# KLR Algebra 1

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# Card for Quantum Groups

For simplicity, we consider only simply-laced case.

- The Lusztig algebra

$$\mathfrak{f} = \Bbbk \langle \theta_i : i \in \mathbb{I} \rangle \Big/ \left\langle \begin{matrix} \theta_i\theta_j = \theta_j\theta_i & i \cdot j = 0 \\ \theta_i^2\theta_j - [2]\theta_i\theta_j\theta_i + \theta_j\theta_i^2 = 0 & i \cdot j = 1 \end{matrix} \right\rangle$$

  $\deg \theta_i = \alpha_i \in$ (root lattice).

- The $q$-twisted coproduct.

$$\begin{gathered} \Delta(1) = 1 \\ \Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i \end{gathered} \qquad \begin{gathered} \Delta(x) = \sum x \otimes x', \Delta(y) = \sum y \otimes y' \\ \Longrightarrow \Delta(xy) = \sum q^{-\deg x \cdot \deg y} xy \otimes x'y' \end{gathered}$$

- An pairing

$$\begin{gathered} \langle 1, 1 \rangle = 1 \\ \langle \theta_i, \theta_j \rangle = \frac{\delta_{ij}}{1-q^2} \end{gathered} \qquad \begin{gathered} \langle x_1x_2, y \rangle = \langle x_1 \otimes x_2, \Delta(y) \rangle \\ \langle x, y_1y_{@} \rangle = \langle \Delta(x), y_1 \otimes y_2 \rangle . \end{gathered}$$

# Card for Quivers

For simplicity, we consider only simply-laced case.

- For a dimension vector $\underline{\mathbf{v}} = (v_i)_{i \in I} \in \mathbb{N}^{\mathbb{I}}$. We define

$$G(\underline{\mathbf{v}}) = \bigoplus_{i \in \mathbb{I}} \mathrm{GL}_{v_i}(\Bbbk) \curvearrowright E(\underline{\mathbf{v}}) = \bigoplus_{i \to j} \mathrm{Hom}_{\Bbbk}(\Bbbk^{v_i}, \Bbbk^{v_j})$$

  Define $\alpha_i = (\cdots 0, \overset{i}{1}, 0 \cdots)$ the standard basis.

- For a sequence of $\underline{\mathbf{i}} = (i_1, \ldots, i_r) \in \mathbb{I}^r$ with $v_i$ many $i$'s (write $\underline{\mathbf{i}} \vdash \underline{\mathbf{v}}$), we define

$$\mathcal{F}\ell(\underline{\mathbf{i}}) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = \Bbbk^{\underline{\mathbf{i}}} : \underline{\mathbf{dim}}(V_* / V_{*-1}) \cong \alpha_* \right\}$$

$$\widetilde{\mathcal{F}\ell}(\underline{\mathbf{i}}) = \left\{ (V_*, f) \in \mathcal{F}\ell(\underline{\mathbf{i}}) \times E(\underline{\mathbf{v}}) : f(V_*) \subseteq V_* \right\}$$

# Card for Cohomology

- Let $G = \mathrm{GL}_n$, and $B$ the subgroup of upper triangular matrices. Then

$$\mathcal{F}\ell(n) = \left\{ 0 = V_0 \subseteq V_1 \cdots \subseteq V_n = \mathbb{C}^n : \dim V_i / V_{i-1} = 1 \right\} = G/B.$$

- The cohomology

$$H^*_G(G/B) = \mathbb{Z}[x_1, \ldots, x_n], \qquad x_i = -c_2(\phi_i/\phi_{i-1}) = c_2(\mathcal{O}(x_i)).$$

  Here $\phi_i$ the tautological bundle of the $i$-th subspace.

- We also need

$$H^*_G(\mathrm{pt}) = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}, \qquad H^*_T(\mathrm{pt}) = \mathbb{Z}[x_1, \ldots, x_n].$$

# Card for Combinatorics

- For $i \in \mathbb{Z}_{>0}$, we define the *BGG Demazure operator* $\partial_i$ on $\mathbb{Z}[x_1, x_2, \ldots]$ by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}} \qquad f \in \mathbb{Z}[x_1, x_2, \ldots].$$

- They satisfy

$$\begin{gathered} \partial_i^2 = 0 \\ \partial_i f = 0 \iff s_i f = f \end{gathered} \qquad \begin{cases} \partial_i \partial_j = \partial_j \partial_i & |i - j| \geq 2 \\ \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \end{cases}$$

  So we can define $\partial_w$ for all $w \in \mathfrak{S}_\infty = \bigcup_{n \gg 0} \mathfrak{S}_n$.

