KLR Algebra 3

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May 20, 2021

Card for Quiver Hecke Algebras

Denote $q_{ij}(u,v) = \begin{cases} 0 & i = j \\ (v-u)^{\#\{i \to j\}}(u-v)^{\#\{j \to i\}} & i \neq j \end{cases}$.

Let $R(\underline{\mathbf{v}})$ be the algebra generated by $x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}, \mathbb{1}_{\underline{\mathbf{i}}}$, with $n = |\underline{\mathbf{v}}|$ and $\underline{\mathbf{i}} \vdash \underline{\mathbf{v}}$ with relations

- $\mathbb{1}_{\underline{\mathbf{i}}} \mathbb{1}_{\underline{\mathbf{j}}} = \delta_{\underline{\mathbf{i}},\underline{\mathbf{j}}} \mathbb{1}_{\underline{\mathbf{i}}}$.
- $x_k x_\ell = x_\ell x_k$
- $\tau_k \tau_\ell = \tau_\ell \tau_k \quad |k-\ell| > 1$.
- $\left(\tau_k x_\ell - x_{s_k(\ell)} \tau_k\right) \mathbb{1}_{\underline{\mathbf{i}}} = \delta_{\underline{\mathbf{i}}_k, \underline{\mathbf{i}}_{k+1}} (\delta_{k+1,\ell} - \delta_{k,\ell}) \mathbb{1}_{\underline{\mathbf{i}}}$
- $\tau_\ell^2 \mathbb{1}_{\underline{\mathbf{i}}} = q_{\ell,\ell+1}(x_\ell, x_{\ell+1}) \mathbb{1}_{\underline{\mathbf{i}}}$
- $\left(\tau_{\ell+1}\tau_\ell\tau_{\ell+1} - \tau_\ell\tau_{\ell+1}\tau_\ell\right) \mathbb{1}_{\underline{\mathbf{i}}} =$
$$= \delta_{\underline{\mathbf{i}}_\ell, \underline{\mathbf{i}}_{\ell+2}} \frac{q_{\underline{\mathbf{i}}_\ell \underline{\mathbf{i}}_{\ell+1}}(x_\ell, x_{\ell+1}) - q_{\underline{\mathbf{i}}_{\ell+2} \underline{\mathbf{i}}_{\ell+1}}(x_{\ell+2}, x_{\ell+1})}{x_\ell - x_{\ell+2}} \mathbb{1}_{\underline{\mathbf{i}}}.$$

This standard notation differs from mine (used in last two slides) by $x_i \leftrightarrow -x_i$.

Card for Quantum Groups

For simplicity, we consider only simply-laced case. Let $U_q(\mathfrak{g})$ be the algebra generated by E_i, F_i, K_i, K_i^{-1} with $i \in \mathbb{I}$ with relations

- $K_i K_i^{-1} = K_i^{-1} K_i = 1$
- $[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$
- $\begin{cases} K_i E_i K_j^{-1} = q^{i \cdot j} E_i \\ K_i F_i K_j^{-1} = q^{-i \cdot j} F_i \end{cases}$
- $\begin{cases} E_i E_j = E_j E_i & i \cdot j = 0 \\ E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 & i \cdot j = 1 \end{cases}$
- $\begin{cases} F_i F_j = F_j F_i & i \cdot j = 0 \\ F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 & i \cdot j = 1 \end{cases}$

Categorification Theorem

Let

$$[R\,\mathrm{Proj}] = \bigoplus_{\underline{\mathbf{v}}} [R(\underline{\mathbf{v}})\,\mathrm{Proj}].$$

We proved the following categorification theorem.

Theorem (Categorification)

$$U_q^-(\mathfrak{g}) \xrightarrow{\sim} [R\,\mathrm{Proj}] \qquad F_i \mapsto [\Theta_i].$$

Here $\Theta_i = R(\alpha_i) = \mathbb{Q}[x_1]\mathbb{1}_i$.

