Overview of Representation theory

Lecture 5 — Representation of lie algebras (I)

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Abelian lie groups

Theorem

Any abelian lie group is a direct product of copies of $\mathbb{R}$, $\mathbb{T}$ and discrete abelian group.

Compact group	cire group $\mathbb{T} = \mathbb{S} = \{z \in \mathbb{C} : \lvert z \rvert = 1\}$	torus $\mathbb{T}^n$
Lie algebra	one-diemensional space $\mathbb{R}$, with $[,] = 0$.	vector space $\mathbb{R}^n$, with $[,] = 0$.
Complex group	multiplicative group $\mathbb{G}_m = \mathbb{C}^\times = \mathbb{C} \setminus 0$	commplex torus $\mathbb{G}_m^n$
Lie algebra	one-diemensional space $\mathbb{C}$, with $[,] = 0$.	vector space $\mathbb{C}^n$, with $[,] = 0$.

Representation of torus

- Let T be a torus, and $T \to \mathrm{GL}(V)$ be a representation, then

$$V = \bigoplus_{\chi \in \mathrm{X}(T)} V_\chi, \qquad \{v \in V : t \cdot v = \chi(t)v\},$$

  where $\mathrm{X}(T) = \mathrm{Hom}_{\text{lie group}}(T, \mathbb{G}_{\mathrm{m}})$ the character group.
- Let $\mathfrak{t}$ be a commutative lie algebra, and $\mathfrak{t} \to \mathfrak{gl}(V)$ be a representation, then

$$V = \bigoplus_{\lambda \in \mathfrak{t}^\vee} V^\lambda, \qquad V^\lambda = \{v \in V : (t - \lambda(t))^{\gg 0} \cdot v = 0\},$$

  where $\mathfrak{t}^\vee$ the dual space of $\mathfrak{t}$.

The lie algebra $\mathfrak{sl}_2$

- $\mathfrak{sl}_2$ is the semisimple lie algebra of minimal dimension.

Compact group	special unitary group $\begin{aligned} \mathrm{SU}(2) &= \{X \in \mathrm{SL}_2 : X\overline{X}^\top = 1\} \\ &= \{\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) : \lvert\alpha\rvert^2 + \lvert\beta\rvert^2 = 1\} \end{aligned}$
Lie algebra	special unitary algebra $\begin{aligned} \mathfrak{su}_2 &= \{X \in \mathfrak{sl}_2 : X + \overline{X}^\top = 0\} \\ &= \{\left(\begin{smallmatrix} a & \beta \\ -\overline{\beta} & -a \end{smallmatrix}\right) : a \in i\mathbb{R}\} \end{aligned}$
Complex group	special linear group $\mathrm{SL}(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : ad - bc = 1\}$
Lie algebra	special linear algebra $\mathfrak{sl}_2 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a + d = 0\}$

Representations of SU(2)

- Denote the diagonal matrices $T = \{\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} : z \in \mathbb{T}\} \subseteq \mathrm{SU}(2)$.
- For any irreducible representation $\mathrm{SU}(2) \to \mathrm{GL}(V)$, then

$$V = \bigoplus_{\chi \in \mathrm{X}(T)} V_\chi, \qquad V_\chi = \{v \in V : t \cdot v = \chi(t)v\},$$
$$\xlongequal[\mathrm{X}(T)\cong\mathbb{Z}]{} \bigoplus_{n\in\mathbb{Z}} V(n), \qquad V(n) = \{v \in V : \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \cdot v = z^n v\}.$$

- SU(2) acts on n-dimensional homogenous polynomials V_n of two variables by $f(x,y) \overset{A}{\mapsto} f((x,y)A)$,

$$V_n = \underbrace{\mathbb{C}y^n}_{V_n(-n)} \oplus \underbrace{\mathbb{C}xy^{n-1}}_{V_n(-n+2)} \oplus \cdots \oplus \underbrace{\mathbb{C}x^{n-1}y}_{V_n(n-2)} \oplus \underbrace{\mathbb{C}x^n}_{V_n(n)}.$$

These are all irreducible, and the full list of irreducible representations.

