Each path can be reduced to a simple path

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Edit: In [1], the answer gives more references for this result. I learn from it that our conclusion **is perfectly covered** by [5] Page 222 31.6. And the property we called 'simple-path-connected' is more often refered as arcwise connected (the path is assumed to be imbedding). Although, Our proof is more elementary and direct. But for sake of the fact that the result has been found, out proof is not as interesting as I thought before. 2018/11/22

Abstract

A question raised reasonably when a freshman learn point-set topology is whether path-connected implies simple-path-connected, that is any two distinct points can be connected with injective path p. For example, this question in MathStackExchange [4], the answer of which gives a ghost reference and a terrible long proof. It is not difficult to consider some examples, such as the path looks like $\propto$. One can easily reduce it into simple path. We will show that for Hausdorff space the reduction is possible.

Firstly, we give some definitions. We say a topological space X is *simple-path-connected* if for any two distinct points $x, y \in X$ there exists an injective continuous map $p : [0, 1] \to X$ such that $p(0) = x, p(1) = y$. For a (original) path $p : [0, 1] \to X$, we say $\tilde{p} : [0, 1] \to X$ is a *reduction* of p if $p(0) = \tilde{p}(0), p(1) = \tilde{p}(1)$ and the image of $\tilde{p}$ is contained in p's.

Our question is whether path-connected implies simple-path-connected or not. If we replace X by the image of p, it is equivalent to that any path can be reduced to be a simple one. One can see the first two nontrivial examples of reduction to simple path in the following figure.

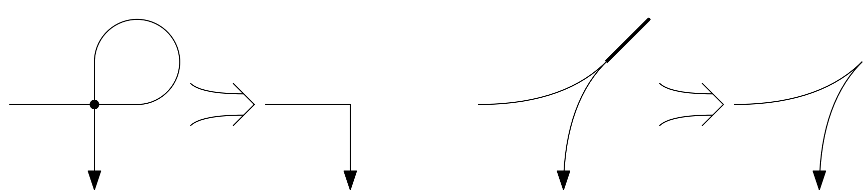

Figure 1: path reduction

As the figure above goes, it seems that it is true that one can reduce each path into a simple one. But it is a pity that it not true in general.

- For example, consider the trivial topology over $\{0,1\}$, then any map $[0,1] \to \{0,1\}$ is continuous, but no injective one exists.
- Another example is the quotient space of $[0,1]$ by gluing $\left\{\frac{1}{2^n}, 1-\frac{1}{2^n}\right\}$ for each $n \geq 1$. Since any $0 < x < 1/2$ is contained in the image of path implies for all n, $0 < \frac{1}{2^n} < x$, the path must pass $\left[\frac{1}{2^n}\right] = \left[1-\frac{1}{2^n}\right]$ twice.

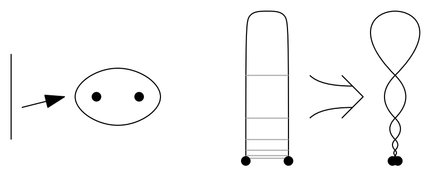

Figure 2: two counter examples

The topology in first example is too weak. The topology in seconde example is a little strong than the first one, since it is T_1 (since any equivalent class is closed). But the second topology is still not Hausdorff—$0,1$ can not be separated by open sets. Our theorem is that one can reduce each path into a simple one if the space is Hausdorff.

Theorem 1 *For any Hausdorff space X, two distinct points, any path connecting them can be reduced to a simple path. In particular, any path-connected space is simple-path-connected.*

Proof. Let $p : [0,1] \to X$ connected $x \neq y$. There is a equivalent relation R over $[0,1]$ by

$$a\,R\,b \iff p(a) = p(b)$$

it induces a injective continuous map $\hat{p} : [0,1]/R \to X$. Let $\pi : [0,1] \to [0,1]/R$ be the natural map.

By replace X by $[0,1]/R$, it suffices to show there is a injective continuous map $\tilde{p} : [0,1] \to [0,1]/R$ such that $\tilde{p}(0) = \tilde{p}(1)$. An easy exercise shows that $[0,1]/R$ is Hausdorff iff R is a closed set in $[0,1] \times [0,1]$, for example, cf [2] Page 606 Exercise A.36.

