Overview of Representation theory

Lecture 6 — Representation of lie algebras (II)

Xiong Rui

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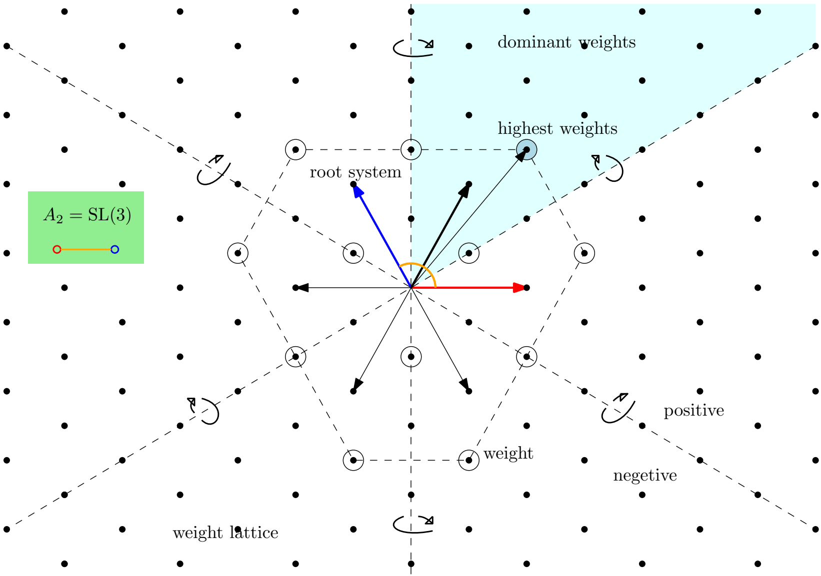

dominant weights
highest weights
root system
$A_2 = \mathrm{SL}(3)$
positive
weight
negetive
weight lattice

Character

- For two $\mathfrak{g}$-representations V and W, we can introduce a $\mathfrak{g}$-representation structure on $V^\vee$ and $V \otimes W$ by

$$X \cdot f = [v \mapsto -f(Xv)], \qquad X \cdot (v \otimes w) = xv \otimes w + v \otimes xw.$$

- For a $\mathfrak{g}$-representation V, define its **character**

$$\mathrm{ch}(V) = \sum_{\lambda \in \Lambda} (\dim V_\lambda) e^\lambda \in \mathbb{Z}[e^\Lambda]$$

the group ring of weight lattice. Then of course,

$$\mathrm{ch}(V^\vee) = \overline{\mathrm{ch}(V)}, \qquad \mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W).$$

Weyl group

- For a root system Δ, define the **Weyl group** generated by reflections of roots.
- For compact group G, and a maximal torus T, we define the **Weyl group** to be

$$N_G(T)/T = \{g \in G : gTg^{-1} = T\}/T.$$

Through the adjoint action of $N_G(T)/T$ on $\mathrm{Lie}(T) \to \mathrm{Lie}(T)$, they coincides.

- The Weyl group acts on $\mathfrak{g}$ by setting the reflection of α acts as

$$\exp(\mathrm{ad}(X_\alpha))\exp(-\mathrm{ad}(Y_\alpha))\exp(\mathrm{ad}(X_\alpha)).$$

First properties of characters

- For any representation V, for any weight $\lambda \in \Lambda$ and Weyl group element $w \in W$

$$\dim V_\lambda = \dim V_{w\lambda}.$$

- The characters forms a basis for $\mathbb{Z}[e^\Lambda]^W$, i.e. the symmetric polynomials.

Theorem (Weyl character formula)

Let λ be a dominant weight, the character of unique representation V_λ of highest weight λ is

$$\mathrm{ch}(V_\lambda) = \frac{\sum_{w\in W}(-1)^w e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^w e^{w\rho}}, \qquad \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha,$$

where $(-1)^w = \det w$ is 1 if w is a product of even simple reflections, is -1 otherwise.

Special linear algebra

- $\mathfrak{sl}_n$ is a simple Lie algebra.

