Lecture 2 — The structures of algebras and groups (II)

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Definitions

- A **topological group** is a group G which is a topological space with

$$G \times G \longrightarrow G \qquad (x, y) \longmapsto x^{-1}y$$

  continuous.

- For a topological group G, a **complex representation** is a continuous group homomorphism

$$\rho : G \to \mathrm{GL}(V), \qquad V \text{ is some finite dimensional complex vector space.}$$

  We will say V is a G-representation or G-module. And write $g \cdot v$ by $(\rho(g))(v)$

Haar measure

Theorem (Haar measure)

For a locally compact group, there is a Borel measure μ over G such that

$$\forall g \in G, \ \textit{Borel set } E, \quad \mu(E) = \mu(gE).$$

This measure is unique up to a nonzero scalar.

- For Lie groups (defined later), this measure is computable by differential forms.
- For discrete groups, this is just the measure of counting.
- For $\mathbb{R}$, this is just the usual measure $\mathrm{d}x$.
- For $\mathbb{R}^{\times}$, this is $\frac{\mathrm{d}x}{x}$.

Compact group is reductive

Theorem (Weyl's unitary trick)

If G is compact, then any continuous group homomorphism $G \to \mathrm{GL}_n$ is conjugated to $G \to \mathrm{U}_n$.

- The proof is easy by construction the invariant unitary form $\langle v, w \rangle = \frac{1}{\mu(G)} \int_G \langle gv, gw \rangle \, \mathrm{d}\mu(g)$.
- So for any representation V, and any submodule $W \subseteq V$, since we have unitary form, $V = W \oplus W^{\perp}$.
- So in our principle,

$$\boxed{G \text{ is compact}} \Rightarrow \boxed{G \text{ is reductive}}.$$

Definitions

- A **Lie group** is a group G with smooth manifold structure with

$$G \times G \longrightarrow G \qquad (x, y) \longmapsto x^{-1}y$$

  smooth.
- A **Lie algebra** is a finite dimensional vector space $\mathfrak{g}$ equipped with a bilinear map called **Lie bracket** $\mathfrak{g} \times \mathfrak{g} \xrightarrow{[\cdot,\cdot]} \mathfrak{g}$ such that
  - $[x, x] = 0$,
  - $[x, y] + [y, x] = 0$, and
  - $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Lie algebras of Lie groups

- The **Lie algebra** $\mathrm{Lie}(G)$ of a Lie group G is

$$\mathrm{Lie}(G) = \text{Left invariant vector fields over } G.$$

It equipped with Lie bracket makes it an abstract Lie algebra

$$[\cdot, \cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G) \qquad (X, Y) \mapsto XY - YX.$$

- Note that the Lie algebra is completely determined by the tangent vector at $1 \in G$, so

$$\mathrm{Lie}(G) = \mathrm{Tan}_1 G.$$

But no good interpolation of Lie bracket over $\mathrm{Tan}_1 G$.

Exponential Map

- There is a lot of way to understand tangent space, but for Lie group, the best to connect it with Lie group is via **exponential map**.
- There is a differential map $\exp : \mathrm{Lie}(G) \to G$ for each Lie group G with the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Lie}(H) & \xrightarrow{\mathrm{Lie}(\varphi)} & \mathrm{Lie}(G) \\
\downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\
H & \xrightarrow[\text{group homomorphism}]{\varphi} & G
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Tan}_1 & = & \mathrm{Lie}(\mathbb{R}) \\
\| & & \downarrow{\scriptstyle \exp} \\
\mathbb{R} & \xrightarrow[\mathrm{id}]{} & \mathbb{R}
\end{array}
$$

such that for each $X \in \mathrm{Tan}_1\, G$,

$$\frac{\mathrm{d}}{\mathrm{dt}} \exp(tX)\Big|_{t=0} = X \in \mathrm{Tan}_1\, G.$$

Examples

- For $\mathbb{R} = (\mathbb{R}, +)$,

$$\mathrm{Lie}(\mathbb{R}) = \mathrm{Tan}_0 = \mathbb{R}, \qquad \exp = \mathrm{id}\,.$$

- For $\mathbb{R}^\times = (\mathbb{R} \setminus 0, \times)$,

$$\mathrm{Lie}(\mathbb{R}) = \mathrm{Tan}_0 = \mathbb{R}, \qquad \exp = [x \mapsto e^x].$$

- For $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$,

$$\mathrm{Lie}(\mathbb{R}) = \mathrm{Tan}_1 = i\mathbb{R}, \qquad \exp = [ix \mapsto e^{ix}].$$

- For $\mathbb{C}^\times = (\mathbb{C} \setminus 0, \times)$,

$$\mathrm{Lie}(\mathbb{C}) = \mathrm{Tan}_1 = \mathbb{C}, \qquad \exp = [z \mapsto e^z].$$

Examples

- For $\mathrm{GL}_n(\mathbb{R})$,

$$\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n := \mathbb{M}_n(\mathbb{R}), \qquad \exp = [A \mapsto e^A].$$

  Lie bracket is given by $[A, B] = AB - BA$.