- Moreover, if we denote $\rho = \rho_n = (n-1, n-2, \ldots, 1, 0)$,

$$\left\{ f \in \mathbb{Z}[x_1, \ldots, x_n] : \begin{array}{l} \text{each monomial } x^\lambda \\ \text{of } f \text{ satisfties } \lambda \leq \rho \end{array} \right\}$$

  is closed under $\partial_w$.

# Nil-Hecke algebras

This is the case for $\mathfrak{sl}_2$ whose $\mathfrak{f}$ is easy.

- Define the $n$-th *nil-Hecke algebra*

$$\mathrm{NH}_n = \begin{array}{l}\text{The algebra generated by}\\ \text{(a) left multiplications of } x_1, \ldots, x_n \text{ and}\\ \text{(b) Demazure operators } \partial_1, \ldots, \partial_{n-1}.\end{array}$$

$$= \frac{\mathbb{Q}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_{n-1} \rangle}{\left\langle \begin{array}{ll} x_i x_j = x_j x_i,\ \partial_i^2 = 0 & \\ \partial_i \partial_j = \partial_j \partial_i & |i-j| \geq 2 \\ \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} & \\ \partial_i x_i = 1 + x_{i+1} \partial_i & \\ \partial_i x_{i+1} = -1 + x_i \partial_i & \\ \partial_i x_j = x_j \partial_i & j \notin \{i, i+1\} \end{array} \right\rangle} \subseteq \mathrm{End}(\mathbb{Q}[x_1, \ldots, x_n])$$

Take $\deg x_i = 2, \deg \partial_i = -2$.

# Structure of $\mathrm{NH}_n$

## Theorem

$$\begin{aligned} \mathrm{NH}_n &= \mathrm{End}_{\Lambda_n}(P_n) \\ &= \mathbb{M}_{n!\times n!}(\Lambda_n) \end{aligned}, \qquad \begin{aligned} P_n &= \mathbb{Q}[x_1, \dots, x_n] \\ \Lambda_n &= \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}. \end{aligned}$$

- For $n = 1$, $\mathrm{NH}_1 = \mathbb{Q}[x_1] = \Lambda_1$.
- For $n = 2$, under the basis $\mathbb{Q}[x_1, x_2] = x_1\Lambda_2 \oplus \Lambda_2$, we have

$$\begin{array}{cccccc} x_1\partial_1 & \partial_1 x_1 & \partial_1 & -\partial_1 x_2 & x_1 & x_2 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} & \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} & \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} & \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} & \begin{pmatrix} x_1+x_2 & 1 \\ -x_1x_2 & 0 \end{pmatrix} & \begin{pmatrix} & -1 \\ x_1x_2 & x_1+x_2 \end{pmatrix} \end{array}$$

$$\begin{aligned} x_2 \cdot (ax_1 + b) &= -\overbrace{b}^{\in\Lambda_2} x_1 + \overbrace{a(x_1x_2) + b(x_1 + x_2)}^{\in\Lambda_2} \\ x_1 \cdot (ax_1 + b) &= \underbrace{(a(x_1 + x_2) + b)}_{\in\Lambda_2} x_1 - \underbrace{a(x_1x_2)}_{\in\Lambda_2} \end{aligned}$$

# Structure of $\mathrm{NH}_n$ (continued)

## Theorem

$$\begin{aligned} \mathrm{NH}_n &= \mathrm{End}_{\Lambda_n}(P_n) \\ &= \mathbb{M}_{n!\times n!}(\Lambda_n) \end{aligned}, \qquad \begin{aligned} P_n &= \mathbb{Q}[x_1, \ldots, x_n] \\ \Lambda_n &= \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}. \end{aligned}$$

- This can be proved by Geometry. But algebraically, it follows from the following three facts (classic invariant theory)

$$P_n = \bigoplus_{\lambda \leq \rho_n} \Lambda_n \cdot x^\lambda, \qquad \#\{\lambda : \lambda \leq \rho_n\} = n!.$$

$$\partial_i(f \cdot g) = f \cdot \partial_i g \qquad \forall f \in \Lambda_n. \implies x_i \text{ and } \partial_i \text{ are all } \Lambda_n\text{-maps.}$$

$$\mathrm{NH}_n = \bigoplus_{\lambda \leq \rho_n, w \in \mathfrak{S}_n} x^\lambda \partial_w \cdot \Lambda_n \qquad \#\{w : w \in \mathfrak{S}_n\} = n!.$$