Lie algebra	$\mathfrak{g}$	$\mathfrak{sl}_n = \{A \in \mathbb{M}_n(\mathbb{C}) : \operatorname{tr} A = 0\}$
Cartan subalgebra	$\mathfrak{h}$	$\{\operatorname{diag}(a_1, \ldots, a_n) : a_1 + \cdots + a_n = 0\}$
roots	Δ	$\{\alpha(i,j) : \operatorname{diag}(a_i) \longmapsto a_i - a_j\}_{i \neq j}$
the $\mathfrak{sl}_2$ triple	$\mathfrak{s}_\alpha$	$\mathfrak{s}_{\alpha(i,j)} = \mathbb{C} \cdot (E_i - E_j) \oplus \mathbb{C} \cdot E_{ij} \oplus \mathbb{C} \cdot E_{ji}$
basis	Φ	$\{\alpha(i, i+1) : \operatorname{diag}(a_i) \longmapsto a_i - a_{i+1}\}_{i=1}^{n-1}$
polarization	Δ^+	$\{\alpha(i,j) : i < j\}$
weight	Λ	$\{(\lambda_i) + C : \operatorname{diag}(a_i) \longmapsto \sum \lambda_i a_i : \lambda_i - \lambda_j \in \mathbb{Z}\}$
dominant weights	$\Lambda_{\geq 0}$	$\{(\lambda_i) + C : \operatorname{diag}(a_i) \longmapsto \sum \lambda_i a_i : \lambda_i \text{ decreasing}\}$

Special linear algebra

- For $\mathfrak{sl}_n$, the Weyl group is $\mathfrak{S}_n$, the reflection of α_{ij} corresponds the the swap $i \leftrightarrow j$.
- Its action on the weight lattice is

$$(\lambda_1, \dots, \lambda_n) + C \xmapsto{\sigma} (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) + C$$

- We can use a Young diagram with length $< n$ to present a dominant weight λ, say, by the unique $(\lambda_1, \dots, \lambda_{n-1}, 0)$ presented it.
- The $\rho = \frac{1}{2}(n-1, n-3, \cdots, -n+1) + C = (n-1, \cdots, 1, , 0)$,

Schur polynomial again

- By the Weyl character formula,

$$\mathrm{ch}(V_\lambda) = \frac{\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma e^{\sigma(\lambda+\rho)}}{\sum_{w\in W}(-1)^\sigma e^{\sigma(\rho)}} = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})} = s_\lambda(x)$$

where s_λ the Schur polynomial (see lecture 4), with

$$x_i = \exp(0,\dots,0,\underbrace{1}_{i},0,\dots,0), \qquad x_1\cdots x_n = 1,$$

Equivalently, it is a function in $\mathrm{diag}(x_1,\dots,x_n)\in \mathrm{SL}(n)$.

$$\boxed{\text{characters of } \mathfrak{sl}_n} = \boxed{\text{Schur polynomials}}$$

The construction

- Let V be the natural representation of $\mathfrak{sl}_n \to \mathfrak{gl}_n = \mathfrak{gl}(V)$. Put the standard basis by $e_1, \dots, e_n$. The highest weight is $(1, \dots, 0)$, with vector e_1.
- Consider
$$V^{\otimes m} = V \otimes \overset{m}{\cdots} \otimes V.$$
It is a representation of $\mathfrak{gl}_n$, but also a representation of $\mathfrak{S}_m$ (write by right action). The action of them commutes.
- Let λ be a partition λ of m, and fix some filling of λ from 1 to $|\lambda|$ (no repetition) and denote again by λ. Denote r_λ and c_λ the row sum and column sum (see lecture 4).
- Consider $V_\lambda = V^{\otimes m} \cdot c_\lambda r_\lambda$.

The construction

- We present a monomial $v_1 \otimes \cdots \otimes v_m$ in Young diagram with v_i at the position filled by i.

$$v_1 \otimes \cdots \otimes v_m \leftrightarrow \begin{array}{ccccccc} v_1 & \otimes & v_2 & \otimes & v_3 & \otimes & v_4 \\ \otimes & & \otimes & & & & \\ v_5 & \otimes & v_6 & & & & \\ \otimes & & & & & & \\ v_7 & & & & & & \end{array}$$

- For a filling of λ with 1 to n, it corresponds to a monomial

$$\begin{array}{|c|c|c|c|} \hline 4 & 2 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \leftrightarrow \begin{array}{ccccccc} e_4 & \otimes & e_2 & \otimes & e_4 & \otimes & e_1 \\ \otimes & & \otimes & & & & \\ e_3 & \otimes & e_2 & & & & \\ \otimes & & & & & & \\ e_1 & & & & & & \end{array}$$

The construction

- For a filling of λ with 1 to n, we consider the monomial of it by M.
- If there is some repetition in some column, then $Mc_\lambda r_\lambda = 0$.
  — Since c_λ is an alternative sum.
- The weight of M is $(\phi_1, \ldots, \phi_n)$ with ϕ_i the number of i used to fill λ.
- The monomial M_0 of the filling i-th row by i is such that $M_0 c_\lambda r_\lambda \neq 0$.
  — Since the coefficient of M_0 itself is 1.