Some Abstract Nonsense

Consider the combinatorial category "*quiver Hecke category*"

$$\mathcal{H}(Q) = \begin{cases} \textbf{Obj}: & \underline{\mathbf{i}} \quad \text{some sequence of } \mathbb{I} \\ \textbf{Mor}: & \underline{\mathbf{i}} \to \underline{\mathbf{j}} \quad \text{elements from } \in \mathbb{1}_{\underline{\mathbf{j}}} R(\underline{\mathbf{v}}) \mathbb{1}_{\underline{\mathbf{i}}} \end{cases}$$

It is the same to the full subcategory of $\{R(\underline{\mathbf{v}})\mathbb{1}_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}}}$. But this is more combinatorial.

This is a monoidal category over $\mathbb{Q}$. Furthermore, the *Karoubi envelope* (=idempotent completion) is equivalent to the direct sum of the category $\bigoplus_{\underline{\mathbf{v}}} R(\underline{\mathbf{v}})\,\mathrm{Proj}$.

$$\mathcal{H}(Q)^{\oplus} \cong \bigoplus_{\underline{\mathbf{v}}} R(\underline{\mathbf{v}})\,\mathrm{Proj}\,.$$

Modified Version

- We use the modified quantum groups

$$\dot{\mathbf{U}}_q(\mathfrak{g}) = \bigoplus_{\mu,\lambda\in\Lambda} \mathbb{1}_\mu \cdot U_q(\mathfrak{g}) \cdot \mathbb{1}_\lambda$$

$(\Lambda = \text{weight lattice})$

$$\begin{cases} \mathbb{1}_\lambda \mathbb{1}_\mu = \delta_{\lambda\mu}\mathbb{1}_\lambda \\ K_i\mathbb{1}_\lambda = \mathbb{1}_\lambda K_i = q^{\langle \alpha_i^\vee, \lambda\rangle}\mathbb{1}_\lambda \\ E_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i}E_i \\ F_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}F_i \end{cases}$$

- For $\dot{\mathbf{U}}_q(\mathfrak{g})$, there is no coproduct. But there is a global symmetric paring, see [Lusztig Chapter 26].

$$(\lambda,\mu) \neq (\lambda',\mu') \Rightarrow \left\langle \mathbb{1}_\mu f \mathbb{1}_\lambda, \mathbb{1}_{\mu'} g \mathbb{1}_{\lambda'} \right\rangle = 0$$

$$\langle x^+\mathbb{1}_\lambda, y^+\mathbb{1}_\mu\rangle = \langle x, y\rangle_{\mathfrak{f}} \qquad \langle xy, z\rangle = \langle y, \rho(x)z\rangle \qquad \langle x, y\rangle = \langle y, x\rangle$$

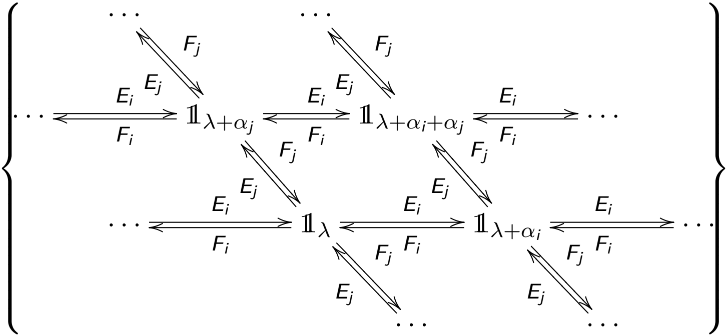
Here

$$\rho : \begin{cases} \mathbb{1}_{\lambda+\alpha_i}E_i\mathbb{1}_\lambda \longmapsto q^{(\cdots)}\mathbb{1}_\lambda F_i \mathbb{1}_{\lambda+\alpha_i} \\ \mathbb{1}_{\lambda+\alpha_i}F_i\mathbb{1}_\lambda \longmapsto q^{(\cdots)}\mathbb{1}_\lambda E_i \mathbb{1}_{\lambda+\alpha_i} \end{cases}$$