# Grothendieck Groups

### Theorem

*In the Grothendieck group of* $\operatorname{Proj}\mathrm{NH}_n$,

$$[\Theta^{(n)}] = \frac{[\mathrm{NH}_n]}{n!} \in K_0(\operatorname{Proj}\mathrm{NH}_n) \qquad \Theta^{(n)} = \mathrm{NH}_n \cdot \partial_{w_0}.$$

- By the structure theorem, it is clear that $P_n$ is the unique indecomposable projective module of $\mathrm{NH}_n$. And $\mathrm{NH}_n \cong n! \cdot P_n$ direct sum of $n!$ copies of $P_n$.
- But in practice, rather than $P_n$, we take the left module

$$\mathrm{NH}_n \cdot \partial_{w_0}, \qquad \begin{array}{l} \because \lambda \leq \rho \Longrightarrow \partial_{w_0} x^\lambda = \delta_{\lambda\rho}. \\ w_0 = \binom{1 \cdots n}{n \cdots 1} \text{ is the longest element in } \mathfrak{S}_n \end{array}$$

Actually, the corresponding idempotent is just $x^\rho \partial_{w_0}$.

# Grothendieck Groups (continued)

## Theorem

*In the Grothendieck group of* $\mathrm{Proj}\,\mathrm{NH}_n$*, taking degrees into consideration,*

$$q^{n(n-1)/2}\cdot[\Theta^{(n)}] = \frac{[\mathrm{NH}_n]}{[n]!} \in K_0(\mathrm{Proj}\,\mathrm{NH}_n) \qquad \Theta^{(n)} = \mathrm{NH}_n \cdot \partial_{w_0}.$$

- We should take degree into consideration. We use $q$ to denote

$$q[M] = [M(-1)] \qquad \deg_{=\bullet} M(-1) = M_{\bullet-1}.$$

- Under the basis $x^\lambda$ with $\lambda \leq \rho$,

$$\partial_{w_0} = \begin{pmatrix} \vdots & \vdots & \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \end{pmatrix} \qquad \mathrm{NH}_n \cdot \partial_{w_0} \cong \begin{pmatrix} \Lambda_n & 0 & \cdots \\ \vdots & \vdots & \\ \Lambda_n & 0 & \cdots \end{pmatrix}$$

- Take $n = 3$ as an example. Under the basis $x_1^2x_2, x_1^2, x_1x_2, x_2, x_1, 1$, then the degree of the entries

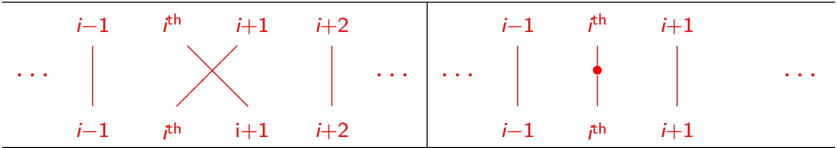
| deg | $x_1^2x_2$ | $x_1^2$ | $x_1x_2$ | $x_2$ | $x_1$ | 1 |
|---|---|---|---|---|---|---|
| $x_1^2x_2$ | 0 | 2 | 2 | 4 | 4 | 6 |
| $x_1^2$ | −2 | 0 | 0 | 2 | 2 | 4 |
| $x_1x_2$ | −2 | 0 | 0 | 2 | 2 | 4 |
| $x_2$ | −4 | −2 | −2 | 0 | 0 | 2 |
| $x_1$ | −4 | −2 | −2 | 0 | 0 | 2 |
| 1 | −6 | −4 | −4 | −2 | −2 | 0 |

$$1 + 2q^2 + 2q^4 + q^6$$
$$= (1 + q^2)(1 + q^2 + q^4)$$

By definition the first column is $\mathrm{NH}_n \cdot \partial_{w_0}$. Thus $[\mathrm{NH}_n] = (1 + q^2)(1 + q^2 + q^4)[\mathrm{NH}_n \cdot \partial_{w_0}]$.

