Lecture 4 — Representation of finite groups (II)

Xiong Rui

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Young diagrams

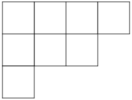
- A **partition** λ of n is a series of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that

$$\lambda_1 + \lambda_2 + \cdots = n.$$

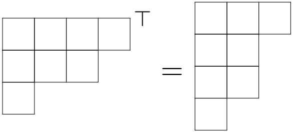
  Denote $|\lambda| = n$ and $\lambda \vdash n$. Call the number of nonzero λ_i the **length**.

- A partition λ can be presented by Young diagrams with i-boxes in i-th row.

$$8 = 4 + 3 + 1,$$



- We can define its **transpose**



Symmetric groups

- Denote $\mathfrak{S}_n$ the n-th symmetric group.
- Let λ be a partition of n, denote

$$\mathbf{c}(\lambda) = \text{the conj.class of } \overbrace{\underbrace{(\cdots)}_{\lambda_1}\cdots\underbrace{(\cdots)}_{\lambda_n}}^{\text{disjoint cycles}},$$

and

$$z(\lambda) = |\mathbf{c}(\lambda)| = \frac{n!}{1^{i_1} i_1! \cdots d^{i_d} i_d!}$$

where $i_j = \#\{k : \lambda_k = j\}$.

Row sum and column sum

- We denote the sign function to be $\sigma \mapsto (-1)^\sigma$.
- Let λ be a partition λ of length n, and fix some filling of λ from 1 to $|\lambda|$ (no repetition) and denote again by λ . We define two elements in $\mathbb{C}[\mathfrak{S}_n]$

$$r_\varphi = \sum_{\sigma\in\mathfrak{S}_n:\sigma \text{ permutes inside each row}} \sigma$$

$$c_\varphi = \sum_{\sigma\in\mathfrak{S}_n:\sigma \text{ permutes inside each column}} (-1)^\sigma\sigma.$$

Row sum and column sum

$$r_\varphi = \sum_{\sigma \in \mathfrak{S}_{\{1,2,3,4\}} \times \mathfrak{S}_{\{5,6,7\}} \times \mathfrak{S}_{\{8\}}} \sigma$$

$$c_\varphi = \sum_{\sigma \in \mathfrak{S}_{\{1,5,8\}} \times \mathfrak{S}_{\{2,6\}} \times \mathfrak{S}_{\{3,7\}} \times \mathfrak{S}_{\{4\}}} (-1)^\sigma \sigma.$$

1	2	3	4
5	6	7	
8			

- If by lexicographical order $\lambda < \mu$, then

$$c_\lambda \cdot \mathbb{C}[\mathfrak{S}_n] \cdot r_\mu = 0.$$

- For the same λ,

$$c_\lambda \cdot \mathbb{C}[\mathfrak{S}_n] \cdot r_\lambda = \mathbb{C} \cdot c_\lambda \cdot r_\lambda.$$

Specht moduless

- One can define **permutation modules** and **specht modules** by

$$M^\lambda = \mathbb{C}[\mathfrak{S}_n] \cdot r_\lambda \quad \supseteq \quad S^\lambda = \mathbb{C}[\mathfrak{S}_n] \cdot c_\lambda \cdot r_\lambda \quad \neq 0.$$

Theorem

$M^\lambda = \mathbb{1}\uparrow_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}}^{\mathfrak{S}_n}$, and S^λ is irreducible.

- If by lexicographical order $\lambda < \mu$, then S^λ does not appear in M^μ, since

$$\mathrm{Hom}_{\mathfrak{S}_n}(S^\lambda, M^\mu) = c_\lambda \cdot r_\lambda \cdot \mathbb{C}[\mathfrak{S}_n] \cdot r_\mu = 0.$$

In particular, $S^\lambda \cong S^\mu$ if and only if the underlying Young diagram of λ and μ are the same.

