Lecture 3 — Representation of finite groups (I)

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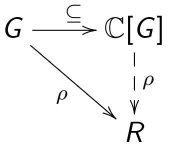
Review

- Let G be a finite group. Define the **group algebra** $\mathbb{C}[G]$ to be

$$\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g, \qquad z_1 g_1 \cdot z_2 g_2 := z_1 z_2 g_1 g_2.$$

The multiplication is called **convolution**. It satisfies

For any multiplicative map $G \to R$ for some $\mathbb{C}$-algebra R, there exists a unique $\hat{\rho} : \mathbb{C}[G] \to R$ extenting ρ.



From the point view of algebra

- So

$$\boxed{\text{Representation of } G} = \boxed{\text{Representation of } \mathbb{C}[G]}.$$

- By the Weyl's unitary trick or the Maschke theorem, $\mathbb{C}[G]$ is semisimple, so by the Wedderburn–Artin theorem

$$\mathbb{C}[G] \xrightarrow{\sim} \prod_{i=1}^{s} \mathbb{M}_{n_i}(\mathbb{C}).$$

$$\boxed{\text{irreducible representations of } G} = \boxed{\text{simple modules of } \mathbb{C}[G]}$$

$$= \left\{ G \to \mathbb{M}_{n_i}(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^{n_i}) \right\}_{i=1}^{s}$$

But we nearly know nothing

- How?

$$\mathbb{C}[G] \xrightarrow{\text{Actually very far}} \prod_{i=1}^{s} \mathbb{M}_{n_i}(\mathbb{C}$$

- But we still can say something.

$$s = \#\{\text{irreducilbe representation}\}$$

$$s = \dim Z(\mathbb{C}[G]) = \#\{\text{conjugation class}\}$$

$$\sum_{i=1}^{s} n_i^2 = \dim \mathbb{C}[G] = |G|$$

Frobenius' character theory

- Let V be a representation, define its character

$$\chi_V : G \longrightarrow \mathbb{C} \qquad g \longmapsto \mathrm{tr}[V \xrightarrow{g} V].$$

This is natural if you know the Morita equivalence.

- If $V = \mathbb{C}^{n_i}$ is one of the irreducible representation, then

$$\begin{array}{ccc} G & \xrightarrow{\quad \chi_V \quad} & \mathbb{C} \\ \downarrow & & \uparrow \mathrm{tr} \\ \mathbb{C}[G] \cong \prod \mathbb{M}_{n_i}(\mathbb{C}^{n_i}) & \xrightarrow[\text{projection}]{} & \mathbb{M}_{n_i}(\mathbb{C}^{n_i}) \end{array}$$

- So for two irreducible representations V_1 and V_2,

$$\chi_{V_1} = \chi_{V_2} \iff V_1 \cong V_2.$$

Trivial representation

- Let $\mathbb{1}$ be the one dimensional trivial representation, i.e. all G acts trivially.
- For any representation V, there is a way to produce invariant vectors by averaging

$$p : V \longrightarrow V \qquad v \longmapsto \frac{1}{|G|}\sum_{g\in G} gv.$$

- This is definitely a projection to the trivial summand. More exactly, if
$$V = k\mathbb{1} \oplus \text{non-trivial summand}.$$
Then $\operatorname{tr} p = \frac{1}{|G|}\sum_{g\in G}\chi_V(g) = k$.

General representation

- For an irreducible representation V_i with character χ_i, how to find the multiplicity of V_i in V?
- We have

$$\text{multiplicity of } V_i \text{ in } V = \dim \operatorname{Hom}_G(V_i, V).$$

- Now that $\operatorname{Hom}(V_i, V)$ is also a representation, by

$$g \cdot f : x \mapsto g \cdot f(g^{-1} \cdot x).$$

$$\operatorname{Hom}_G(V_i, V) = \{f \in \operatorname{Hom}(V_i, V) : g \cdot f = f\}.$$

- Note that $\operatorname{tr}[A \mapsto BAC] = \operatorname{tr} B \cdot \operatorname{tr} C$. So

$$\text{multiplicity of } V_i \text{ in } V = \frac{1}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \chi_V(g).$$

The outline

- We can introduce a unitary product

$$\langle \cdot, \cdot \rangle : \langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} \overline{f(x)} g(x).$$

- For characters, since every $g \in G$ having finite order, so with eigenvalues root of unity, so $\overline{\chi}(g) = \chi(g^{-1})$. By our computation

$$\langle \chi_V, \chi_U \rangle = \dim \mathrm{Hom}_G(V, U).$$

- We consider the space of class functions,

$$\{G \xrightarrow{f} \mathbb{C} : f(xy) = f(yx)\}, \qquad \langle \cdot, \cdot \rangle$$

Then by what we did, the character of irreducible representations form a set of orthogonal basis of it.