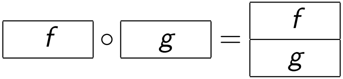
# Diagram Notations

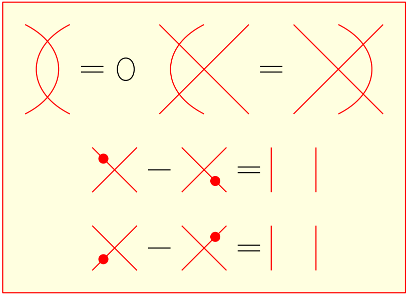
Well, let us stop using the presentation, but use diagrams.




| $\partial_i$ | $x_i$ |
|---|---|
| $\deg = -2$ | $\deg = 2$ |

The diagram is read from down to up





Exercise

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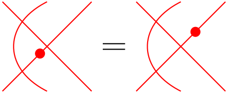

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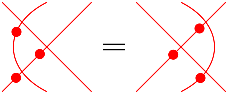
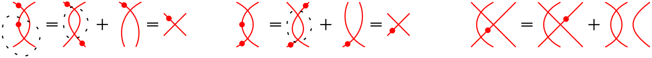
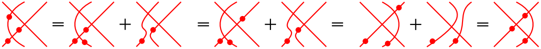
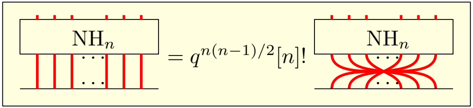
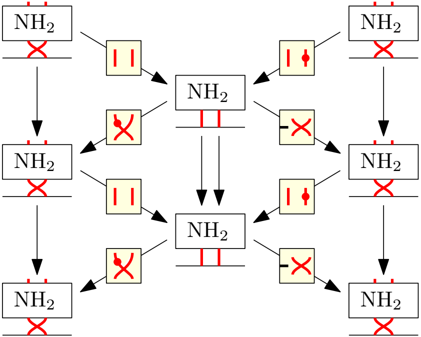


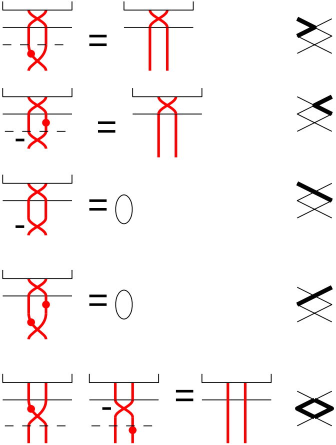







Special case: $n = 2$



# Geometric Picture

- Actually, for one vertex quiver, $\overset{1}{\circ}$. We can assume a dimension vector $\underline{\mathbf{v}} = n$, and then $\underline{\mathbf{i}} = {\color{red}1} \overset{n}{\cdots} {\color{red}1}$.

$$\underbrace{H^{BM}_{G(\underline{\mathbf{v}})}\left(\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}}) \underset{E(\underline{\mathbf{v}})}{\times} \widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}})\right)}_{=\mathrm{NH}_n = H^{BM}_G(G/B \times G/B)} \curvearrowright \underbrace{H^*_{G(\underline{\mathbf{v}})}(\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}}))}_{=P_n = H^*_G(G/B)}$$

  Here $\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}}) = \bigsqcup_{\underline{\mathbf{i}} \vdash \underline{\mathbf{v}}} \widetilde{\mathcal{F}\ell}(\underline{\mathbf{i}})$.

- The closure of the $G$-orbit $\{(xB, yB) : x^{-1}y \in BwB\}$ for $w \in \mathfrak{S}_n$ acts as $\partial_w$.

# KLR algebra

- Given a dimension vector $\underline{\mathbf{v}}$. Denote $n = |\underline{\mathbf{v}}|$, we define

$$P(\underline{\mathbf{v}}) = \bigoplus_{\underline{\mathbf{i}} \vdash \underline{\mathbf{v}}} \mathbb{Q}[x_1, \dots, x_n] \cdot \mathbb{1}_{\underline{\mathbf{i}}}.$$

  Here $\mathbb{1}_{\underline{\mathbf{i}}}$ is just a symbol of projection to the summand.
- For $1 \leq \ell \leq n$, We define an operator on $P(\underline{\mathbf{v}})$

$$\tau_\ell\big(f \cdot \mathbb{1}_{\underline{\mathbf{i}}}\big) = \begin{cases} (\partial_\ell f) \cdot \mathbb{1}_{\underline{\mathbf{i}}} & s_\ell(\underline{\mathbf{i}}) = \underline{\mathbf{i}} \\ (x_{\ell+1} - x_\ell)^{h(\underline{\mathbf{i}}_\ell, \underline{\mathbf{i}}_{\ell+1})} \big(s_\ell f\big) \cdot \mathbb{1}_{s_\ell(\underline{\mathbf{i}})} & s_\ell(\underline{\mathbf{i}}) \neq \underline{\mathbf{i}}. \end{cases}$$