Representations of $\mathfrak{sl}_2$

- Denote $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $X = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$, $Y = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$, and $\mathfrak{t} = \mathbb{C}H$. Then

$$\mathfrak{sl}_2 = \underbrace{\mathbb{C}H}_{[H,*]=0} \oplus \underbrace{\mathbb{C}X}_{[H,*]=2\cdot *} \oplus \underbrace{\mathbb{C}Y}_{[H,*]=-2\cdot *} \qquad [X, Y] = H = -[Y, X].$$

- For any irreducible representation $\mathfrak{sl}(2) \to \mathfrak{gl}(V)$, then

$$V = \bigoplus_{\lambda \in \mathfrak{t}^\vee} V_\lambda, \qquad V_\chi = \{v \in V : t \cdot v = \lambda(t)v\}$$

- Any irreducible representation takes the form

$$\begin{tikzcd}
0 & \mathbb{C}v_{-n} \arrow[l, "Y"] \arrow[loop below, "H"] \arrow[r, bend left, "X"] \arrow[r, phantom, "\oplus"] & \mathbb{C}v_{-n+2} \arrow[l, bend left, "Y"] \arrow[loop below, "H"] \arrow[r, bend left, "X"] \arrow[r, phantom, "\oplus"] & \cdots \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[r, phantom, "\oplus"] & \mathbb{C}v_{n-2} \arrow[l, bend left, "Y"] \arrow[loop below, "H"] \arrow[r, bend left, "X"] \arrow[r, phantom, "\oplus"] & \mathbb{C}v_n \arrow[l, bend left, "Y"] \arrow[loop below, "H"] \arrow[r, "X"] & 0
\end{tikzcd}$$

Maximal torus theory

- For a compact Lie group G, a subgroup $T \subseteq G$ is called a **maximal torus** if if is a torus and maximal among such subgroups.
- The adjoint action $\mathrm{Lie}(G) \stackrel{\mathrm{ad}_t}{\to} \mathrm{Lie}(G)$ of $t \in T$ induced from $G \stackrel{t*t^{-1}}{\to} G$ gives rise a decomposition

$$\mathrm{Lie}(G)_{\mathbb{C}} = \bigoplus_{\chi \in \mathrm{X}(T)} \mathfrak{g}_\chi \qquad \mathfrak{g}_\chi = \{X \in \mathrm{Lie}(G)_{\mathbb{C}} : \mathrm{ad}_t X = \chi(t)X\},$$

  with $\mathfrak{g}_0 = \mathrm{Lie}(T)$.
- Call each $\chi \in \mathrm{X}(T) \setminus 0$ such that $\mathfrak{g}_\chi \neq 0$ a **root** of G.

Maximal torus theory

- For a semisimple complex lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ is called a **cartan subalgebra** if it is commutative and maximal among such subalgebra.
- The adjoint action $\mathfrak{g} \stackrel{[H,-]}{\to} \mathrm{Lie}(G)$ of $H \in \mathfrak{h}$ gives rise a decomposition (assuming semisimplicity and using regular element)

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^\vee} \mathfrak{g}_\lambda \qquad \mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X\},$$

  with $\mathfrak{g}_0 = \mathfrak{h}$. Note that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.
- Call each $\lambda \in \mathfrak{h}^\vee \setminus 0$ such that $\mathfrak{g}_\lambda \neq 0$ a **root** of $\mathfrak{g}$.

The $\mathfrak{sl}_2$ triple

Theorem

For a complex semisimple Lie algebra $\mathfrak{g}$, and a root α. If we denote $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$, then

$$\mathfrak{sl}_2 = \mathbb{C}H \oplus \mathbb{C}X \oplus \mathbb{C}Y \cong \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} := \mathfrak{s}_\alpha$$

as Lie algebra. We will denote $H_\alpha, X_\alpha, Y_\alpha \in \mathfrak{g}$ the elements corresponding to H, X, Y respectively.
For a compact Lie group G, and a root α, there is a lie group homomorphism

$$SU(2) \to G$$

such that the induced $\mathfrak{sl}_2 \to \mathrm{Lie}(G)_\mathbb{C}$ gives the isomorphism above.

Root system

- For each root α, $\mathfrak{g}$ is a $\mathfrak{s}_\alpha$-representation by adjoint action. Denote $H_\alpha \in \mathfrak{h}$ corresponds to the $H \in \mathfrak{sl}_2$. For any root β, consider the $\mathfrak{s}_\alpha$-representation on the "straight line"

$$\cdots \oplus \underbrace{\mathfrak{g}_{\beta-\alpha}}_{[H_\alpha,*]=(\beta-\alpha)(H_\alpha)*} \oplus \underbrace{\mathfrak{g}_{\beta}}_{[H_\alpha,*]=\beta(H_\alpha)*} \oplus \underbrace{\mathfrak{g}_{\beta+\alpha}}_{[H_\alpha,*]=(\beta+\alpha)(H_\alpha)*} \oplus \cdots$$