The construction

- If there is no repetition in some column, then $M \cdot c_\lambda r_\lambda \neq 0$, and M_0 is the only highest weight.
  — Since we can first make the first row, then the second, etc.

$$\ytableaushort{1242,23,4} \xrightarrow{E_{12}} \ytableaushort{1142,23,4} + \ytableaushort{1241,23,4} + \underbrace{\ytableaushort{1242,13,4}}_{=0}$$

$$\xrightarrow{E_{12}} \ytableaushort{1141,23,4} + 0 + \ytableaushort{1141,23,4} + 0$$

$$\xrightarrow{E_{14}} 2\,\ytableaushort{1111,23,4} + 0 \xrightarrow{E_{32}}\xrightarrow{E_{43}} 2\,\ytableaushort{1111,22,3}$$

- As a result, $V^{\otimes m} \cdot c_\lambda r_\lambda$ is the $\mathfrak{sl}_n$-irreducible representation of highest weight λ.

Schur–Weyl Duality

Theorem (Schur–Weyl)

$$V^{\otimes m} = \bigoplus_{\lambda \vdash m, length < n} V_\lambda \otimes S^\lambda.$$

where S^λ *is a right* $\mathfrak{S}_m$ *module via* $\sigma \mapsto \sigma^{-1}$.

$$\begin{aligned} V^{\otimes m} &= \bigoplus_{\lambda \vdash m} \mathrm{Hom}^{\mathfrak{S}_m}(S^\lambda, V^{\otimes m}) \otimes S^\lambda \\ &= \bigoplus_{\lambda} \mathrm{Hom}^{\mathfrak{S}_m}(r_\lambda c_\lambda \mathbb{C}[G], V^{\otimes m}) \otimes S^\lambda \\ &= \bigoplus_{\lambda} \mathrm{Hom}^{\mathfrak{S}_m}(c_\lambda r_\lambda \mathbb{C}[G], V^{\otimes m}) \otimes S^\lambda \\ &= \bigoplus_{\lambda} V^{\otimes m} c_\lambda r_\lambda \otimes S^\lambda = \bigoplus_{\lambda} V_\lambda \otimes S^\lambda \end{aligned}$$

Schur–Weyl Duality

- Actually, there is a general fact about duality.

Theorem

Let V be a semisimple module of some algebra $R \subseteq \mathrm{End}_k(V)$. Then $R' = \mathrm{End}_R(V)$ is semisimple, and $R = \mathrm{End}^{R'}(V)$. Moreover, V has the decomposition as R, R'-bimodule

$$V = \bigoplus_{U \text{ irr rep of } R} U \otimes \mathrm{Hom}_R(U, V),$$

with $\mathrm{Hom}_R(U, V)$ pairwise nonisomorphic irreducible R'-modules.

Schur–Weyl Duality

- In our case, it is easy to see that

$$\begin{aligned}\mathrm{End}(V^{\otimes m})^{\mathfrak{S}_m} &= (\mathrm{End}(V)^{\otimes m})^{\mathfrak{S}_m} = S^n(\mathrm{End}(V)) \\ &= \mathrm{span}\{g^m = g \otimes \cdots \otimes g : g \in \mathrm{GL}_n\}\end{aligned}$$

Hint: we can solve $x^k y^{m-k}$ in $(x + \lambda y)^m$ by Vandermonde determinant of different λ.

- Example,

$$V \otimes V = S^2 V \oplus \bigwedge^2 V.$$

More Duality theorem

- Duality is a very deep topic in representation theory.

Theorem (Howe duality)

There is a duality of $\mathrm{GL}(n)$ *and* $\mathrm{GL}(k)$ *over* $S(\mathbb{C}^n \otimes \mathbb{C}^k)$, *and*

$$S(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda \text{ length} < \min(k,n)} V_\lambda(n) \otimes V_\lambda(k).$$

- It can be derived from Schur–Weyl duality. An explanation of why $\mathrm{GL}(n)$ and $\mathrm{GL}(k)$ dually commute is given by Weyl algebra.

