Algebraic $\mathcal{D}$-modules

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Generalities

Let $\Bbbk$ be an algebraic closed field of characteristic zero. Let X be a smooth algebraic variety.

Denote $\mathcal{O}_X$ the sheaf of regular functions. Denote $\mathcal{M}_{qc}(X)$ (resp. $\mathcal{M}_q(X)$) the category of quasi-coherent (resp. coherent) sheaves.

Differential Operators. We say $\varphi : \mathcal{O}(U) \to \mathcal{O}(U)$ is a **differential operator** over U of **degree** 0 if φ is a multiplication of some $f \in \mathcal{O}(U)$; of **degree** k if $[\varphi, \psi]$ is a differential operator of degree $k-1$ for all differential operator of degree 0. Denote $\mathcal{D}(X)$ the differential operator over X. Denote $\mathcal{D}_X$ be the sheaf of differential operators.

As an algebra, $\mathcal{D}_X$ is generated by the sheaf of vector fields Θ_X over $\mathcal{O}_X$ with relation $f \cdot \xi = f\xi$, $\xi \cdot f - f \cdot \xi = \xi f$ and $\xi \cdot \eta - \eta \cdot \xi = [\xi, \eta]$ for any $\xi, \eta \in \Theta_X$ and $f \in \mathcal{O}_X$.

Coherent $\mathcal{D}$-modules. A sheaf is said to be a **quasi-coherent** (resp. **coherent**) $\mathcal{D}_X$-module if it is a sheaf of $\mathcal{D}_X$-module (resp. finitely generated). Denote $\mathcal{M}_{qc}(\mathcal{D}_X)$ (resp. $\mathcal{M}_q(X)$) the category of the category of quasi-coherent (resp. coherent) $\mathcal{D}_X$-modules. When we say $\mathcal{D}_X$-module, it is automatically quasi-coherent.

Denote the sheaf correspondent to tangent bundle by Θ_X. Note that $\mathcal{D}_X$ is naturally filtered by degrees, with associated graded ring commutative and isomorphic to $S^*(\Theta_X)$ over $\mathcal{O}_X$ as $\mathcal{O}_X$-module. As a result, $\mathcal{D}_X$ is noetherian.

Denote $\mathcal{D}_X^{\mathrm{op}}$ the inverse ring of $\mathcal{D}_X$. We identify the sheaf of right $\mathcal{D}_X$-modules as sheaf of $\mathcal{D}_X^{\mathrm{op}}$-modules, and call $\mathcal{D}_X$-modules to be left $\mathcal{D}_X$-modules if it is necessary to dintinguish. We define similarly $\mathcal{M}_{qc}(\mathcal{D}_X^{\mathrm{op}})$, $\mathcal{M}_c(\mathcal{D}_X^{\mathrm{op}})$, $\mathcal{D}_{qc}(\mathcal{D}_X^{\mathrm{op}})$, and $\mathcal{D}_c(\mathcal{D}_X^{\mathrm{op}})$.

Left and Right $\mathcal{D}$-modules. The sheaf $\mathcal{O}_X$ of regular functions is naturally a (left) $\mathcal{D}$-module.

The sheaf Ω_X of highest differential forms over X is a right $\mathcal{D}$-module locally by $\Omega \cdot \xi = -L_\xi \omega$ for ξ a vector field, where L_ξ is the Lie derivative. In general, for any left (quasi-)coherent $\mathcal{D}_X$-module $\mathcal{F}$, we can assign a right (quasi-)coherent $\mathcal{D}_X$-module $\Omega(\mathcal{F}) = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{F}$ locally by

$$(\omega \otimes s) \cdot \xi = \omega \cdot \xi \otimes s - \omega \otimes \xi \cdot s = -L_\xi \omega \otimes s - \omega \otimes \xi \cdot s.$$

This is an equivalence of $\mathcal{D}_x(\mathcal{D}_X)$ and $\mathcal{D}_x(\mathcal{D}_X^{\mathrm{op}})$ for $x \in \{c, qc\}$. This follows from the fact $\mathcal{D}_X^{\mathrm{op}} \cong \Omega \otimes \mathcal{D}_X \otimes \Omega^{-1}$ as sheaf of algebras. Locally, the isomorphism is given by

$$\sum_{\mathbf{j}} f_{\mathbf{j}}(x_i) g_{\mathbf{j}}\big(\tfrac{\partial}{\partial x_i}\big) \mapsto dx \otimes \sum_{\mathbf{j}} g_{\mathbf{j}}\big(-\tfrac{\partial}{\partial x_i}\big) f_{\mathbf{j}}(x_i) \otimes dx^{-1}$$

where dx any differential form $dx_1 \wedge \cdots \wedge dx_n$ and dx^{-1} the dual basis for dx.