  Here $h(i, j) = \#\{i \to j\} \in \{0, 1\}$. (We will see that $\tau_\ell$ does not satisfy braid relation)

# Two Vertex

Now consider the quiver

$$Q : \overset{1}{\circ} \longrightarrow \overset{2}{\circ}$$

Then, for example, for $\underline{\mathbf{v}} = (1, 2)$,

$$P(\underline{\mathbf{v}}) = \mathbb{Q}[x_1, x_2, x_3]\mathbb{1}_{122} \oplus \mathbb{Q}[x_1, x_2, x_3]\mathbb{1}_{212} \oplus \mathbb{Q}[x_1, x_2, x_3]\mathbb{1}_{221}.$$

Then $\tau_1, \tau_2$ acts as

| $\tau f \cdot \mathbb{1}_{\underline{\mathbf{i}}} = (?f) \cdot \mathbb{1}_{s(\underline{\mathbf{i}})}$ | $\underline{\mathbf{i}} = 122$ | $\underline{\mathbf{i}} = 212$ | $\underline{\mathbf{i}} = 221$ |
|---|---|---|---|
| $\tau = \tau_1$ | $(x_2 - x_1)s_1$ | $s_1$ | $\partial_1$ |
| $\tau = \tau_2$ | $\partial_2$ | $(x_3 - x_2)s_2$ | $s_2$ |

(Note that the 1 in the index of $\tau_1$ is different from 1 in $\underline{\mathbf{i}} = 122$; 1 is just the label of vertex in $Q$. )

# KLR algebra (continued)

- Define the $\underline{\mathbf{v}}$-th *quiver-Hecke algebra* = *KLR algebra*

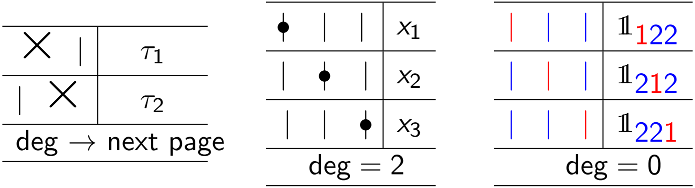
$$R(\underline{\mathbf{v}}) = \begin{array}{l} \text{The algebra generated by} \\ \text{(a) left multiplications of } x_1, \ldots, x_n, \\ \text{(b) formal projections } \mathbb{1}_{\underline{\mathbf{i}}} \text{ with } \underline{\mathbf{i}} \vdash \underline{\mathbf{v}} \text{ and} \\ \text{(c) “Demazure operators” } \tau_1, \ldots, \tau_{n-1}, \end{array}$$

$$= \frac{\mathbb{Q}\left\langle x_1, \ldots, x_n, \tau_1, \ldots, \tau_{n-1}, \mathbb{1}_{\underline{\mathbf{i}}} : \underline{\mathbf{i}} \vdash \underline{\mathbf{v}} \right\rangle}{\left\langle \begin{array}{c} \text{a very very very very} \\ \text{very very very very} \\ \text{very very very very} \\ \text{very very very very} \\ \text{very very very very} \\ \text{very very very very} \\ \text{very very very long list} \end{array} \right\rangle} \subseteq \mathrm{End}(P(\underline{\mathbf{v}}))$$

Actually, $R(\underline{\mathbf{v}}) \subseteq \mathrm{End}_{\Lambda_n}(P(\underline{\mathbf{v}}))$.

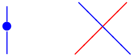
# Diagram Notations

We will use colored notation. For example




We take the convention that

$$= x_1 \mathbb{1}_{122}$$

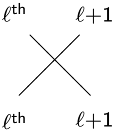


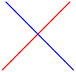
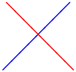
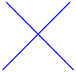
$$= \mathbb{1}_{221}(x_1\tau_2)\mathbb{1}_{212}$$

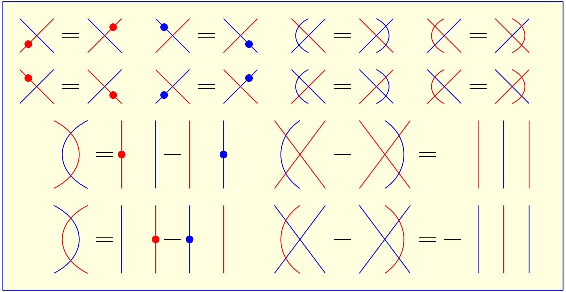
$$= (x_1\tau_2)\mathbb{1}_{212} = \mathbb{1}_{221}(x_1\tau_2).$$

# Diagram Notations (continued)

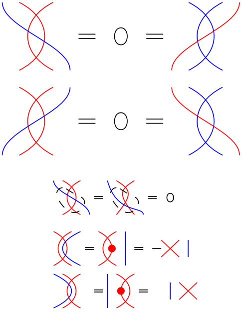
The degree here is normalized (we will see that the relation is homogeneous).