We can think $U_q(\mathfrak{g})$ a category with only one point

$U_q(\mathfrak{g}) =$ 

But $\dot{\mathbf{U}}_q(\mathfrak{g})$ a category with objects to be elements of Λ

$\dot{\mathbf{U}}_q(\mathfrak{g}) =$ 

Remark

We think $\dot{\mathbf{U}}_q(\mathfrak{g})$ as an algebra, also as a category

- If we just want to compute this category $\dot{\mathbf{U}}_q(\mathfrak{g})$, there is some diagrammatic approach. (In this "categorification", relations are equalities, not isomorphisms)
- For $\mathfrak{gl}_n$, known as *ladder diagram*, with $\mathfrak{sl}_N$-web, it gives a diagrammatic approach to the Howe duality. It also provides knot invariants. REF REF
- But today, we will categorify $\dot{\mathbf{U}}_q(\mathfrak{g})$. (In this "categorification", relations are isomorphisms, not equalities)

Our Main Purpose

We will construct the category $\mathcal{U}_q(\mathfrak{g})$.

- Its object is all the monomials in $\dot{\mathbf{U}}_q(\mathfrak{g})$

$$\mathbf{Obj}(\mathcal{U}_q(\mathfrak{g})) = \left\{ E_i^a F_j^b \cdots 1_\lambda \right\}.$$

- The morphism will be constructed such that

$$\langle X, Y \rangle = \underline{\mathbf{dim}}\, \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{g})}(X, Y).$$

and categorify

$$\begin{cases} 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda \\ E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i \\ F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i \end{cases}$$

$$\begin{cases} E_i E_j = E_j E_i & i \cdot j = 0 \\ E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 & i \cdot j = 1 \end{cases}$$

$$\begin{cases} F_i F_j = F_j F_i & i \cdot j = 0 \\ F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 & i \cdot j = 1 \end{cases}$$

$$\begin{cases} E_i F_j = F_j E_i & i \neq j, \\ E_i F_i \cdot 1_\lambda = F_i E_i \cdot 1_\lambda + [\langle \alpha_i^\vee, \lambda \rangle] \cdot 1_\lambda & i \neq j, \end{cases}$$

Objects

We use the following notations

$$\mathbb{1}_\lambda = {}_\lambda \qquad E_i = \overset{i}{\uparrow} \qquad F_i = \overset{i}{\downarrow}.$$

Since in $\dot{\mathbf{U}}(\mathfrak{g})$, E_i and F_i does not exists singly, so more precisely,

$$\lambda' = \lambda + \alpha_i \qquad \mathbb{1}_{\lambda'} E_i \mathbb{1}_\lambda = {}_{\lambda'} \overset{i}{\uparrow} {}_\lambda \qquad \mathbb{1}_\lambda F_i \mathbb{1}_{\lambda'} = {}_\lambda \overset{i}{\downarrow} {}_{\lambda'}.$$

For example,

$$F_1 F_2 E_1 \mathbb{1}_\lambda = {}_{\lambda'''} \downarrow {}_{\lambda''} \downarrow {}_{\lambda'} \uparrow {}_\lambda \qquad \begin{aligned} \lambda' &= \lambda + \alpha_1 \\ \lambda'' &= \lambda' - \alpha_2 \\ \lambda''' &= \lambda'' - \alpha_1 \end{aligned}$$

Serre Relations

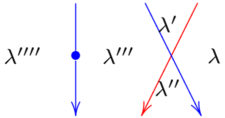
We impose

> The relations in $\mathcal{H}(Q)$ holds for $\uparrow$ and $\downarrow$ respectively

That is, we add the following into $\mathrm{Mor}(\mathcal{U}(\mathfrak{g}))$

(to be colored)