Connections. Denote the sheaf of k-forms by Ω_X^k.

For a quasi-coherent sheaf $\mathcal{F} \in \mathcal{M}_{qc}(X)$, a **connection** over $\mathcal{F}$ is a morphism of $\Bbbk$-module $\nabla : \mathcal{F} \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}$ with $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$ for all $f \in \mathcal{O}_X$ and $s \in \mathcal{F}$. It can be naturally extended to $\nabla : \Omega_X^{k-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{F}$ with $\nabla(\omega \wedge s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes \nabla s$ for $\omega \in \Omega_X^i$, and $s \in \mathcal{F}$. Define its curvature to be ∇^2, it can be shown that it is an element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega_X^2 \otimes \mathcal{F})$, say, a tensor. A connection is said to be **flat** if the **curvature** ∇^2 vanishes. In this case, the **de Rham complex**,

$$\mathrm{dR}^\bullet(\mathcal{F}) : 0 \to \mathcal{F} \to \Omega^1 \otimes \mathcal{F} \to \cdots \to \Omega \otimes \mathcal{F} \to 0$$

is a complex.

For a vector field $\xi \in \Theta_X$, denote $\nabla_\xi : \mathcal{F} \to \mathcal{F}$ defined by $\nabla_\xi(s) = (\nabla s)(1 \otimes \xi)$. Then $\nabla_{f\xi} s = f \nabla_\xi s$ and $\nabla_\xi(fs) = (\xi \cdot f)s + f \cdot \nabla_\xi s$ for all $f \in \mathcal{O}_X$, $\xi \in \Theta_X$ and $s \in \mathcal{F}$ by definition. The curvature can be computed to be $\nabla^2 s(\xi, \eta) = \nabla_{[\xi,\eta]} s - [\nabla_\xi, \nabla_\eta] s$ for $s \in \mathcal{F}$, $\xi, \eta \in \Theta_X$. So ∇ is **flat** if and only if $\nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta]$ in $\mathcal{H}\mathrm{om}_{\Bbbk}(\mathcal{F}, \mathcal{F})$.

Assume ∇ is a flat connection over $\mathcal{F}$, then $\mathcal{F}$ is a left $\mathcal{D}_X$-module defined by $\xi \cdot s = \nabla_\xi s$. Conversely, any $\mathcal{D}_X$-module is induced by a flat connection. In particular, assume a left $\mathcal{D}_X$-module $\mathcal{F}$ is coherent over $\mathcal{O}_X$, then it is a locally free sheaf (vector bundle).

Dually, one can define the dual notation exchanging the corresponding property by $\nabla_{f\xi} s = \nabla(fs)$, but it is seldom used.

Let $\mathcal{F}, \mathcal{G} \in \mathcal{M}_{qc}(X)$ with connections both denoted by ∇. Then we can define a connection over $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ by $\nabla(s \otimes t) = \nabla s \otimes t + s \otimes \nabla t$ where $s \in \mathcal{F}$ and $t \in \mathcal{G}$. Similarly, we can define a connection over $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by $\nabla_\xi(\phi)(s) = -\phi(\nabla_\xi s) + \nabla_\xi(\phi(s))$ where $\xi \in \Theta$, $s \in \mathcal{F}$ and $\phi \in \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Restricting to flat connections, this construction extends to left $\mathcal{D}_X$-modules. That is, for left $\mathcal{D}_X$-module $\mathcal{F}$ and $\mathcal{G}$, define a left $\mathcal{D}_X$-module structure over $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ by $\xi \cdot (s \otimes t) = \xi \cdot s \otimes t + s \otimes \xi \cdot t$; a left $\mathcal{D}_X$-module structure over $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by $\big(\xi \cdot (\phi)\big)(s) = -\phi(\xi \cdot s) + \xi \cdot \phi(s)$. All notations the same as above.