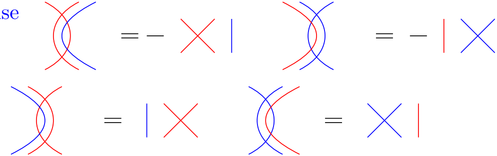

| | | |
|---|---|---|
|  | deg $= 1$ | $\tau_\ell \mathbb{1}_{\dots 12 \dots} : P(\underline{\mathbf{v}})\mathbb{1}_{\dots 12 \dots} \longrightarrow P(\underline{\mathbf{v}})\mathbb{1}_{\dots 21 \dots}$<br>$f \cdot \mathbb{1}_{\dots} \longmapsto (x_{\ell+1} - x_\ell)(s_\ell f) \cdot \mathbb{1}_{\dots}$ |
|  | deg $= 1$ | $\tau_\ell \mathbb{1}_{\dots 21 \dots} : P(\underline{\mathbf{v}})\mathbb{1}_{\dots 21 \dots} \longrightarrow P(\underline{\mathbf{v}})\mathbb{1}_{\dots 12 \dots}$<br>$f \cdot \mathbb{1}_{\dots} \longmapsto (s_\ell f) \cdot \mathbb{1}_{\dots}$ |
|  | deg $= -2$ | $\tau_\ell \mathbb{1}_{\dots 11 \dots} : P(\underline{\mathbf{v}})\mathbb{1}_{\dots 11 \dots} \longrightarrow P(\underline{\mathbf{v}})\mathbb{1}_{\dots 11 \dots}$<br>$f \cdot \mathbb{1}_{\dots} \longmapsto (\partial_\ell f) \cdot \mathbb{1}_{\dots}$ |
|  | deg $= -2$ | $\tau_\ell \mathbb{1}_{\dots 22 \dots} : P(\underline{\mathbf{v}})\mathbb{1}_{\dots 22 \dots} \longrightarrow P(\underline{\mathbf{v}})\mathbb{1}_{\dots 22 \dots}$<br>$f \cdot \mathbb{1}_{\dots} \longmapsto (\partial_\ell f) \cdot \mathbb{1}_{\dots}$ |



Exercise



Exercise



(We omit the relation involving only one color which is the same as NH)

# Serre Relations

## Theorem (Quantum Serre Relatioin)

*we have a split exact sequence in* Proj $R(\underline{\mathbf{v}})$ *with* $\underline{\mathbf{v}} = (1, 2)$

$$0 \to R(\underline{\mathbf{v}})\tau_2 \mathbb{1}_{122} \xrightarrow{\deg=1} R(\underline{\mathbf{v}})\mathbb{1}_{212} \xrightarrow{\deg=-1} R(\underline{\mathbf{v}})\tau_1 \mathbb{1}_{221} \to 0$$