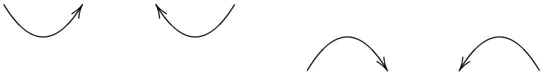
For example,



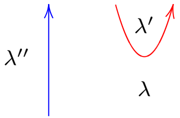
is a morphism from $\lambda''''\downarrow\lambda'''\downarrow\lambda''\downarrow\lambda = F_2F_1F_2\mathbb{1}_\lambda$
to $\lambda''''\downarrow\lambda'''\downarrow\lambda'\downarrow\lambda = F_2F_2F_1\mathbb{1}_\lambda$

Adjointness

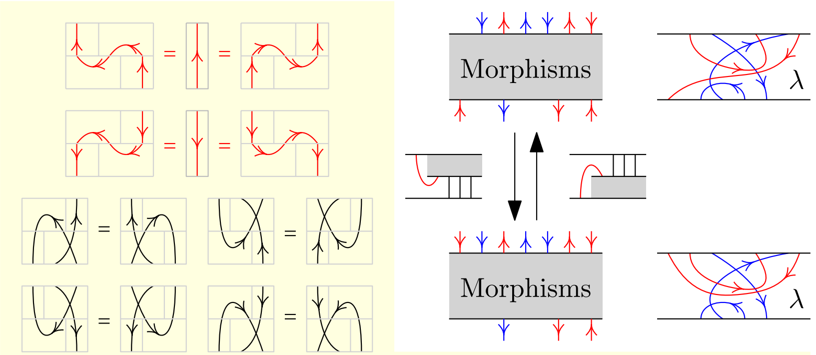
Since under the pairing $\langle -, - \rangle$, E_i and F_i are in principle adjoint. To categorify them, we consider the unit and counit

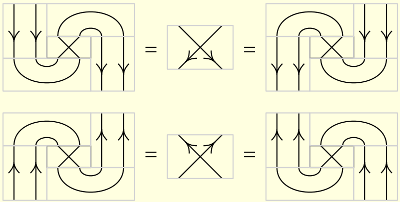


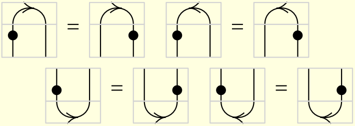
For example,



is a morphism from $\lambda'' \uparrow \lambda = E_2 \mathbb{1}_\lambda$ to $\lambda'' \uparrow \lambda \downarrow \lambda' \uparrow \lambda = E_2 F_1 E_1 \mathbb{1}_\lambda$





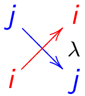


Well, there are more relations...

The rest relation

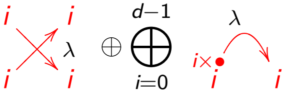
Then we "force" the following morphisms to be invertible.

- When $i \neq j$, or $i = j$ with $\langle \alpha_i^\vee, \lambda \rangle = 0$



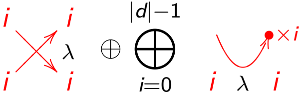
$$: E_i F_j \mathbb{1}_\lambda \longrightarrow F_j E_i \mathbb{1}_\lambda$$

- When $d := \langle \alpha_i^\vee, \lambda \rangle \geq 0$,



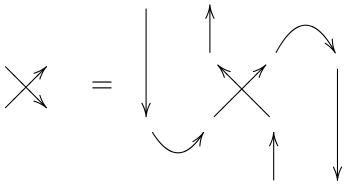
$$: E_i F_i \longrightarrow F_i E_i \mathbb{1}_\lambda \oplus \mathbb{1}_\lambda \oplus \overset{d}{\cdots} \oplus \mathbb{1}_\lambda$$

- When $d := \langle \alpha_i^\vee, \lambda \rangle \leq 0$,



$$: E_i F_i \mathbb{1}_\lambda \oplus \overset{|d|}{\cdots} \oplus \mathbb{1}_\lambda \longrightarrow F_i E_i \mathbb{1}_\lambda \oplus$$