Parallel Translation. Let $\mathcal{F}$ be a coherent sheaf with connection ∇. We say a section $s \in \mathcal{F}$ is **horizontal** or **parallel** if for any $\xi \in \Theta$, $\nabla_\xi s = 0$. Denote the set of horizontal section over Y to be $\mathcal{F}^\nabla(U)$. Note that the sheaf $U \mapsto \mathcal{F}^\nabla(U)$ is a locally constant sheaf. A fortiori, $\mathcal{F}^\nabla = \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F})$.

If we remove the assumption of being algebraic, consider the case of smooth or analytic, and $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. For any curve C, if there is any coherent sheaf $\mathcal{F}$ with connection, it is necessary flat. For any $x \in C$, denote $\mathfrak{m}_x$ the only maximal ideal of $\mathcal{O}_x$. By the existence and uniqueness of ODEs, the map $\mathcal{F}_x^\nabla \to \mathcal{F}_x \to \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ is an isomorphism.

In general, by pulling back of $\mathcal{F}$ along curves, we define the **parallel translation** along paths. Then, the condition of being parallel is equivalent to say the section is translation-invariant; the condition of being flat for connection is to say it does not depends on the choice of homotopy class of paths.

Linear PDEs. Let $\mathcal{F}$ be a coherent $\mathcal{D}_X$-module. We can take a resolution by

$$\mathcal{D}_X^n \xrightarrow{\ *\ } \mathcal{D}_X^m \longrightarrow \mathcal{F} \longrightarrow 0,$$

Assume the first map $*$ is given by $e_j \mapsto \sum_{j=1}^n D_{ij} e_i$, with e_i the standard basis for some $D_{ij} \in \mathcal{D}(X)$. Consider the linear partial differential equations $\sum_{j=1}^n D_{ij} f_i = 0$ for $i = 1, \ldots, m$ over U. Denote $\mathrm{Sol}(U)$ the set of solutions of them. Then $\mathrm{Sol}(U)$ is clearly forms a sheaf. One can check directly that $\mathrm{Hom}_{\mathcal{D}(U)}(\mathcal{F}(U), \mathcal{O}(U)) = \mathrm{Sol}(U)$. In other word, $\mathrm{Sol} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \mathcal{O}_X)$.

If we remove the assumption of being algebraic, consider the case of smooth in which case $\Bbbk = \mathbb{R}$. Consider the sheaf of distributions $\mathcal{D}\mathrm{ist}$ over X, then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{D}\mathrm{ist})$ is just the local weak solution in classic analysis.

Direct Images and Inverse Images

Let $f : X \to Y$ be a morphism of algebraic varieties. For a quasi-coherent sheaf $\mathcal{F}$ over X, we define **direct image/push forward** $f_*\mathcal{F} = [U \mapsto \mathcal{F}(f^{-1}(U))]$. For a quasi-coherent sheaf $\mathcal{G}$ over Y, we define **inverse image/pull back** $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$, where $f^{-1}\mathcal{G}$ the sheaf associated to $U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$.

Inverse Image. Assume there is a connection ∇ over $\mathcal{G}$, then we can define a connection over $f^*\mathcal{G}$ by $\nabla(f \otimes s) = df \otimes s + f\nabla s$ with $f \in \mathcal{O}_X$, $s \in f^{-1}\mathcal{G}$ which is presented by a section $s \in \mathcal{G}$. This is known as **inverse image / pull back** of the connection. In other word, $\nabla_\xi(f \otimes s) = \xi f \otimes s + f\nabla_{df(\xi)}s$. Under local coordinate $\{y_i\}$ of Y, $\nabla_\xi(f \otimes s) = \xi f \otimes s + f\sum_i(\xi y_i) \otimes \frac{\partial}{\partial y_i} \cdot s$. The curvature commutes with pull back, so the pull back of flat connection is still flat.

Restrict the case of flat connections, we defined for each left $\mathcal{D}_Y$-module $\mathcal{G}$ a **$\mathcal{D}$-module inverse image** $f^\Delta\mathcal{G}$. As sheaf, it coincides with inverse image of $\mathcal{G}$, and the $\mathcal{D}_X$-module structure over it is given by $\xi(f \otimes s) = \xi f \otimes s + f\sum(\xi y_i) \otimes \frac{\partial}{\partial y_i} \cdot s$, with notations above. It is clear $(f \circ g)^\Delta = g^* \circ f^\Delta$. Denote $\mathcal{D}_{X\to Y} = f^\Delta \mathcal{D}_X$. Then a fortiori, $f^\Delta\mathcal{G} = \mathcal{D}_{Y\to X} \otimes_{f^{-1}(\mathcal{D}_X)} f^{-1}\mathcal{G}$.