The character

- It is not very hard to compute the character of M^λ,

$$\begin{aligned}\psi_\lambda(\mathbf{c}(\mu)) &= \tfrac{1}{|\mathfrak{S}_\lambda|}\#\{y : y^{-1}cy \in \mathfrak{S}_\lambda\} \\ &= \cdots \\ &= \text{(computation)} \\ &= \cdots \\ &= \text{coefficient of } x_1^{\lambda_1}\cdots x_n^{\lambda_n} \text{ in} \\ &\qquad (x_1^{\mu_1}+\cdots+x_n^{\mu_1})\cdots(x_1^{\mu_*}+\cdots+x_n^{\mu_*}).\end{aligned}$$

where $c \in \mathbf{c}(\mu)$, and $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$. Lastly

$$\mu = \mu_1 \geq \cdots \geq \underbrace{\mu_*}_{>0} \geq 0 \geq 0.$$

Symmetric polynomials

- A polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is called **symmetric** if it is fixed by the index-permutation-action of $\mathfrak{S}_n$.
- Define the **elementary symmetric polynomials** $\{e_k : k = 1, \ldots, n\}$,

$$e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}.$$

- Define the **complete symmetric polynomials** $\{h_k : k = 1, \ldots, n, \ldots\}$,

$$h_k = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k}.$$

Symmetric polynomials

Theorem (Fundamental theorem of symmetric polynomials)

Every symmetric function is a unique polynomial of elementary symmetric polynomials.

- For a partition λ of length $\leq n$, define the **complete symmetric polynomials**

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_n}.$$

If $\lambda^\top = \mu$, define the **elementary symmetric polynomials**

$$e_\lambda = e_{\mu_1} \cdots e_{\mu_*}.$$

define the **monomial symmetric polynomials**

$$m_\lambda = \sum \text{the orbit of the monomial } x_1^{\lambda_1} \cdots x_n^{\lambda_n} \text{ under } \mathfrak{S}_n$$

Schur polynomials

- For a partition λ of length $\leq n$, define the **Schur polynomial**

$$s_\lambda = \frac{\det_{i,j}(x_j^{\lambda_i+n-i})}{\det_{i,j}(x_j^{n-i})} = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ \vdots & \ddots & \vdots \\ x_1^{\lambda_n+n-n} & \cdots & x_n^{\lambda_n+n-n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ \vdots & \ddots & \vdots \\ x_1^{0} & \cdots & x_n^{0} \end{vmatrix}}.$$

- It turns out that $\{s_\lambda : \text{length of } \lambda \leq n\}$ forms a basis too.

Power sum

- Define for $i \geq 1$, the **power sum**

$$p_i = x_1^i + \cdots x_n^i,$$

and $p_0 = 1$ for convention.
- For a partition λ of length $\leq n$, define

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_n}.$$

- It turns out that $\{p_\lambda : \text{length of } \lambda \leq n\}$ forms a basis too.

An inner product

- By a more or less interesting computation

$$\prod_{1\le i,j\le n}\frac{1}{1-x_iy_j} = \sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda \frac{z(\lambda)}{n!}\cdot p_\lambda(x)p_\lambda(y).$$

- One can introduce a bilinear form such that

$$\langle h_\lambda, m_\mu\rangle = \langle s_\lambda, s_\mu\rangle = \frac{z(\lambda)}{n!}\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu}.$$

The character of M^λ

- So

$$\psi_\lambda(\mathbf{c}(\mu)) = \text{coefficient of } x_1^{\lambda_1}\cdots x_n^{\lambda_n} \text{ in } p_\mu.$$

$$\iff p_\mu = \sum_\lambda \psi_\lambda(\mathbf{c}(\mu))\cdot m_\lambda \iff \psi_\lambda(\mathbf{c}(\mu)) = \langle p_\mu, h_\lambda\rangle\,.$$

- For any class function φ,

$$\exists!\text{degree-}n \text{ symmetric polynomial } f,\ \text{such that } \varphi(\mathbf{c}(\mu)) = \langle p_\mu, f\rangle\,.$$

- If $\varphi \leftrightarrow f$ and $\psi \leftrightarrow g$, then

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\overline{\varphi(\sigma)}\cdot\psi(\sigma) = \langle f, g\rangle\,.$$

Jacobi–Trudy identity

Theorem (Jacobi–Trudy identity, Giambelli formula)

$$s_\lambda = \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+n-1} \\ \vdots & \ddots & \vdots \\ h_{\lambda_n-n+1} & \cdots & h_{\lambda_n} \end{vmatrix}.$$

Convention: $h_0 = 1$, *and* $h_{<0} = 0$.

