_i, \operatorname{rad} Re_j) \\
&= e_i \operatorname{rad} Re_j \qquad\qquad f \mapsto f(e_i) \\
\mathrm{Hom}_{\overline{R}}(\overline{P}_i, \overline{P}_j) &= e_i(\operatorname{rad} R / \operatorname{rad} R^2)e_j.
\end{aligned}
$$

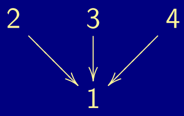
- Actually,

$$
\begin{aligned}
\mathrm{Ext}_R(S_i, S_j) &= \mathrm{Hom}_R(\operatorname{rad} P_i / \operatorname{rad}^2 P_i, S_j) \\
&= \mathrm{Hom}_R(S_j, \operatorname{rad} P_i / \operatorname{rad}^2 P_i) \\
&= \mathrm{Hom}_R(P_j, \operatorname{rad} P_i / \operatorname{rad}^2 P_i) \\
&= \mathrm{Hom}_{\overline{R}}(\overline{P}_i, \operatorname{rad} \overline{P}_j)
\end{aligned}
$$

- Major of professional books work over right modules, so the definition is inverse.

Quiver algebra

- Example



$$R = \begin{pmatrix} k & & & \\ k & k & & \\ k & & k & \\ k & & & k \end{pmatrix}$$

$$R = k[t]$$

$$R = k\langle X, Y \rangle$$

The module of path algebra

- Giving a module of a path algebra $k\Gamma$, is nearly given a diagram of linear spaces.
- Let V be a right module of $k\Gamma$. Consider

$$V_i = Ve_i, \qquad e_i = \text{the path of staying at vertex } i$$

  For each edge $i \xrightarrow{\gamma} j$, since $e_i \gamma e_j = \gamma$, it defines a map $V_i \to V_j$. So we get a diagram of the sharp Γ.
- Conversely, any diagram of the sharp Γ makes the sum of vector spaces a $k\Gamma$ module.

Finite type

- The main aim of representation of associative algebras is to classifying all indecomposable modules (so all modules).

Theorem (Drozd, Crawley-Boevey)

Every finite-dimensional algebra is either of finite, tame or wild type, and these presentation types are mutually exclusive.

- An algebra is called of **finite presentation type**, if there are only finite many indecomposable modules.
- An algebra R is said to be of **tame presentation type**, (roughly speaking) if the indecomposable modules can be parameterized by indecomposable modules of $k[t]$, the polynomial algebra.
- An algebra R is said to be **wild presentation type**, (roughly speaking) if the indecomposable modules can be parameterized by indecomposable modules of $k\langle X, Y\rangle$, or equivalently, cannot determined by Turing machine.

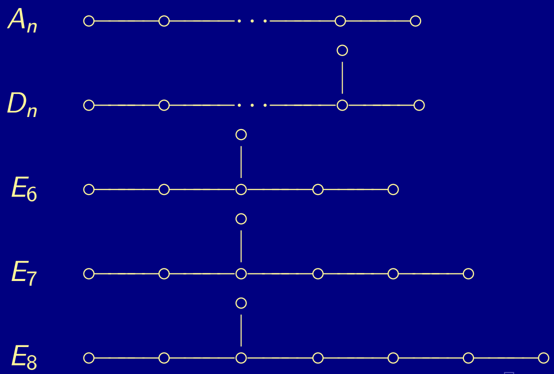
Remarks

- It is no hope to describe all finite presentation type algebras.
- There are very limited type of algebras we know when it is of finite/tame type.
- Path algebras for quivers
  — answer: Dynkin diagrams.
- Algebra with radical 2-nilpotent, i.e. $\operatorname{rad} R^2 = 0$
  — answer: also Dynkin diagrams.
- Group algebras of finite groups
  — answer: the size of its Sylow subgroup.
- **To understand the reason of Dynkin diagram appearing again is one of the main purpose in our last lectures.**

When a path algebra to be of finite type

Theorem (Gabriel)

A path algebra $k\Gamma$ is of finite presentation type if and only if Γ is a disjoint union of simply-laced Dynkin diagrams after forgetting the directions of edges.



References for associative algebras

- Pierce. Associative algebras.
- Auslander, Reiten, Smalø. Representation theory of Artin algebras.
- Benson. Representations and cohomology.
- Assem, Simson, Skowroński. Elements of the Representation Theory of Associative Algebras Volume 1 Techniques of Representation Theory.

Thanks