Direct Image. Let $\mathcal{F}$ be a right $\mathcal{D}_X$-module, we define the **direct image** of it to be $f_+\mathcal{F} = f_*(\mathcal{F} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$. Then we can transfer them into left $\mathcal{D}_X$-module. To be exact, define $\mathcal{D}_{Y\leftarrow X} = \Omega(f^*\Omega(\mathcal{D}_X))$, and for left $\mathcal{D}_X$-module $\mathcal{F}$, define the $\mathcal{D}$-module **direct image** $f_+\mathcal{F} = f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{F})$. When $X \to Y$ is a closed embedding, then $\mathcal{D}_{Y\leftarrow X}$ is generated by $\mathcal{D}_Y$ and the sheaf of normal vector fields $\Theta_Y|_X$. In particular, when f is an open embedding, $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X$, so $f_+\mathcal{F}$ is $f_*\mathcal{F}$ as $\mathcal{O}_X$-module. However, in general $(f \circ g)_+ \neq f_+ \circ g_+$.

Derived Categories. Denote the derived category of bounded complexes of quasi-coherent sheaves by $\mathcal{D}_{qc}(X)$. Denote the full subcategory in $\mathcal{D}_{qc}(X)$ consisting of complexes with cohomology coherent by $\mathcal{D}_c(X)$. It is also the derived category of bounded complexes of coherent sheaves.

Denote the derived category of bounded complexes of quasi-coherent $\mathcal{D}_X$-modules by $\mathcal{D}_{qc}(\mathcal{D}_X)$. Denote the full subcategory in $\mathcal{D}_{qc}(\mathcal{D}_X)$ consisting of complexes with cohomology coherent by $\mathcal{D}_c(\mathcal{D}_X)$. It is also the derived category of bounded complexes of coherent $\mathcal{D}_X$-modules. The category of quasi-coherent sheaf or quasi-coherent $\mathcal{D}_X$-module have finite projective dimensions, so the derived functor can be defined.

By an algebra argument, for $\mathcal{F}^\bullet \in \mathcal{D}_c(\mathcal{D}_X)$ and $\mathcal{G}^\bullet \in \mathcal{D}_{qc}(\mathcal{D}_X)$, we have

$$R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{F}^\bullet, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} \mathcal{G}^\bullet.$$

Functors. For $\mathcal{G}^\bullet \in \mathcal{D}_{qc}(Y)$, define the **derived inverse image** by

$$f^!\mathcal{G}^\bullet = Lf^\Delta\mathcal{G}^\bullet[d] = \mathcal{D}_{X\leftarrow Y} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{G}[d] \in \mathcal{D}_{qc}(X)$$

where $d = \dim X - \dim Y$. Then $(f \circ g)^! = g^! \circ f^!$.

For $\mathcal{F}^\bullet \in \mathcal{D}_{qc}(X)$, define the **derived inverse image** by

$$f_*\mathcal{F}^\bullet = Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{F}^\bullet) \in \mathcal{D}_{qc}(Y).$$

Then $(f \circ g)_* = f_* \circ g_*$.

Define the **duality functor** $\mathcal{D}_{qc}(\mathcal{D}_X) \to \mathcal{D}_{qc}(\mathcal{D}_X)^{op}$ by

$$\begin{aligned}\mathbf{D}(\mathcal{F}^\bullet) &= R\mathcal{H}\mathrm{om}^\bullet_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[n] \\ &= R\mathcal{H}\mathrm{om}^\bullet_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{-1})[n],\end{aligned}$$

where $n = \dim X$. Then $\mathbf{D}^2(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet$. Thus we can define $f^* = \mathbf{D}f^!\mathbf{D}$ and $f_! = \mathbf{D}f_*\mathbf{D}$.