In summary

- In summary

Finding representations → Using Character theory → Until finding all representations

What happen for ...

- For compact group, do we have the Wedderburn-Artin theorem?

Theorem (Peter-Weyl)

For a compact group G, all irreducible representations are finite dimensional, and

$$L^2(G) = \widehat{\bigoplus} \mathrm{End}(V) \qquad \text{(direct sum of Hilbert spaces).}$$

No algebra structure asserted.

- For a compact group, do we have Frobenius' character theory?

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) \overset{\text{exchang to}}{\leftrightarrow} \frac{1}{\mu(G)}\int_G \chi \mathrm{d}\mu.$$

What happen for ...

- If the representation is over $\mathbb{R}$-space V. One can consider $V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification.

Theorem

For an irreducible representation V, it is a complexification of some real representation if and only if

$$\chi_V \text{ takes real value,} \qquad \frac{1}{|G|}\sum_{g\in G}\frac{\chi(g)^2-\chi(g)}{2}=0.$$

i.e. $V \cong V^{\vee}$ *and* $(\wedge^2 V)^G = 0$.

What happen for ...

- If the representation is over k, with char $k \mid |G|$. Then $k[G]$ is not semisimple. However, $k[G]/\operatorname{rad}$ is. Assume k is algebraic closed. We have

Theorem (Brauer)

The number of irreducible representations of G over k equals to the number of conjugation classes whose elements are of order prime to p.

- The **Brauer** characters (some lift of characters to some characteristic zero field) forms a basis of functions over the conjugation classes whose elements are of order prime to p.

Induction and restriction

- Let $H \subseteq G$ be a subgroup.
- For a G representation V, we can view it as a representation of H, call it **restriction**, and denote it by

$$\operatorname{res}_H^G V = V{\downarrow}_H^G.$$

- For an H representation W, then there is an isomorphism

$$\operatorname{Hom}_H(k[G], W) \xrightarrow{\sim} k[G] \otimes_H W \qquad f \mapsto \sum_{x \in G} x^{-1} \otimes f(x).$$

We call it induced representation, and denote it by

$$\operatorname{ind}_H^G V = V{\uparrow}_H^G.$$

Induction and restriction

- We have

$$\mathrm{Hom}_H(V, U\downarrow^G_H) = \mathrm{Hom}_G(V\uparrow^G_H, U)$$
$$\mathrm{Hom}_H(U\downarrow^G_H, V) = \mathrm{Hom}_G(U, V\uparrow^G_H)$$

  which is known as **Frobenius reciprocity**.

- By a direct computation,

$$\begin{aligned}\chi(x)\uparrow^G_H &= \sum_{yH\in G/H: y^{-1}xy\in H} \chi(y^{-1}xy). \\ &= \frac{1}{|H|}\sum_{y\in G: y^{-1}xy\in H} \chi(y^{-1}xy).\end{aligned}$$

The tensor product

- For two groups G and H, V and U two representations respectively, then this defines a natural $G \times H$ representation $V \otimes U$ by
$$(g, h) \cdot v \otimes u = gv \otimes hu.$$
- For a group G and two representations V and U, then $V \otimes U$ is also a G-representation, through diagonal map $G \to G \times G$, or exactly
$$g \cdot v \otimes u = gv \otimes gu.$$
- If $\{V_i\}$ and $\{U_j\}$ the lists of irreducible representations respectively. Then $\{V_i \otimes U_j\}$ is the lists for $G \times H$. Since

  the conjugation class of product $=$ the product of conjugation class.

The duality

- For a representation V over G, $V^\vee$ is a representation of G^{op}, by

$$g \cdot f : x \mapsto f(gx).$$

It is also a G-representation through the involution $G \overset{x \mapsto x^{-1}}{\to} G^{op}$, or exactly

$$g \cdot f : x \mapsto f(g^{-1}x).$$

The Hopf structure

- For G-representations U, V, W, and the trivial representation $\Bbbk$, the natural isomorphisms

$$\mathrm{Hom}(U, V) \cong U^\vee \otimes V, \qquad U \otimes V \cong V \otimes U,$$

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

$$U \otimes \Bbbk = U = \Bbbk \otimes U, \qquad \mathrm{Hom}(U, \Bbbk) = U^\vee.$$

are also G-isomorphisms.