- For $\mathrm{SL}_n(\mathbb{R})$,

$$\mathrm{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n := \{A \in \mathfrak{gl}_n : \operatorname{tr} A = 0\}, \qquad \exp = [A \mapsto e^A].$$

  Lie bracket is given by $[A, B] = AB - BA$.

From Lie groups to Lie algebras

- It is easy to see that the connected component of 1 is a closed lie subgroup. The universal covering of a connected lie group equipped with a lie group structure.

Theorem

Let G, H be two connected Lie groups, then

$$\mathrm{Hom}_{Lie\ group}(G, H) \to \mathrm{Hom}_{Lie\ algebra}(\mathrm{Lie}(G), \mathrm{Lie}(H))$$

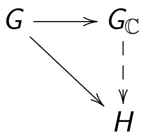
is injective. When G is simply connected, then it is a bijection in which case the inverse in induced by exponential map.

$$\begin{aligned} \text{Lie groups} \quad & \rightsquigarrow \text{connected Lie groups} \\ & \rightsquigarrow \text{simply connected Lie groups} \\ & \rightsquigarrow \text{Lie algebras.} \end{aligned}$$

Complexification

- For a real Lie algebra $\mathfrak{g}$ define its **complexification** $\mathfrak{g}_{\mathbb{C}}$ to be $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.
- For a Lie group G, defines its **complexification** to be lie group $G_{\mathbb{C}} \supseteq G$ such that

  For any Lie group homomorphism $G \xrightarrow{\varphi} H$ with H some analytic complex Lie group H, it can be uniquely extended to a analytic group homomorphism $G_{\mathbb{C}} \to H$.



Theorem

For compact Lie group G, the complexification of it exists, and $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(G)_{\mathbb{C}}$.

From compact Lie groups to complex Lie algebras

- We have the similar theorem.

Theorem

Let G, H be two connected complex analytic Lie groups, then

$$\mathrm{Hom}_{analytic\ Lie\ group}(G,H) \to \mathrm{Hom}_{Lie\ algebra}(\mathrm{Lie}(G),\mathrm{Lie}(H))$$

is injective. When G is simply connected, then it is a bijection in which case the inverse in induced by exponential map.

$$\begin{aligned}\text{compact Lie groups} &\rightsquigarrow \text{complex analytic Lie groups}\\ &\rightsquigarrow \text{complex Lie algebras.}\end{aligned}$$

Representations of Lie groups and Lie algebras

- For a Lie group G, a **complex representation** is a Lie group homomorphism
$$G \to \mathrm{GL}(V), \qquad V \text{ is some finite dimensional complex vector space.}$$
We will say V is a G-representation or G-module.
- For a Lie algebra $\mathfrak{g}$, a **complex representation** is a Lie algebra homomorphism
$$\mathfrak{g} \to \mathfrak{gl}(V), \qquad V \text{ is some finite dimensional complex vector space.}$$
Equivalently, for $X, Y \in \mathfrak{g}$,
$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = [X, Y] \cdot v.$$

Summary

$$\text{Lie groups} \overset{\text{restriction}}{\rightsquigarrow} \text{connected Lie groups}$$

$$\text{connected Lie groups} \overset{\text{universal covering}}{\rightsquigarrow} \text{simply connected Lie groups}$$

$$\text{simply connected Lie groups} \underset{\text{exponential map}}{\overset{\text{tangent map}}{\leftrightsquigarrow}} \text{Lie algebras}$$

$$\text{compact Lie groups} \overset{\text{complexification}}{\rightsquigarrow} \text{complex Lie algebras}$$

Representation of compact Lie groups	$\rightsquigarrow$	Representation of the complexification of its lie algebra

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Reductive and Semisimple Lie algebras

- We call a lie algebra $\mathfrak{g}$ to be **nilpotent**, if the series

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad \mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}].$$

  is zero for some n.
- We call a Lie algebra $\mathfrak{g}$ to be **reductive**, if it contains no nonzero **nilpotent ideal**.
- We call a Lie algebra $\mathfrak{g}$ to be **semisimple**, if it contains no nonzero commutative ideal (i.e. $[\cdot, \cdot] = 0$), equivalently no solvable ideal.

Theorem

For a complex reductive lie algebra $\mathfrak{g}$, it is product of a semisimple lie algebra and a commutative lie algebra.

Reductive and Semisimple Lie algebras

- We call an analytic lie group G to be **unipotent**, if it consists only unipotent element due to Jordan decomposition.
- We call a Lie group G to be **reductive**, if it contains no nontrivial unipotent normal subgroup.
- We call a Lie group G to be **semisimple**, if it contains no nontrivial commutative normal subgroup, equivalently no solvable normal subgroup.