By the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}_\beta \neq 0 \Longrightarrow \mathfrak{g}_{\beta+n\alpha} \neq 0$ where the n such that $-\beta(H_\alpha) = \beta(H_\alpha) + n\alpha(H_\alpha)$. Note that $\alpha(H_\alpha) = 2$, so

$$\beta \text{ is a root} \Longrightarrow \text{so is } \beta - \beta(H_\alpha)\alpha.$$

Root system

- An **abstract root system** Δ is a finite subset of $\mathbb{R}$-space V without zero spanning V, and

  For all $\alpha \in \Delta$, there exists $\alpha^\vee \in V^\vee$ such that Δ is invariant under the hyperplane reflection $\left[v \mapsto v - \langle \alpha^\vee, v \rangle \alpha\right]$, and $\alpha^\vee(\Delta) \in \mathbb{Z}$.

- We also assume Δ to be **reduced**, that is, $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha\}$, because the root systems of semisimple lie algebras suit this.
- We can introduce an inner product such that the reflections are invariant.

Polarization and basis

- Geometrically, the hyperplane of $\ker \alpha^\vee$ cut the unit sphere into several congruent pieces of hyper triangles. Pick one of such piece S (called **Weyl chambre**), define its Coxeter diagram

$$\text{vertices} = \text{hyperplanes along } S, \qquad H \overset{2\cos\angle HH'}{\rightarrow} H'$$

When it is labelled by 2, i.e. $H \perp H'$, we omit the arrow.

- We call the corresponding choice of root Φ a **basis** for Δ. Equivalently, its Coxeter diagram

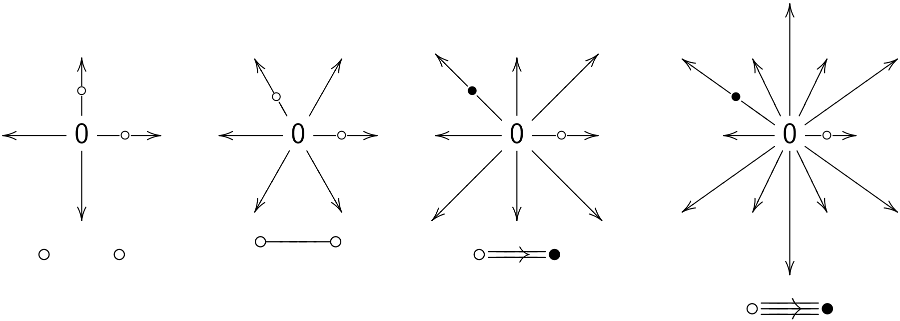
$$\text{vertices} = \Phi, \qquad \alpha \overset{-2\cos\angle\alpha\beta}{\rightarrow} \beta$$

This defines a **polarization**, say $\Delta = \Delta^+ \sqcup \Delta^-$ by

$$\Delta^+ = \{\alpha \in \Delta, \langle\alpha, \Phi\rangle > 0\}, \quad \Delta^- = \{\alpha \in \Delta, \langle\alpha, \Phi\rangle < 0\}.$$

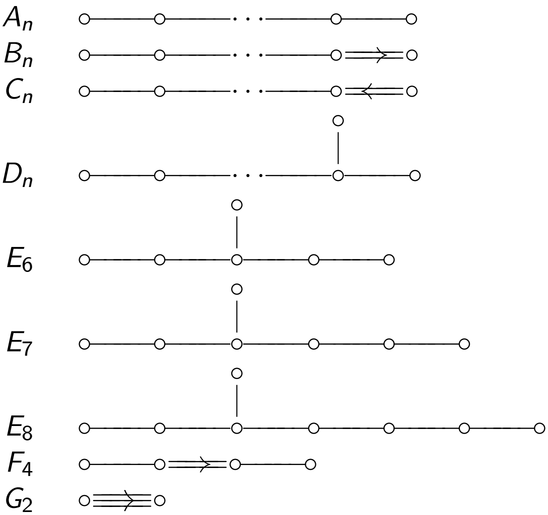
Dimension 2 case

- Two of roots have very limited case, here we list all of them, and we modify the arrow between them by



This is called the **Dynkin diagram** of the root system.

All connected Dynkin diagrams



Root system determines the lie algebra

Theorem

The Cartan subalgebras are all conjugated, in particular, the root system does not depend on the choice of Cartan subalgebra.