Where

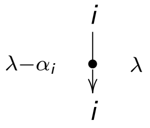
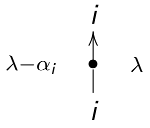
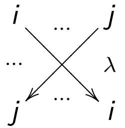
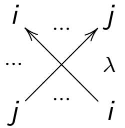
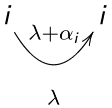
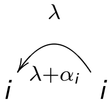
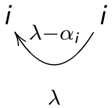
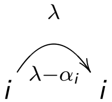


Actually, it is proved that if we assume the morphisms to be invertible, then the inverse can also be expressed by a long list of diagrams and relations. REF

Be careful, it even has imaginary “minus •” on bubbles.

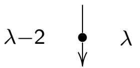
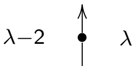
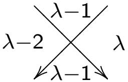
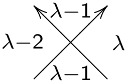
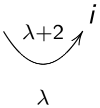
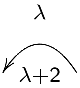
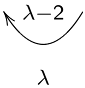
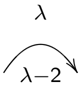
Degrees

Let us determine the degrees

				
deg	2	2	$-i \cdot j$	$-i \cdot j$
				
deg	$1 + \langle \alpha_i, \lambda \rangle$	$1 + \langle \alpha_i, \lambda \rangle$	$1 - \langle \alpha_i, \lambda \rangle$	$1 - \langle \alpha_i, \lambda \rangle$

Example of $\mathfrak{sl}_2$

In this case, let us assume $\Lambda = \mathbb{Z}$, and $\alpha_1 = 2 \in \Lambda$. Then $\langle \alpha_1^\vee, b \rangle = b$. Then

				
deg	2	2	-2	-2
				
deg	$1 + \lambda$	$1 + \lambda$	$1 - \lambda$	$1 - \lambda$

Summary

- $\begin{cases} \mathbb{1}_\lambda \mathbb{1}_\mu = \delta_{\lambda\mu}\mathbb{1}_\lambda \\ E_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i}E_i \quad F_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}F_i \end{cases}$ By notation
- $\begin{cases} E_iE_j = E_jE_i & i\cdot j = 0 \\ E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0 & i \cdot j = 1 \end{cases}$ $\begin{cases} F_iF_j = F_jF_i & i\cdot j = 0 \\ F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0 & i \cdot j = 1 \end{cases}$ By quiver-Hecke
- $\begin{cases} E_iF_j = F_jE_i & i \neq j, \\ E_iF_i \cdot \mathbb{1}_\lambda = F_iE_i \cdot \mathbb{1}_\lambda + [\langle \alpha_i^\vee, \lambda\rangle] \cdot \mathbb{1}_\lambda & i \neq j, \end{cases}$ By forcing
- Actually, there is a combinatorial formula for monomial, summing over all homotopy class, see [KL, section 2.2].

From the construction above, $[\mathcal{U}_q(\mathfrak{g})] = \dot{\mathbf{U}}_q(\mathfrak{g})$.

The 2-category $\mathcal{U}_q(\mathfrak{g})$

But actually, this construction here is too "artificial". Actually, the purpose of $\mathcal{U}_q(\mathfrak{g})$ here is different from $R(\underline{\mathbf{v}})$.
The significance here is

$$\text{think } \dot{\mathbf{U}}_q(\mathfrak{g}) \text{ as an algebra} \Longrightarrow \text{Category } \mathcal{U}_q(\mathfrak{g}) \text{ categorify it.}$$

$$\text{think } \dot{\mathbf{U}}_q(\mathfrak{g}) \text{ as a category} \Longrightarrow \text{2-Category } \mathcal{U}_q(\mathfrak{g}) \text{ categorify it.}$$