More Duality theorem

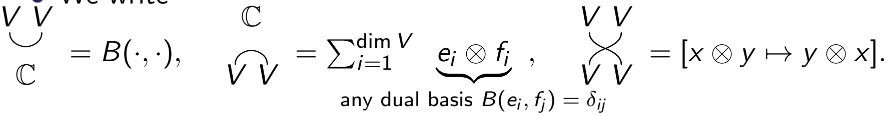
- Schur—Weyl duality is for GL(n), so are there any analogue for SO(n), O(n) or Sp(n)?
- The general problem is to determine

$$(\mathrm{End}(V^{\otimes m}))^G = V^{\otimes m} \otimes (V^\vee)^{\otimes m} \overset{\text{self dual by quadratic form}}{\cong} V^{\otimes 2m}.$$

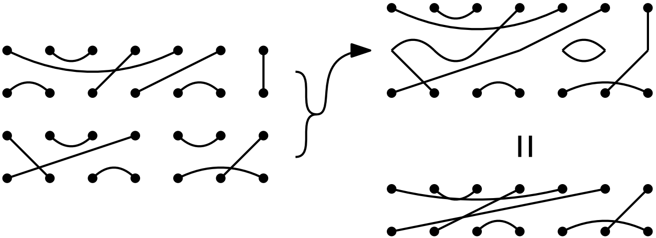
- It turns out for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(n)$, the invariant of linear functor of $V^{\otimes m}$ is generated by pairing (when m is not even, there is no invariant).
- For $G = \mathrm{SO}(n)$, determinant is another invariant.

Brauer algebra

- We write



- So for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(n)$, the right analogue to replace $\mathfrak{S}_n$ is the **Brauer algebra**.



Construction

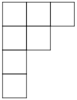
- Consider the natural representation V of $G = \mathrm{SO}(n)$ or $\mathrm{Sp}(n)$. Then

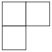
$$V^{\otimes m} c_\lambda r_\lambda \cap \bigcap_{i<j} \ker \left[V^{\otimes m} \overset{\text{pair } i \text{ and } j}{\longrightarrow} V^{\otimes (m-2)} \right]$$

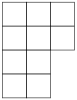
  is an irreducible analytic representation of G (equivalently of $\mathfrak{g} = \mathrm{Lie}(G)$).
- What is interesting, this constructs all irreducible representations for $\mathrm{Sp}(n)$, sp_n, $\mathrm{SO}(n)$. But not for $\mathfrak{so}_n$, since it is not simply connected. To get the remain irreducible representations, one should consider the **spin representations**.

Littlewood–Richardson again

- Due to the character reason, the multiplicity of V_λ in $V_\mu \otimes V_\nu$ is given by Littlewood–Richardson coefficient. Actually, more beautiful combinatorial explanation is given.
- The main data is $\lambda_i - \lambda_{i+1}$ for a partition λ.


$(1, 1, 0, 1)$


$(1, 1, 0, 0)$


$(0, 1, 0, 2)$

Bernstein–Zelevinsky triangle

Theorem (Bernstein and Zelevinsky)

The multiplicity of V_λ in $V_\mu \otimes V_\nu$ is the number of way to fulfill the Bernstein triangles with nonnegative integers.

For each side, the sum of two elements of each triangles left-to-right gives $\xi_i - \xi_{i+1}$, where $\xi = \lambda$ for left side, μ for right side and ν for lower side. And for each hexagon $\begin{smallmatrix} & b & & c & \\ a & & & & x \\ & z & & y & \end{smallmatrix}$, we have $a + b = x + y, a + c = x + z, b + c = y + z$.

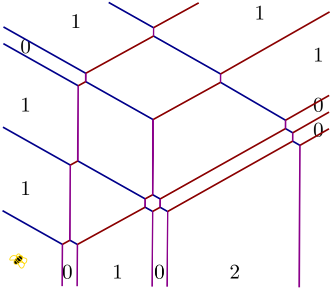
```
              1
             0 0
          0       0
         0 1     1 1
      1       1       0
     0 1     0 0     2 0
  1       0       0       0
 0 0     1 0     0 0     2 0
```

Tao–Knutson Honey comb

Theorem (Tao and Knutson)

The multiplicity of V_λ in $V_\mu \otimes V_\nu$ is the number of Honey comb corresponds to μ, ν, λ.

For each side, the spread of two lines left-to-right gives $\xi_i - \xi_{i+1}$, where $\xi = \lambda$ for left side, μ for right side and ν for lower side.

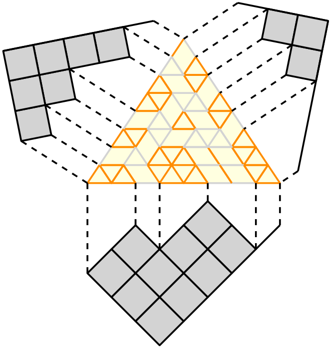


Tao–Knutson Puzzle

Theorem (Tao and Knutson)

The multiplicity of V_λ in $V_\mu \otimes V_\nu$ is the number of tiling the triangles with puzzles  *(rotation permitted but not reflection).*

The color of sides of puzzles is compatible, with the corresponding sides the projection of the boundry of Young diagrams.



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Thanks