We have

$$\begin{aligned}R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) &= \left(\Omega_X \otimes^L_{\mathcal{O}_X} \mathbf{D}\mathcal{F}^\bullet\right) \otimes^L_{\mathcal{D}_X} \mathcal{G}^\bullet[-\dim X] \\ &= \Omega_X \otimes^L_{\mathcal{D}_X} \left(\mathbf{D}\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{G}^\bullet\right)[-\dim X] \\ &= R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{O}_X, \mathbf{D}\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}).\end{aligned}$$

The **Kashiwara theorem** claims that for closed embedding $i : X \to Y$, $i^\Delta : \mathcal{M}(\mathcal{D}_X) \to \mathcal{M}_X(\mathcal{D}_Y)$ is an equivalence of category with $\mathcal{M}_X(\mathcal{D}_Y)$ the category of $\mathcal{D}_Y$-module supported over X. The inverse is the restriction of $i^!$ by view each $\mathcal{D}_X$-module as a complex centralized at zero degree.

Spencer Resolution. We can pick a right $\mathcal{D}_X$-module resolution

$$\cdots \to \Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X (\to \Omega_X) \to 0$$

by $d(\omega \otimes P) = d\omega \otimes P + \sum_i dx_i \wedge \omega \otimes \frac{\partial}{\partial x_i}P$, where $\omega \in \Omega_X^k$, $P \in \mathcal{D}_X$ and $\{x_i\}$ a local coordinate. Dualize it to a left $\mathcal{D}_X$-module, we see a left $\mathcal{D}_X$-module resolution of $\mathcal{O}_X$

$$\cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^1\Theta_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^0\Theta_X (\to \mathcal{O}_X) \to 0.$$

Explicitly, it is given by

$$\begin{aligned}&d(P \otimes \xi_1 \wedge \cdots \wedge \xi_k) \\ &\quad = \textstyle\sum_{i=1}^k (-1)^{i+1} P\xi_i \otimes \xi_1 \wedge \cdots \widehat{\xi_i} \cdots \wedge \xi_k \\ &\qquad + \textstyle\sum_{1 \le i < j \le k} (-1)^{i+j} P \otimes [\xi_i, \xi_j] \otimes \xi_1 \wedge \cdots \wedge \xi_k.\end{aligned}$$

So $R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F}^\bullet) = \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{F}^\bullet[-\dim X]$.

Holonomic $\mathcal{D}$-modules

Consider the cotangent bundle T^*X, with projection $\pi : T^*X \to X$. For a local coordinate $\{x_i\}$, the local vector field $\frac{\partial}{\partial x_i}$ defines local function over T^*X, and it will be denoted alternatively by ξ_i. It is clear $\{\xi_i, x_i\}$ form a local coordinate of T^*X.

Sympletic Structure. Over T^*X, there is a **tautological form** $\lambda_X \in \Omega^1(T^*X)$, such that for any form $\alpha \in \Omega^1(X)$, viewing as a map $X \to T^*X$, the pull back $\alpha^*(\lambda)$ is α itself. This uniquely determines λ_X, since for any point $x \in X$, and $y \in T^*_xX$, the union of images of T_xX under $d\alpha$ for all $\alpha(x) = y$ is whole $T_y(T^*X)$. Define $\omega_X = d\lambda_X \in \Omega^2(T^*X)$, and call it the **standard sympletic form**. Locally, under coordinate $\{x_i\}$, it is given by $\lambda_X = \sum \xi_i dx_i$, thus $\omega_X = \sum d\xi_i \wedge dx_i$.

For smooth map $f : X \to Y$, the map $df : T_xX \to X_{f(x)}Y$ for each point x induces two maps $T^*Y \xleftarrow{p} X \times_Y T^*Y \xrightarrow{q} T^*X$. Then two pull back of tautological forms coincide, say $q^*\lambda_X = p^*\lambda_Y$. This is functorial in the following sense.

$$\begin{array}{ccccc} X \times_Z T^*Z & \longrightarrow & X \times_Y T^*Y & \longrightarrow & T^*X \\ \downarrow & & \downarrow & & \\ Y \times_Z T^*Z & \longrightarrow & T^*Y & & \\ \downarrow & & & & \\ T^*Z & & & & \end{array}$$

Symbols. It is known that the associated graded algebra of $\mathcal{D}_X$ is $S^*\Theta = \pi_*(\mathcal{O}_{T^*X})$. For a differential operator $P \in \mathcal{D}(X)$, define its **symbol** $\sigma(P)$ to be the image in $\mathcal{O}(T^*X)$. Locally, under coordinate $\{x_i\}$, it is given by

$$\sum_{\mathbf{j}} f_{\mathbf{j}}(x_i) g_{\mathbf{j}}\big(\tfrac{\partial}{\partial x_i}\big) \longmapsto \sum_{\mathbf{j}} f_{\mathbf{j}}(x_i) g_{\mathbf{j}}\big(\xi_i\big),$$

where $\xi_i : T^*U \to \Bbbk$ sending df to $\frac{\partial f}{\partial x_i}$ as a member of coordinate.