(When mathematicians tried to categorify, Rouqier had started to categorify categories)

$$\mathcal{U}_q(\mathfrak{g}) = \begin{cases} \textbf{Obj}: & \lambda \quad \in \Lambda \\ \textbf{Mor}: & \Big\downarrow_{\mu} \quad = \begin{cases} \textbf{Obj} & \text{monomials in } E_i, F_i \\ \textbf{Mor}: & \text{spaces generated by diagrams} \end{cases} \end{cases}$$

Representations

- If we think $U_q(\mathfrak{g})$ as a one-point category with endormorphism ring $U_q(\mathfrak{g})$, then

$$U_q(\mathfrak{g})\text{-Rep} = \text{Fun}(U_q(\mathfrak{g}), \mathbb{k}\text{-Vec}).$$

  Equivalently, we find one vector space with an action E_i, F_i, K_i satisfying the relations.

- Similarly,

$$\dot{\mathbf{U}}_q(\mathfrak{g})\text{-Rep} = \text{Fun}(\dot{\mathbf{U}}_q(\mathfrak{g}), \mathbb{k}\text{-Vec}).$$

  Equivalently, we find a famility of space parameterized by Λ, (i.e. weight space) with maps E_i, F_i between them.

2-Representations

- Then, we can define 2-representation

$$\mathcal{U}_q(\mathfrak{g})\text{-Rep} = 2\text{-}\mathrm{Fun}(\dot{\mathbf{U}}_q(\mathfrak{g}), \mathbb{k}\text{-Cat}).$$

  Equivalently, we find a familiity of categories parameterized by Λ, (i.e. weight space) with functor E_i, F_i between them; natural transformation (diagrams) between functors.

- But, can we really find 2-representations?
  Yes! From geometry. Springer theory, Nakajima quiver variety, perverse sheaves, ... This is known as "categorificatioin of representations". REF REF
  To do this, may be only do for a half of quantum group (quiver-Hecke algebra) is easier. This is known as "cyclotomic categorification conjecture". REF

References and Preview

- Brundan. On the definition of Kac-Moody 2-category. [arXiv]
- Kamnitzer. Categorification of Lie algebras [d'apres Rouquier, Khovanov-Lauda]. [arXiv]
- Rouquier. 2-Kac-Moody algebras. [arXiv]
- Khovanov, Lauda. A diagrammatic approach to categorification of quantum groups III [arXiv].

General Remarks on Categorification

As far as I see, there are two kinds of *categorification*

Realization algebra as a "Grothendieck group" of category

$$\mathcal{H}eck_q(W) \cong \mathbf{SSPerv}_B(G/B)$$

$$U_q(\mathfrak{g})^+_{\deg=\underline{\mathbf{v}}} \cong \mathbf{SSPerv}_{G(\underline{\mathbf{v}})}(E(\underline{\mathbf{v}}))$$

$$\mathcal{H}all(Q) \cong U_q(\mathfrak{g})^+\Big|_{q=\mathbf{q}^{1/2}}.$$

. . . .

Computation of a known category combinatorially (invariant theory)

$$\mathbb{SSB}\mathrm{im} \cong \mathbf{SSPerv}_B(G/B)$$

$$R(\underline{\mathbf{v}})\text{-Proj} \cong \mathbf{SSPerv}_{G(\underline{\mathbf{v}})}(E(\underline{\mathbf{v}}))$$

$$\mathcal{TL}^{\oplus} \cong U_q(\mathfrak{sl}_2)\text{-Rep}$$

. . . .

Of course, we do both of them —"diagrammatic algebra".