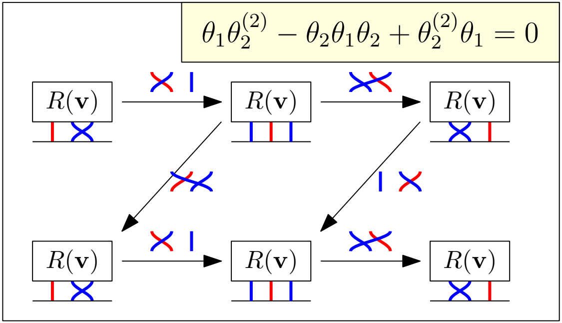
Actually,

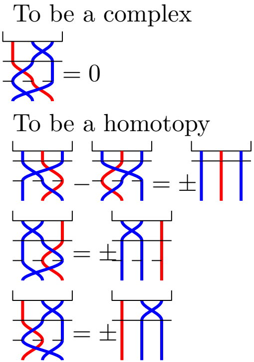
$$\begin{aligned}
P(\underline{\mathbf{v}})\mathbb{1}_{122} &\cong R(\underline{\mathbf{v}})\tau_2\mathbb{1}_{122} &= R(\underline{\mathbf{v}}) \otimes_{NH_1 \otimes NH_2} \Theta \otimes \Theta^{(2)} \\
P(\underline{\mathbf{v}})\mathbb{1}_{212} &\cong R(\underline{\mathbf{v}})\mathbb{1}_{212} &= R(\underline{\mathbf{v}}) \otimes_{NH_1 \otimes NH_1 \otimes NH_1} \Theta \otimes \Theta \otimes \Theta \\
P(\underline{\mathbf{v}})\mathbb{1}_{221} &\cong R(\underline{\mathbf{v}})\tau_1\mathbb{1}_{221} &= R(\underline{\mathbf{v}}) \otimes_{NH_2 \otimes NH_1} \Theta^{(2)} \otimes \Theta
\end{aligned}$$

$\Theta = \Theta^{(1)}$. The right isomorphisms will be explained in detail next time.

# Serre Relations (continues)

Proof: The map in the diagram is the “right multiplication the element labelled on the arrow”



# Reference and Preview

Note that the notations in the references are slightly different. Algebraists and a part of geometors prefer to take $x_i = -c_2(\mathcal{O}(x_i)) = c_2(\phi_i/\phi_{i-1})$. So their Demazure operator is different from us by a sign.

- Khovanov, Lauda. A diagrammatic approach to categorification of quantum groups I [arXiv], II [arXiv], III [arXiv].
- Brundan. Quiver Hecke algebras and categorification. [arXiv]

Next time, we will show how to construct $\mathfrak{f}$ from Grothendieck group of Proj $R(\underline{\mathbf{v}})$ (categorification theorem), the precise statement.

But now, let me state some geometry.

# Appendix: Geometric Picture

- The picture for general quiver is the same

$$\underbrace{H^{BM}_{G(\underline{\mathbf{v}})}\left(\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}}) \underset{E(\underline{\mathbf{v}})}{\times} \widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}})\right)}_{=R(\underline{\mathbf{v}})} \curvearrowright \underbrace{H^*_{G(\underline{\mathbf{v}})}\left(\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}})\right)}_{=P(\underline{\mathbf{v}})}$$

  Here $\widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}}) = \bigsqcup_{\underline{\mathbf{i}}\vdash\underline{\mathbf{v}}} \widetilde{\mathcal{F}\ell}(\underline{\mathbf{i}}) \subseteq \mathcal{F}\ell(|\underline{\mathbf{v}}|)$. Actually, the normalized of degree comes here — they do not have the same dimension.

- The closure of $\{(xB, yB) \in \mathcal{F}\ell(|\underline{\mathbf{v}}|) \times \mathcal{F}\ell(|\underline{\mathbf{v}}|) : x^{-1}y \in Bs_\ell B\} \cap \widetilde{\mathcal{F}\ell}(\underline{\mathbf{v}})$ acts by $\tau_\ell$ mod $P(\underline{\mathbf{v}})$. We can compute them using a Demazure type argument.

Take $\underline{\mathbf{i}} = {\color{red}1}{\color{blue}2}{\color{blue}2}{\color{red}1}{\color{blue}2} \vdash \underline{\mathbf{v}} = ({\color{red}2}, {\color{blue}3})$ as an example. Denote

$$G(\underline{\mathbf{i}}) = \begin{pmatrix} {\color{red}*} & & & {\color{red}*} & \\ & {\color{blue}*} & {\color{blue}*} & & {\color{blue}*} \\ & {\color{blue}*} & {\color{blue}*} & & {\color{blue}*} \\ {\color{red}*} & & & {\color{red}*} & \\ & {\color{blue}*} & {\color{blue}*} & & {\color{blue}*} \end{pmatrix} \subseteq \mathrm{GL}_5, \qquad E(\underline{\mathbf{i}}) = \begin{pmatrix} 0 & & & & \\ * & 0 & & * & \\ * & & 0 & * & \\ & & & 0 & \\ * & & & * & 0 \end{pmatrix} \subseteq \mathfrak{gl}_5.$$

(Note that, by a permutation, $G(\underline{\mathbf{i}}) \cong G(\underline{\mathbf{v}})$ and $E(\underline{\mathbf{i}}) \cong E(\underline{\mathbf{v}})$). We define

$$B(\underline{\mathbf{i}}) = G(\underline{\mathbf{i}}) \cap B = \begin{pmatrix} {\color{red}*} & & & {\color{red}*} & \\ & {\color{blue}*} & {\color{blue}*} & & {\color{blue}*} \\ & & {\color{blue}*} & & {\color{blue}*} \\ & & & {\color{red}*} & \\ & & & & {\color{blue}*} \end{pmatrix}, \qquad \mathfrak{n}(\underline{\mathbf{i}}) = E(\underline{\mathbf{i}}) \cap \mathfrak{n} = \begin{pmatrix} 0 & & & & \\ & 0 & & * & \\ & & 0 & * & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}.$$