The Hecke algebra

- Let B, C be two subgroups of G, consider

$$\begin{aligned}\mathrm{Hom}_G(\mathbb{1}\uparrow_B^G, \mathbb{1}\uparrow_C^G) &= \mathrm{Hom}_C(\mathbb{1}\uparrow_B^G\downarrow_C, \mathbb{1}) \\ &= \mathrm{Hom}_C(k[G] \otimes_B \mathbb{1}, \mathbb{1}) \\ &= \mathrm{Hom}(\mathbb{1} \otimes_C k[G] \otimes_B \mathbb{1}, \mathbb{1}) \\ &= \{G \xrightarrow{f} k : {}^{c\in C, b\in B \Rightarrow} f(cxb) = f(x)\} \\ &= \{C\backslash G/B \xrightarrow{f} k\}\end{aligned}$$

- For a such $C\backslash G/B \xrightarrow{f} k$, it corresponds to

$$\mathbb{1}\uparrow_B^G \longrightarrow \mathbb{1}\uparrow_C^G \qquad x \otimes 1 \mapsto \frac{1}{|C|}\sum_{g\in G} f(g^{-1}x)g^{-1} \otimes 1.$$

The Hecke algebra

- If $\mathbb{1}\uparrow_B^G \xrightarrow{\varphi} \mathbb{1}\uparrow_C^G$ and $\mathbb{1}\uparrow_C^G \xrightarrow{\psi} \mathbb{1}\uparrow_D^G$ corresponds to f and g respectively, then

$$\psi \circ \varphi \text{ corresponds to } g * f(x) = \frac{1}{|C|} \sum_{yz=x} g(y) f(z),$$

the convolution.

- If we denote $e_B = \frac{1}{B} \sum_{b \in B} b$, then there is an isomorphism

$$\{C \backslash G / B \xrightarrow{f} k\} \longrightarrow e_C \cdot \mathbb{k}[G] \cdot e_B \qquad f \mapsto \sum f(x) x.$$

With

$$\Big(\sum g(x)x\Big)\Big(\sum f(x)x\Big) = \frac{1}{|C|}\Big(\sum (g * f)(z) z\Big).$$

The Hecke algebra

- When $B = C$, the algebra

$$\big(\, \mathrm{End}_G(\mathbb{1}\uparrow_B^G), \quad \text{composition} \,\big)$$

is called the **Hecke algebra** of B.

- It is isomorphism to

$$\big(\, \{C\backslash G/B \xrightarrow{f} k\}, \text{convolution} \,\big)$$

with such f corresponds to $x \mapsto \frac{1}{|B|}\sum_{g\in G} f(g^{-1}x)g^{-1} \otimes 1$.

- By $f \mapsto \frac{1}{|B|}\sum f(x)x$, it is also isomorphism to

$$\big(\, e_B \cdot \mathbb{k}[G] \cdot e_B, \quad \text{usual product} \,\big).$$

The Hecke algebra

- For $x \in B\backslash G/B$, denote T_x the characteristic function of x. Then for $x, y, z \in B\backslash G/B$,

$$\begin{aligned} T_x T_y(z) &= \frac{1}{|B|}\#\{ab = z : a \in BxB, b \in ByB\} \\ &= \frac{1}{|B|}\#\{b \in G : zb^{-1} \in BxB, b \in ByB\} \\ &= \frac{|Bx^{-1}Bz \cap ByB|}{|B|} = |B\backslash(Bx^{-1}Bz \cap ByB)|. \end{aligned}$$

Hecke algebra

- In the case of $G = \mathrm{GL}_n(\mathbb{F}_q)$, and B the upper triangle matrix, then we know that

$$G = \bigsqcup_{w \text{ permutation matrix}} BwB.$$

- Denote $s_i = (i, i+1)$, and $T_i = T_{s_i}$, we have

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \\ T_i T_j = T_j T_i, & |i-j| \geq 2; \\ T_i^2 = (q-1) T_i + q T_e. \end{cases}$$

The first relation is called the Yang-Baxter equation, or the Braid relation.

Hecke algebra

- Generally, one can define this for any Chevalley group $\mathbb{G}$ over finite field. Let $G = \mathbb{G}(\mathbb{F}_q)$, and B the split Borel subgroup. Then we have the **Bruhat decomposition**

$$G = \bigsqcup_{w \in W} BwB, \qquad W = \text{Weyl group}.$$

  The relation can be read from its root system.
- Abstractly, one can define the Hecke algebra for any Coxeter system, and in which case q is not a concrete number but a parameter.

References

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Thanks