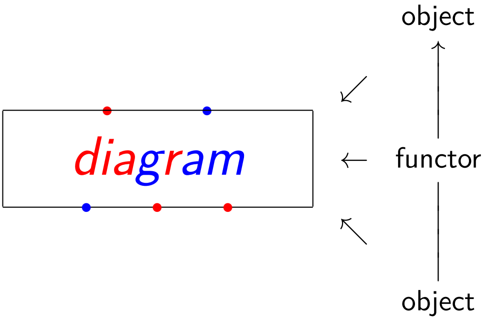
When realizing algebras, usually

element	object
$+$	$\oplus$ or formal sum
$\cdot$	$\otimes$, composition etc
$=$	$\cong$, exact sequence, etc
q	degree shift
pairing	$\underline{\mathbf{dim}}\,\mathrm{Hom}^\bullet(-,-)$
involution	duality

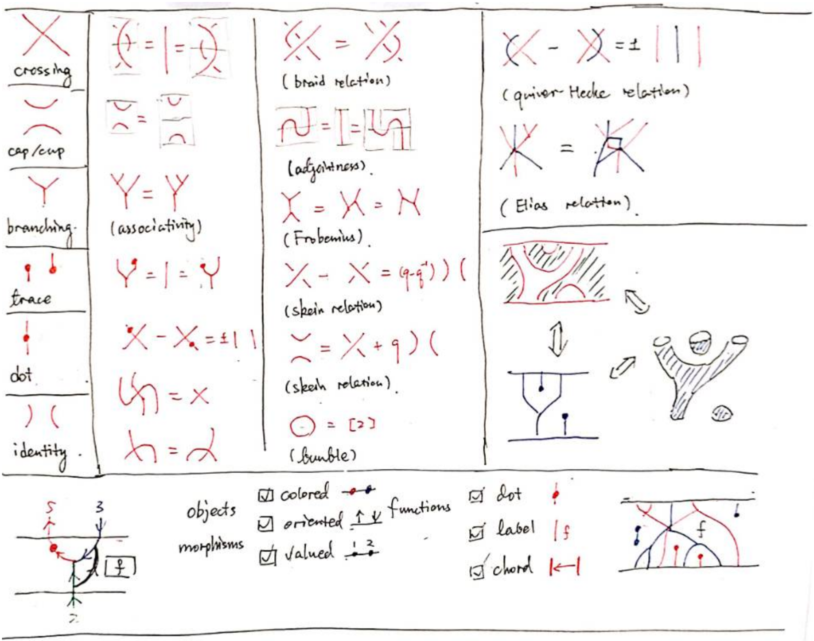
- **Subalgebra/subspace?** We may consider subcategory, for example, the category of modules over a quotient algebra.
- **Subalgebra fixed by a group G action?** We can consider the functor category from G to $\mathcal{C}$, $\mathrm{Fun}(G, \mathcal{C})$. For example, lusztig uses similar idea to construct non-simply-laced quantum groups.
- **Quotient algebra/space?** Well, it adds more relation so we should consider the localiztion of the category. For example, cyclotomic KLR algebra.
- **Module $\approx$ representation?** We can consider the 2-category generated by the functor $X \otimes -$ and natural transform $\phi \otimes -$ for all object X and functor ϕ. For example, the 2-category $U\mathfrak{g}$.

When Computation of Categories, usually ...

- We only computes a category $\mathcal{C}$ whose *Karoubi envelope* (=idempotent completion) is what we want.



Usually, the diagram is a "cobordism" of the boundary.


crossing
cap/cup
branching
trace
dot
identity
(associativity)
(braid relation)
(adjointness)
(Frobenius)
(skein relation)
(skein relation)
= [2]
(bubble)
(quiver Hecke relation)
(Elias relation)
objects
morphisms
colored
oriented
valued
functions
dot
label
chord
f

Refenrence for Diagrammatic Categorification

- REF The category of $\mathfrak{sl}_2$-representations (=Temperley–Lieb category). [bilibili]
- REF The category of $\mathfrak{sl}_3$-representations.
- REF The category of $\mathfrak{sl}_N$-representations (=$\mathfrak{sl}_N$-web).
- REF Heisenberg algebra. [bilibili]
- REF Realization of $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ (=ladder diagram)
- REF Realization of $\dot{\mathbf{U}}_q(\mathfrak{g})$ (=KLR algebra). [bilibili]
- REF Realization of Hecke algebras (=Soergel bimodules). [bilibili, Youtube]