More general, let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles over X. A sheaf map $\mathcal{E} \to \mathcal{F}$ is said to be a **differential operator** if it can be expressed by a matrix with coefficients in $\mathcal{D}_X$ locally after some trivialization. It is in general not a $\mathcal{O}_X$-module morphism (morphism of vector bundles). The associated graded sheaf of all differential operators from $\mathcal{E} \to \mathcal{F}$ is $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_X} S^*(\Theta) = \pi_* \operatorname{Hom}_{\mathcal{O}_{T^*X}}(\pi^*\mathcal{E}, \pi^*\mathcal{F})$. For a differential operator $P : \mathcal{E} \to \mathcal{F}$ over X, we can define the its **symbol** to be the corresponding $\mathcal{O}_{T^*X}$-morphism $\pi^*\mathcal{E} \to \pi^*\mathcal{F}$. Locally, it is simply the indices-wise symbol.

Singular Support. We denote $\mathcal{O}^i_X$ the differential operators of degree $\le i$. For a quasi-coherent $\mathcal{D}_X$-module $\mathcal{F}$, we say it has a **good filtration** if there is a filtration $\mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots$ such that $\mathcal{D}^i_X \cdot \mathcal{F}^j \subseteq \mathcal{F}^{i+j}$, and $\mathcal{F}^i / \mathcal{F}^{i-1}$ is naturally a coherent $\mathcal{O}^i_X \,/\, \mathcal{O}^{i-1}_X$-module. A $\mathcal{D}_X$-module has a good filtration if and only if it is coherent $\mathcal{D}_X$-module.

Consider the associated graded module of is a coherent sheaf over T^*X with respect to some good filtration, and we call the support of this sheaf **singular support** and denote it by $\mathrm{SS}(\mathcal{F}) \subseteq T^*X$. It turns out that singular support does not depend on the choice of good filtration. We define the **defect** of a coherent $\mathcal{D}_X$-module $\mathcal{F}$ to be $\operatorname{def}\mathcal{F} = \dim \mathrm{SS}(\mathcal{F}) - \dim X$. One can prove that $\dim \mathrm{SS}(\mathcal{F}) \ge \dim X$ (each irreducible component), and $\mathrm{SS}(\mathcal{F})$ is actually a union of coisotropic subvarieties. Furthermore, $\mathcal{F}$ is coherent $\mathcal{O}_X$-module if and only if $\mathrm{SS}(\mathcal{F})$ is simply the zero section of T^*X.

Roos theorem ensures that for coherent $\mathcal{D}_X$-module $\mathcal{F}$, $H^i(\mathbf{D}\mathcal{F}) = 0$ unless $-\operatorname{def}\mathcal{F} \le i \le 0$, and $\dim \mathrm{SS}(H^i(\mathbf{D}\mathcal{F})) \le \dim X - i$.

Holonomic $\mathcal{D}$-modules. A coherent $\mathcal{D}_X$-module $\mathcal{F}$ is called **holonomic** if $\mathcal{F} = 0$ or $\dim \mathrm{SS}(\mathcal{F}) = \dim X$. An algebraic argument ensures that holonomic $\mathcal{D}_X$-module is stable under subquotient, extensions, and artinian. It is clear from above that the duality functor $\mathbf{D}$ maps holonomic $\mathcal{D}_X$-module to holonomic $\mathcal{D}_X$-module (rather than a complex).

Let $\mathcal{D}_h(\mathcal{D}_X)$ be the full subcategory of complexes with cohomology homonomic in $\mathcal{D}_{qc}(\mathcal{D}_X)$. It is nontrivial but true that it is also the category of bounded complexes of holonomic $\mathcal{D}_X$-module.