- So

$$s_\lambda = \sum_{\mu \geq \lambda} K_{\mu\lambda} \cdot h_\mu \quad \xRightarrow{\text{upper triangle matrix}} \quad h_\lambda = \sum_{\mu \geq \lambda} K_{\mu\lambda} \cdot s_\mu$$

  with $K_{\lambda\lambda} = 1 > 0$.
- In particular,

$$h_\lambda \quad \xRightarrow{\text{GramSchmidt process along } \geq} \quad s_\lambda$$

The character of S^λ

- Since $\mathrm{Hom}_{\mathfrak{S}_n}(S^\mu, M^\lambda) = 0$ when $\mu < \lambda$,

$$M^\lambda = \bigoplus_{\mu \geq \lambda} k_{\mu\lambda} S^\mu \qquad k_{\mu\mu} \geq 1.$$

- In particular,

$$\psi_\lambda \quad \xRightarrow{\text{GramSchmidt process along } \geq} \quad \chi_\lambda$$

- So we get

$$\chi_\lambda(\mathbf{c}(\mu)) = \langle p_\mu, s_\lambda \rangle .$$

The character of S^λ

- By the definition of Schur polynomial, we get the following.

Theorem (Frobenius character formula)

$$\chi_\lambda(\mathbf{c}(\mu)) = \textit{coefficient of } x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n} \textit{ in } \Delta \cdot p_\mu$$

where $\Delta = \prod_{i<j}(x_i - x_j) = \det(x_i^{n-j})$.

- We also get the following which can be used to do computation.

Theorem

The linear space of symmetric polynomials of degree n is isomorphic to the space of class function over $\mathbb{S}_n$, with $s_\lambda \leftrightarrow S^\lambda$, and $h_\lambda \leftrightarrow M^\lambda$.

Hook length

- One can show that

$$\{\sigma \cdot c_\lambda \cdot r_\lambda : \sigma\lambda \text{ is a standard Young tableau}\}$$

  forms a basis of S^λ.
- An amazing formula is

$$\dim S^\lambda = \#\{\text{standard Young tableaux of } \lambda\} = \frac{n!}{\prod_{\square \in \lambda} \Gamma(\square)}$$

  where $\Gamma(\square)$ is the length of "hook".

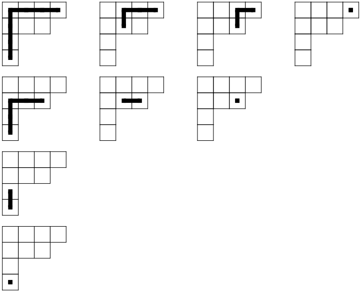
Hook length

$$\dim S^{(4,3,1,1)} = \frac{9!}{7 \times 4 \times 3 \times 1 \times 5 \times 2 \times 1 \times 2 \times 1} = 216.$$

$\Gamma =$

7	4	3	1
5	2	1	
2			
1			

$=$



Branching rule

- Generally, it is interesting to ask, if V, U are two representations of $\mathfrak{S}_n$ and $\mathfrak{S}_m$, what is

$$W = V \otimes U{\uparrow}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{m+n}}.$$

  The result sounds amazing, if we denote the corresponding symmetric polynomial by f_* (in enough variables), then simply $f_W = f_V \cdot f_U$. Note that it suffices to check for $M^\lambda = \mathbb{1}{\uparrow}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$.
- The coefficient

$$\langle s_\lambda s_\mu, s_\nu \rangle = \dim \mathrm{Hom}_{\mathfrak{S}_{m+n}}(S^\lambda \otimes S^\mu{\uparrow}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{m+n}}, S^\nu)$$

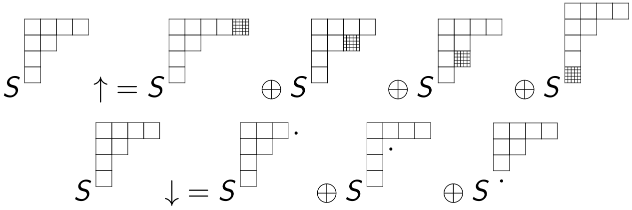
  is called **Littlewood–Richardson coefficients**, which can be computed by **Littlewood–Richardson rule**.

Branching rule

- In particular, if S^λ be one of irreducible representation of $\mathfrak{S}_n$, how to decompose $S^\lambda\uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}$?

Theorem (Pieri Rule)

$$S^\lambda\uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} = \sum_{\mu=\lambda\leftarrow\square} S^\mu, \qquad S^\lambda\downarrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \sum_{\mu=\lambda\backslash\square} S^\mu.$$



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Thanks