Theorem

For an analytic lie group G,

$$G \text{ is reductive} \Longrightarrow \mathrm{Lie}(G) \text{ is reductive},$$
$$G \text{ is semisimple} \iff \mathrm{Lie}(G) \text{ is semisimple}.$$

Reductive and Semisimple Lie algebras

Theorem

Let G be a compact group, its complexfication is reductive.

Theorem

Let G be a complex analytic group

G is reductive by our philosophy. There is no nontrivial unipotent normal subgroup.	$\Longleftrightarrow$	*G is reducitve by our dream. All G-representations are semisimple.*

Reductive and Semisimple Lie algebras

Theorem (Weyl's theorem)

Let $\mathfrak{g}$ be a complex lie algebra

$\mathfrak{g}$ *is semisimple by our philosophy. There is no nonezero commutative ideal.*	$\Longleftrightarrow$	$\mathfrak{g}$ *is semisimple by our dream. All* $\mathfrak{g}$*-representations are semisimple.*

- We say a lie algebra is **simple** if it is not commutative and has no nonzero ideal.

Theorem

For a complex semisimple lie algebra $\mathfrak{g}$, $\mathfrak{g}$ *is a direct product of simple lie algebras.*

The classification of simple lie algebras

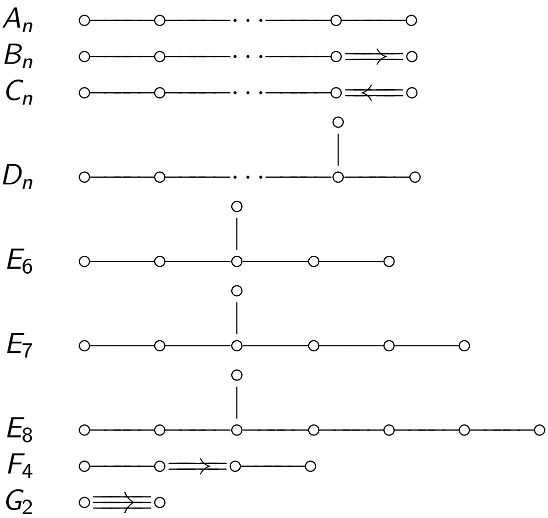
Theorem (Classification of simple lie algebras)

The simple lie algebras are classified.

- *Type A_n: $\mathfrak{sl}_{n+1}$ for $n \geq 1$.*
- *Type B_n: $\mathfrak{so}_{2n+1}$ for $n \geq 1$.*
- *Type C_n: $\mathfrak{sp}_n$ for $n \geq 1$.*
- *Type D_n: $\mathfrak{so}_{2n}$ for $n \geq 1$.*
- *Finite many exceptional types, E_6, E_7, E_8, F_4, G_2, say $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$.*

- They are one-to-one correspondent to the Dynkin diagram next page.
- Being isomorphic if and only if the diagram is the same.
- **To understand the reason is one of the main purpose in our later lectures.**

The classification of simple lie algebras



Realization

Theorem (Lie)

Every abstract (complex) Lie algebra can be realized as a Lie algebra of some (complex analytic) Lie groups.

Theorem (Chevalley)

The category of complex reductive analytic lie groups is equivalent to the category of compact lie groups.

Theorem

Let G be a Lie group, then

$$\{\text{connected Lie subgroup } H \subseteq G\} \leftrightarrow \{\text{subalgebra } \mathfrak{h} \subseteq \mathrm{Lie}(G)\}.$$

Where Lie subgroup H means a subgroup H such that $\mathrm{Lie}(H) \to \mathrm{Lie}(G)$ is injective.

The classification of compact lie groups

Theorem

For a compact Lie group G, its universal covering $\tilde{G}$ is a product of $\mathbb{R}$ and a compact group whose Lie algebra is semisimple.

- If $\tilde{G} \to G$ is a covering of Lie groups, then

$$\mathrm{Lie}(\tilde{G}) = \mathrm{Lie}(G).$$

and the kernel is a central discrete subgroup.

The classification of compact lie groups

Theorem (Classification of semisimple compact lie groups)

The compact group whose Lie algebra is simple is classified.

- *Type A_n:* $\mathrm{SU}(n+1)$, $\mathrm{PSU}(n+1)$ *for* $n \geq 1$.
- *Type B_n:* Spin_{2n+1}, SO_{2n+1} *for* $n \geq 1$.
- *Type C_n:* Sp_n, PSp_n *for* $n \geq 1$.
- *Type D_n:* Spin_{2n}, SO_{2n}, PSO_{2n} *for* $n \geq 1$. *and when* $n \mid 2$, *there is another* HSpin_{2n}.
- *Finite many exceptional groups,* E_6, E_7, E_8, F_4, G_2, *say* $E_6, E_6^{\mathrm{ad}}, E_7, E_7^{\mathrm{ad}}, E_8, F_4, G_2$.

References for lie groups and lie algebras

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Thanks