Then

$$\mathcal{F}\ell(\underline{\mathbf{i}}) = G(\underline{\mathbf{i}})/B(\underline{\mathbf{i}}) \qquad \widetilde{\mathcal{F}\ell}(\underline{\mathbf{i}}) = G(\underline{\mathbf{i}}) \times_{B(\underline{\mathbf{i}})} \mathfrak{n}(\underline{\mathbf{i}}).$$

Let us denote $P_\ell$ and $\mathfrak{p}_\ell$ the standard parabolic subgroup/subalgebra

$$P_\ell(\underline{\mathbf{i}}) = G(\underline{\mathbf{i}}) \cap P_\ell \qquad \mathfrak{p}_\ell(\underline{\mathbf{i}}) = E(\underline{\mathbf{i}}) \cap \mathfrak{p}_\ell.$$

There is a Demazure operator

$$\begin{array}{ccc} H_G^{*+2d}\left(G(\underline{\mathbf{i}}) \underset{B(\underline{\mathbf{i}})}{\times} \mathfrak{n}(\underline{\mathbf{i}})\right) & \longrightarrow & H_G^{*}\left(G(\underline{\mathbf{i}}) \underset{P_\ell(\underline{\mathbf{i}})}{\times} \mathfrak{p}_\ell(\underline{\mathbf{i}})\right) \\ & \swarrow & \\ H_G^{*}\left(G(\underline{\mathbf{i}}') \underset{P_\ell(\underline{\mathbf{i}}')}{\times} \mathfrak{p}_\ell(\underline{\mathbf{i}}')\right) & \longrightarrow & H_G^{*}\left(G(\underline{\mathbf{i}}') \underset{B(\underline{\mathbf{i}}')}{\times} \mathfrak{n}(\underline{\mathbf{i}}')\right) \end{array}$$

where $\underline{\mathbf{i}}' = s_\ell(\underline{\mathbf{i}})$. Actually, push forward is given by

$$f \mapsto \begin{cases} \partial_\ell f & \underline{\mathbf{i}} = s_\ell(\underline{\mathbf{i}}) \\ (x_\ell - x_{\ell+1})^{h(\underline{\mathbf{i}}_\ell, \underline{\mathbf{i}}_{\ell+1})} f, & \underline{\mathbf{i}} \neq s_\ell(\underline{\mathbf{i}}). \end{cases}$$

Actually, $\swarrow$ acts by $s_i$; pull back is just the inclusion.

It is a general fact that (by localization theorem)

$$\underbrace{\operatorname{Proj} R(\underline{\mathbf{v}})}_{\text{graded}} \cong \underbrace{\mathbf{SSPerv}^{\text{geo,wt}=0}_{G(\underline{\mathbf{v}})}(E(\underline{\mathbf{v}})) \xrightarrow{\text{taking } K_0} \mathfrak{f}_{\deg=\underline{\mathbf{v}}}}_{\text{known by Lusztig}} .$$

Well, actually, they are given by a Morita type equivalence. Thus in particular, the indecomposable projective modules correspond to weight 0 perverse sheaves and to the canonical basis. This is the sketch of (my) proof of Rouquier conjecture. Actually, this is also what Soergel did for Hecke algebras.

The following paper proved the same result.

- Varagnolo and E. Vasserot. Canonical bases and KLR-algebras.

But I do not believe their proof.

# "Homework"

- Do the exercises in the diagrams.
- Show that in $\mathrm{NH}_n$, $\partial_i f = (\partial_i f) + (s_i f)\partial_i$.
- Show that in $\mathrm{NH}_n$,

$$\partial_{w_0} = \frac{\sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} w}{\prod_{i<j} (x_i - x_j)}.$$

  Hint: Use the fact $\partial_i \partial_{w_0} = 0$ to show $\partial_{w_0} f \in \Lambda_n$.
- Show that

$$\{\partial_\ell x_\ell : \ell\}, \qquad \{\partial_\ell x_\ell - 1 : \ell\}, \qquad \{\partial_\ell x_\ell x_{\ell+1} : \ell\}$$

  all satisfy the braid relation.