Before our proof, we define some notations

- the boundary of cube $\partial\big([a,b] \times [c,d]\big) = \{a,b\} \times [c,d] \cup [a,b] \times \{c,d\}$
- the boundary of R, $\partial_R\big([a,b] \times [c,d]\big) = \partial\big([a,b] \times [c,d]\big) \cap R$.
- diagonal of $A \subseteq [0,1]$, $\Delta A = \{(a,a) : a \in A\}$.
- the product cap R of $A, B \subseteq [0,1]$, $A \times_R B = (A \times B) \cap R$.

Let $x_0 = \max\{x \in [0,1] : (x,0) \in R\}, x_1 = \min\{x \in [0,1] : (x,1) \in R\}$. since $(0,1) \notin R$, $x_0 \neq x_1$, then by replace $0,1$ by x_0, x_1 and multiply a suitable scale, one can assume that

$$\partial_R[0,1]^2 = \Delta\{0,1\} \tag{$*$}$$

Let $b_0 = a_0 = 0, a_1 = b_1 = 1$. Let

$$d_{\frac{1}{2}} = \sup \left\{ d > 0 : (1-d, 1] \times_R [0, d] = \varnothing \right\} > 0$$

Since the diagonal is contained in R, so $d_{\frac{1}{2}} \le 1/2$. Pick $\left(a_{\frac{1}{2}}, b_{\frac{1}{2}}\right) \in R$ be the point over the boundary of $\left(1 - d_{\frac{1}{2}}, 1\right] \times \left[0, d_{\frac{1}{2}}\right)$ such that

- if $a_{\frac{1}{2}} = 1 - d_{\frac{1}{2}}$, then take $b_{\frac{1}{2}}$ to be minimal,
- if $b_{\frac{1}{2}} = d_{\frac{1}{2}}$, then take $b_{\frac{1}{2}}$ to be maximal.

Then

- $a_0 < b_{\frac{1}{2}} \le a_{\frac{1}{2}} < b_1$ and $\max\{b_1 - a_{\frac{1}{2}}, b_{\frac{1}{2}} - a_0\} < d_{\frac{1}{2}} < \frac{1}{2}$.

  Since by $(*)$, $0 \ne b_{\frac{1}{2}}$, and $a_{\frac{1}{2}} \ne 1$.

- $x \in [a_0, b_{1/2}], y \in [a_{\frac{1}{2}}, b_1]$ such that $x\,R\,y \iff x = b_{\frac{1}{2}}, y = a_{\frac{1}{2}}$. That is

  $$[a_{\frac{1}{2}}, b_1] \times_R [a_0, b_{\frac{1}{2}}] = \{(a_{\frac{1}{2}}, b_{\frac{1}{2}})\}$$

  Since the assumption of $\left(a_{\frac{1}{2}}, b_{\frac{1}{2}}\right)$.

- $\partial_R[a_0, b_{\frac{1}{2}}]^2 = \Delta\{a_0, b_{\frac{1}{2}}\}, \partial_R[a_{\frac{1}{2}}, b_1]^2 = \Delta\{a_{\frac{1}{2}}, b_1\}$.

  Since $\{a_{\frac{1}{2}}\} \times_R [a_0, b_{\frac{1}{2}}] = \{\left(a_{\frac{1}{2}}, b_{\frac{1}{2}}\right)\}$, thus $\{b_{\frac{1}{2}}\} \times [a_0, b_{\frac{1}{2}}] = \{\left(b_{\frac{1}{2}}, b_{\frac{1}{2}}\right)\}$, since $c\,R\,b_{\frac{1}{2}} \iff c\,R\,{}_{\frac{1}{2}}$. then by $(*)$ and reflection $\partial_R[a_0, b_{\frac{1}{2}}]^2 = \Delta\{a_0, b_{\frac{1}{2}}\}$.

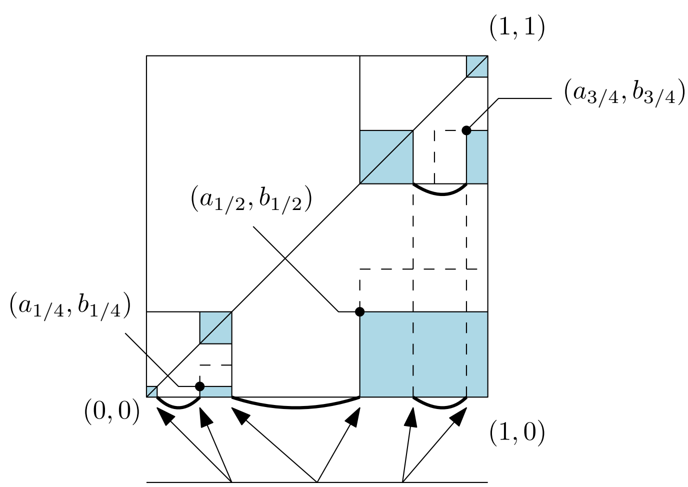

Figure 3: The process of the proof

For $n \ge 1$, assume that

$$d_{\frac{i}{2^n}} > 0 \qquad \left(a_{\frac{i}{2^n}}, b_{\frac{i}{2^n}}\right) \qquad i = 0, \ldots, 2^n$$

