Lecture 1 — The structures of algebras and groups (I)

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What is representation?

All of Mathematics is some kind of representation theory.

— *I. M. Gelfand*[1]

- a representation $\approx$ a linear space being acted
- a group representation $\approx$ a linear space with symmetry

[1] A common misunderstanding is that Gelfand referred the representation to the Gelfand representation in functional analysis. But Gelfand is also an algebraist, for example he is one of the G in BGG theory.

Why representation?

- Some structure which is not easy to be understood
  $\rightsquigarrow$ find a linear point of view
- Some structure which can be understood
  $\rightsquigarrow$ how it acts?
- Example: $\mathfrak{S}_n$ acts on $V \otimes \overset{n}{\cdots} \otimes V$ by permuting indices.

The purpose of representation

- the classification

  some classification theory

- How to put the spaces familiar to us into this theory?
- How our classification theory reflects the operators on spaces?
- Example: $\mathrm{GL}(V)$ acts on $V \otimes \overset{n}{\cdots} \otimes V$ by diagonal action.

Why linearity?

- There is a number of theory which is specific for linear algebra
  - the dimension
  - the eigenvalue
  - the trace and the norm (determinant)
  - the quadratic form
  - ...
- There is a lot of linear spaces being acted.

Simple modules

- For a nonezero module M, if the only submodule is 0 and itself, then it is called **simple** or **irreducible**.
- Note that simple modules are all of the form R/M for some left maximal ideal so it is cyclic.

Theorem (Schur)

$\mathrm{End}(M)$ *is a division ring.*

- Furthermore, if M is "small" and $\mathbb{C} \subseteq \mathrm{End}(M)$, (for example the ring is a $\mathbb{C}$-algebra), then $\mathbb{C} = \mathrm{End}(M)$.
- This makes M a linear space which is easier to be understood.

Simple modules makes up modules

- If a module M is Noetherian and artinian (finite-dimensional linear space), then there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that M_{i+1}/M_i is simple.

Theorem (Jordan–Hölder)

The simple modules are unique with multiplicity.

$$\overbrace{\underbrace{\boxed{\text{simple modules}}}_{\text{Schur}}}^{\text{cyclic}} \xrightarrow[*]{\text{J–H}} \boxed{\text{all ``small'' modules}}.$$

But we lose some information

- Naïvely, do we have

$$\text{small modules} \xleftrightarrow{1:1} \text{multiple-sets of simple modules?}$$

- No, compare

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}v & \longrightarrow & \mathbb{C}v \oplus \mathbb{C}w & \longrightarrow & \mathbb{C}w & \longrightarrow & 0 \\
 & & \| & & \downarrow {\scriptsize \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}} & & \| & & \\
0 & \longrightarrow & \mathbb{C}v & \longrightarrow & \mathbb{C}v \oplus \mathbb{C}w & \longrightarrow & \mathbb{C}w & \longrightarrow & 0
\end{array}$$

What is semisimple?

- We call a module is **semisimple**, if it is a direct sum of simple modules.
- Our dream — it is one to one, that is,

$$\text{all “small” modules are semisimple}$$

$$\approx \text{for all } B \subseteq A, \text{ we have } A = A/B \oplus B.$$

$$\approx \text{all short exact sequences split.}$$

- We will call a ring **semisimple** if it satisfies our dream. (to be defined exactly later)
- We will call a lie algebra **reductive** if it satisfies our dream. (to be defined exactly later)

A philosophy

- To realize our dream, in philosophy,

$$\text{semisimple or reductive} \approx \text{no morphism like } \begin{pmatrix} a & 1 \\ & a \end{pmatrix}.$$

- In principle,

$$\begin{pmatrix} a & 1 \\ & a \end{pmatrix} \text{ in algebra} \approx \begin{pmatrix} a & 1 \\ & a \end{pmatrix} \text{ appears in representations.}$$

- So, the philosophy is

$$\text{semisimple or reductive} \approx \text{no nilpotent stuff}$$

Definitions

- An **associative algebra** R over a field k, or just k-**algebra**, is a ring with

$$k \subseteq Z(R) \underset{\text{centre}}{\subseteq} R \ni 1.$$

  We will assume all the k-algebras mentioned are finite-dimensional.

- We will call a k-algebra which is a division ring a division algebra over k.

Semisimple algebras

- For an ideal $I \subseteq R$, we say it is **nilpotent** if $I^n = 0$ for some n.

Theorem

Let R be a k-algebra.

R is semisimple by our philosophy. There is no nonezero nilpotent ideal.	$\Longleftrightarrow$	*R is semisimple by our dream. All R-modules are semisimple.*

Matrix algebra

Let D be a k-division algebra. For integer $n > 0$, define the matrix algebra $R = \mathbb{M}_n(D)$ by usual product.

- It is simple (no nonzero ideal).
- It is semisimple.
- The only simple R-module is D^n.

Wedderburn–Artin theorem

Theorem (Wedderburn–Artin)

Each semisimple k-algebra R is a finite direct product of matrix algebras over some k-division rings.