It turns out $f^!$, f_* preserve holonomicity, thus so are $f_!$ and f^*. Under the category of $\mathcal{D}_h(\mathcal{D}_X)$, and a morphism $f : X \to Y$, we have

1. There is a canonic morphism $f_! \to f_*$, which is isomorphic for proper map f.
2. If $f : X \to Y$ is smooth, then $f^! = f^*[2d]$ with $d = \dim X - \dim Y$.
3. $\operatorname{Hom}_{\mathcal{D}_h}(f_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \operatorname{Hom}_{\mathcal{D}_h}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$.
4. $\operatorname{Hom}_{\mathcal{D}_h}(\mathcal{G}^\bullet, f_*\mathcal{F}^\bullet) = \operatorname{Hom}_{\mathcal{D}_h}(f^*\mathcal{G}^\bullet, \mathcal{F}^\bullet)$.

By an induction on dimensions, one can show that for a holonomic sheaf $\mathcal{F}$, there is some dense open subset U such that $\mathcal{F}|_U$ is $\mathcal{O}_U$-coherent.

Minimal Extensions. Let Y be a locally closed smooth subvariety of X. Denote $i : Y \to X$ the inclusion. Assume i is affine, that is preimage of affine set is affine, then $\mathcal{D}_{X\leftarrow Y}$ is locally free over $\mathcal{D}_Y$, thus i_* is exact. For any holonomic sheaf $\mathcal{F}$, $i_*\mathcal{F}$ is still holonomic, and so is $i_!\mathcal{F}$. Define the **minimal extension** $\mathcal{L}(Y, \mathcal{F})$ to be the image of $i_!\mathcal{F} \to i_*\mathcal{F}$.

When $\mathcal{F}$ is an irreducible $\mathcal{O}_Y$-coherent $\mathcal{D}_X$-module, $\mathcal{L}(Y, \mathcal{F})$ is an irreducible $\mathcal{D}_X$-module. It is the unique irreducible submodule of $i_*\mathcal{F}$ and the unique irreducible quotient module of $i_!\mathcal{F}$. Furthermore, any irreducible holonomic module is of this form $\mathcal{L}(Y, \mathcal{F})$. Two of them $\mathcal{L}(Y_1, \mathcal{F}_1) = \mathcal{L}(Y_2, \mathcal{F}_2)$ if and only if $\overline{Y_1} = \overline{Y_2}$ and $\mathcal{F}_1 = \mathcal{F}_2$ after restricting to some open subset U both in Y and Y'.

Curves. Let C be a smooth curve. For a point $p \in C$, denote $\mathfrak{m}_p$ the unique ideal of $\mathcal{O}_p$, and $\mathcal{K}_p$ the fraction field of $\mathcal{O}_p$. There is a unique completion $\overline{C}$ containing C as an open dense subset. For any $p \in \overline{C} \backslash C$, $(j_*\, \mathcal{O}_C)_p = \mathcal{K}_p$ where $j : C \to \overline{C}$ the inclusion.

Let M be a finite dimensional $\mathcal{K}_p$-module. A **meromorphic connection** is a $\Bbbk$-linear map $\nabla : M \to \Omega^1_C \otimes_{\mathcal{O}_p} M$ with $\nabla(fs) = df \otimes s + f\nabla s$ for all $f \in \mathcal{K}_p$ and $s \in M$. It is called **regular** if there is an $\mathfrak{m}_p\nabla$-invariant $\mathcal{O}_p$-lattice L. That is, there is an $\mathcal{O}_p$-finitely generated submodule L such that $M = \mathcal{K}_p L$ and $\mathfrak{m}_p\nabla(L) \subseteq \Omega^1_C \otimes_{\mathcal{O}_p} L$.

Let $\mathcal{F}$ be an $\mathcal{O}_C$-coherent $\mathcal{D}_C$-module (that is, a flat connection). For any $p \in \overline{C} \setminus C$, we say $\mathcal{F}$ has a **regular singularity** at p if $(j_+\mathcal{F})_p$ is a regular meromorphic connection We say $\mathcal{F}$ is **regular**, if it has regular singularity at each point $p \in \overline{C} \setminus C$.

Under local coordinate z with $z(p) = 0$, $\mathcal{O}_p = \Bbbk[x, x^{-1}]$, and $\mathcal{K}_p = \Bbbk(k)$. We can recognize $\Omega^1_p = \mathcal{O}_p$, under which df is recognized with $\frac{\mathrm{d}f}{\mathrm{d}z}$. So the condition of being $\mathcal{O}_p$-coherent and $\mathfrak{m}_p\nabla$-invariant is equivalent to that of being $\mathcal{D}^\nu_p$-invariant, where $\mathcal{D}^\nu_p$ is the subsheaf of subalgebra of $\mathcal{D}_{\overline{C}}$ generated by $\mathcal{O}_{\overline{C}}$ and $z\frac{\mathrm{d}}{\mathrm{d}z}$. So $\mathcal{F}$ has a **regular singularity** at p if $(j_+\mathcal{F})_p$ is a union of $\mathcal{D}^\nu$-coherent modules.