Theorem

The Dynkin diagram does not depend on the choice of basis.

Theorem (Serre)

The Dynkin diagram of the Root system of semisimple lie algebra determines the lie algebra structure. Actually, there exists a semisimple lie algebra for each Dynkin diagram.

Representations

- Consider an irreducible representation $\mathfrak{g} \to \mathfrak{gl}(V)$. Then

$$V = \bigoplus_{\lambda \in \mathfrak{h}^\vee} V_\lambda, \quad V_\lambda = \{v \in V : H \cdot v = \lambda(H)v\}$$

  since $[\mathfrak{g}_\alpha, V_\lambda] = V_{\lambda+\alpha}$, the right hand side is a $\mathfrak{g}$-representation. Call each λ such that $V_\lambda \neq 0$ a **weight** of V, and $v \in V_\lambda \setminus 0$ **weight vector**.
  It is also an $\mathfrak{s}_\alpha$-representation, so $\lambda(H_\alpha) \in \mathbb{Z}$.
- Denote the **weight lattice** $\Lambda = \{\lambda \in \mathfrak{h}^\vee : \lambda(H_\alpha) \in \mathbb{Z}\}$ the candidate of weight vectors.

Representations

- Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a polarization. For a representation V, call the weight λ such that $[\mathfrak{g}_\alpha, V_\lambda] = 0$ for all $\alpha \in \Delta^+$ a **highest weight**, the corresponding weight vector **highest weight vector**.
  Note that by the representation theory for $\mathfrak{s}_\alpha$, $\lambda(H_\alpha) \geq 0$ for all Δ^+.
- We call a λ in weight lattice Λ a **dominant weight**, if $\lambda(H_\alpha) \geq 0$ for all Δ^+.

Theorem

For each dominant weight λ, there exists a unique $\mathfrak{g}$-representation V_λ of highest maximal λ. They form the full list of representations of $\mathfrak{g}$.

Borel subalgebra

- Define the **Borel subalgebra**

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta^+} \mathfrak{g}_\alpha.$$

Then the highest weight space forms a representation of $\mathfrak{b}$ with $\mathfrak{g}_\alpha$ acts trivially.

- For any $\lambda \in \mathfrak{h}^\vee$, we can define a $\mathfrak{b}$-representation $\underline{\mathbb{C}}\lambda$, an one dimensional space spanning by v by

$$H \cdot v = \lambda(H)v, \qquad \mathfrak{g}_{\Delta^+} \cdot v = 0.$$

Verma modules

- It turns out that it is very useful to consider the Verma modules. For each weight $\lambda \in \Lambda$, the **Verma module** is defined to be a module with highest weight vector v_0 to λ,

for any other module V with a highest weight vector v to λ, there is a unique $\mathfrak{g}$-module map $M^\lambda \to V$ sending v_0 to v.

$$\begin{array}{ccc} v_0 & \in & M^\lambda \\ \downarrow & & \downarrow \\ v & \in & V \end{array}$$

Actually, $M^\lambda = \underline{\mathbb{C}}\lambda\uparrow_{\mathfrak{b}}^{\mathfrak{g}}$.

Borel subgroup

- One can define the **Borel subgroup** B of a complex analytic group G to be the maximal connected solvable subgroup. When the group is semisimple, $\text{Lie}(B) = \mathfrak{b}$.

The picture illustrates Grothendieck's vision of a pinned reductive group: the body is a maximal torus T, the wings are the opposite Borel subgroups B, and the pins rigidify the situation.

Borel–Weil theorem

- For a compact Lie group G and maximal torus T, the **flag manifold/variety** G/T has a holomorphism/variety structure, actually, $G/T \cong G_{\mathbb{C}}/B$ with B the Borel subgroup. For each $\lambda \in \Lambda$,
$$G \times_T \underline{\mathbb{C}}\lambda = G \times_B \underline{\mathbb{C}}\lambda$$
forms a holomorphic/algebraic line bundle over $G/T \cong G_{\mathbb{C}}/B$. Then the group of holomorphic/algebraic global sections
$$\Gamma(G \times_T \underline{\mathbb{C}}) = \begin{cases} V_\lambda^\vee, & \lambda \text{ is dominant,} \\ 0, & \text{otherwise.} \end{cases}$$
- For example, $\mathrm{SU}(2)/T = \mathrm{SL}(2)/\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\} \cong \mathbb{P}^1$.

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Thanks