- If k is algebraic closed, then the only k-division ring ($\dim < \infty$) is itself. So just a product of $\mathbb{M}_n(k)$.

The proof of Wedderburn–Artin theorem

- Decompose R itself, as left R-module (so-called regular module), into simple modules,

$$R = {}_{\mathrm{reg}}R = n_1 M_1 \oplus \cdots \oplus n_k M_k$$

with M_i list of pairwise non-isomorphic simple modules, and nM stands for n copies of module M, i.e. M^n.

- Then

$$\begin{aligned} R &\cong \mathrm{End}_R(R)^{op} & r \mapsto [s \mapsto sr] \\ &= \mathrm{End}_R\big(\textstyle\bigoplus_{i=1}^k n_i M_i\big)^{op} & \\ &= \textstyle\prod_{i=1}^k \mathrm{End}_R(n_i M_i)^{op} & \mathrm{Hom}_R(M_i, M_{j\neq i}) = 0 \\ &= \textstyle\prod_{i=1}^k M_{n_i}(\mathrm{End}_R(M_i))^{op}. & \end{aligned}$$

Summary

- So we have

R is semisimple by our philosophy. There is no nonezero nilpotent ideal.	$\Longleftrightarrow$	R is semisimple by our dream. All R-modules are semisimple.

$\Longleftrightarrow$ R is semisimple as an algebra. It is a direct product of simple algebras. $\Longleftrightarrow$

Morita equivalence

- For two rings, if the category of finitely generated modules over them are equivalent as abelian categories, then two rings are called **Morita equivalent**.
- For a k-division ring D, there is a Morita equivalence to $\mathbb{M}_n(D)$

$$\left\{\begin{array}{l}\text{finite dimensional } D\text{-}\\ \text{linear spaces}\end{array}\right\} \cong \left\{\begin{array}{l}\text{finite dimensional}\\ \mathbb{M}_n(D)\text{-modules}\end{array}\right\}.$$

- Actually, the one dimensional space corresponds to the only simple module.

Classification of division algebras

- By the Wedderburn–Artin theorem,

$$\begin{array}{l}\text{classification of} \\ \text{division algebras}\end{array} = \begin{array}{l}\text{classification of} \\ \text{simple algebras}\end{array}$$

- For algebraic closed field, only division algebra is itself.
- For $\mathbb{R}$, it is well-known that the only division algebra is $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. (Frobenius theorem)
- For finite field, it is well-known that only finite division ring is a field. (Wedderburn theorem)

Central simple algebras

- An algebra R over field k is called **central** if its centre $Z(R) = k$.
- By the Wedderburn–Artin theorem, for an k-simple algebra R

$$k \overset{\text{field extension}}{\subseteq} Z(R) \overset{\text{central division ring}}{\subseteq} D \overset{\text{matrix algebra}}{\subseteq} \mathbb{M}_n(D).$$

- Let D be a central division algebra over k,

$$k \subseteq \underbrace{L}_{\text{maximal subfield}} \subseteq D, \qquad [L:K] = [D:L].$$

All such L splits D, i.e.

$$D \otimes_k L = \mathbb{M}_n(L), \qquad n = [L:K] = [D:L].$$

Central simple algebras

- For a field extension K/k,

  (semi-)simple k-algebra $\xrightarrow{-\otimes_k K}$ may not still be (semi-)simple.

  (semi-)simple algebra $\xrightarrow{-\otimes \text{simple}}$ may not still be (semi-)simple.

- But central simple algebra acts well

$$\text{central simple } k\text{-algebra} \xrightarrow{-\otimes_k K} \text{central simple } K\text{-algebra}.$$

$$\text{(central) simple} \xrightarrow{-\otimes \text{central simple}} \text{(central) simple}$$

These are also “if and only if” theorems.

Brauer group

- Define the **Brauer group**

$$\mathrm{Br}(k) = \bigoplus \mathbb{Z} \begin{bmatrix} \text{all central simple} \\ \text{algebras over } k \end{bmatrix} \Big/ \cdots$$

  where the relation is generated by
  - $A = B$ if there is integer m, n such that $\mathbb{M}_n(A) = \mathbb{M}_n(B)$ (i.e. the corresponding division rings are isomorphic);
  - $A = B + C$ if $A = B \otimes_k C$.
- The Brauer group can be also explained as Galois cohomology

$$\mathrm{Br}(k) = \varinjlim_{\text{finite Galois extension } K/k} H^2(\mathrm{Gal}(K/k), K^\times).$$

Summary

- For general field k,

$$\text{simple } k\text{-algebras} = \text{matrix algebras over } \underbrace{k\text{-division ring } D}.$$

$$k\text{-division ring } D = \bigcup_{K/k} \underbrace{K\text{-central division ring } D}.$$

$$K\text{-central division ring } D \overset{\text{classified by}}{\longrightarrow} \mathrm{Br}(K).$$

- In particular, for algebraic closed field k,

$$\text{simple } k\text{-algebras} = \text{matrix algebras over } k.$$

References for associative algebras

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Thanks