is constructed such that

- $a_0 < b_{\frac{1}{2^n}} < a_{\frac{1}{2^n}} < \ldots < b_{\frac{2^n-1}{2^n}} < a_{\frac{2^n-1}{2^n}} < b_1$ and $b_{\frac{i+1}{2^n}} - a_{\frac{i}{2^n}} < d_{\frac{i}{2^n}} < \frac{1}{2^n}$.

- For any i, then

$$\left[a_{\frac{i}{2^n}}, b_{\frac{i+1}{2^n}}\right] \times_R \left[a_{\frac{i-1}{2^n}}, b_{\frac{i}{2^n}}\right] = \{(a_{\frac{i}{2^n}}, b_{\frac{i}{2^n}})\}$$

- For any i, then

$$\partial_R \left[a_{\frac{i}{2^n}}, b_{\frac{i}{2^n}}\right]^2 = \Delta\{a_{\frac{i}{2^n}}, b_{\frac{i}{2^n}}\}$$

continue the process above to $\left[a_{\frac{i}{2^n}}, b_{\frac{i+1}{2^n}}\right]$, to define d_*, $(a_*, b_*) \in R$, where $* = \frac{1}{2}\left(\frac{i}{2^n} + \frac{i+1}{2^n}\right) = \frac{2i+1}{2^{n+1}}$. One see that d_*, $(a_*, b_*) \in R$ satisfy the above three conditions after replacing n by $n+1$.

Finally, let $F = \left\{\frac{\ell}{2^n} : n \geq 1, 0 \leq \ell \leq 2^n\right\}$, we have construct

$$\{(a_t, b_t) : t \in F\}$$

Let

$$C = I \setminus \bigcup_{t \in F} (b_t, a_t)$$

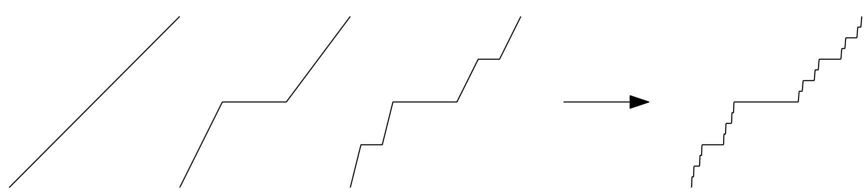

Figure 4: Cantor-like function

It is a Cantor-like set, one can construct Cantor-like function $f : [0,1] \to [0,1]$ satisfy the following property

- f is monotone increasing.

- for any $t \in F$ and $a_t \leq x \leq b_t$, $f(a_t) = f(x) = f(b_t)$.

See for example, Stein [3] Page 125. Note that $x, y \in C$ satisfy

$$x\,R\,y \iff \exists t \in F, \text{ such that } x = b_t, y = a_t \text{ or } x = a_t, y = b_t$$

By restring f over C, one get a bijection $\hat{f} : C/R \to [0,1]$. Note that the restriction of R on C is $(C \times C) \cap R$ which is closed also, so C/R is Hausdorff, thus $\hat{f}$ is homeomorphism. Finally, note that the natural map

$$C/R \to [0,1]/R$$

is continuous and injective. So we finally construct an injective map $[0,1] \to [0,1]/R$. $\square$

References

[1] Brian M. Scott (https://math.stackexchange.com/users/12042/brian-m s-cott). Every path has a simple subpath;. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/155572 (version: 2015-09-25).

[2] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[3] Elias M. Stein and Rami Shakarchi. *Real analysis*, volume 3 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces.

[4] Hagen von Eitzen (https://math.stackexchange.com/users/39174/hagen-von eitzen). Does path-connected imply simple path-connected? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1522642 (version: 2015-12-16).

[5] Stephen Willard. *General topology*. Dover Publications, Inc., Mineola, NY, 2004. Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581].