For a holonomic $\mathcal{D}_C$-module $\mathcal{F}$, it is said to be **regular** if $\mathcal{F}|_U$ is $\mathcal{O}_U$-coherent and regular in the above sense.

Regular $\mathcal{D}$-modules. In general case, a holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is called **regular** if the restriction of it to any curve is regular. The **curve criterion** asserts that a holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is regular if any irreducible subquotient of it is of the form $\mathcal{L}(Y, \mathcal{E})$ with $\mathcal{E}$ a $\mathcal{O}_Y$-coherent and regular $\mathcal{D}_Y$-module. We denote $\mathcal{D}_{rh}(\mathcal{D}_X)$ the full subcategory of complexes with cohomology regular in $\mathcal{D}_h(\mathcal{D}_X)$. It turns out $f^!$, f_* and $\mathbf{D}$ preserve regularity, thus so are $f_!$ and f^*.

Examples. Consider the case $X = \mathbb{C}$. Let z be a local coordinate, and ζ the coordinate for cotangent bundle corresponding to $\frac{\mathrm{d}}{\mathrm{d}z}$. The $\mathcal{D}_X$-module $\mathcal{F} = \mathcal{D}_X \,/\, \mathcal{D}_X \cdot z\frac{\mathrm{d}}{\mathrm{d}z}$ has good filtration such that the corresponding $\mathcal{O}_{T^*X}$-module is $\mathcal{O} \,/\, \mathcal{O} \cdot z\zeta$. As a result, $\mathrm{SS}(\mathcal{F})$ is the union of zero section of $T^*\mathbb{C}$ and the fibre of $T^*\mathbb{C}$ at $0 \in \mathbb{C}$. So $\mathcal{F}$ is holonomic.

On the other hand, in the analytic case, the solution $z\frac{\mathrm{d}}{\mathrm{d}z}f = 0$ is given by $f = c \log z$ in any simple-connected subspace in $\{z \ne 0\}$.

Perverse Sheaves

In this section, we assume $\Bbbk = \mathbb{C}$, and we denote X^{an} the underlying space of algebraic variety X equipped the complex topology. We exchange $\mathcal{O}_X$ by the sheaf of analytic functions, $\mathcal{D}_X$ the sheaf of analytic differential operators, etc.

Let X be a topological space temporarily. Denote $\underline{\mathbb{Z}}_X$ the constant sheaf over X, and $\mathcal{M}(\underline{\mathbb{Z}}_X)$ the category of all sheaves over X^{an}. Denote the derived category of bounded complexes in $\mathcal{M}(\underline{\mathbb{Z}}_X)$ by $\mathcal{D}(\underline{\mathbb{Z}}_X)$. For any continuous map $f : X \to Y$, we can define functors $f_!, f_* : \mathcal{D}(X^{\text{an}}) \to \mathcal{D}(Y^{\text{an}})$ and $f^*, f^! : \mathcal{D}(Y^{\text{an}}) \to \mathcal{D}(X^{\text{an}})$.

1. There is a canonic morphism $f_! \to f_*$, which is isomorphic for proper map f.
2. If $f : X \to Y$ is smooth, then $f^! = f^*[2d]$ with $d = \dim X - \dim Y$.
3. $\text{Hom}_Y(f_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \text{Hom}_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$.
4. $\text{Hom}_Y(\mathcal{G}^\bullet, f_*\mathcal{F}^\bullet) = \text{Hom}_X(f^*\mathcal{G}^\bullet, \mathcal{F}^\bullet)$.

Moreover, there is a **Verdier duality functor D** : $\mathcal{D}(X^{\text{an}}) \to \mathcal{D}(X^{\text{an}})^{op}$.

Constructible Sheaves. Let $\underline{\mathbb{C}}_X$ be the constant sheaf over X^{an}. We denote $\mathcal{M}(\underline{\mathbb{C}}_X)$ the sheaf of $\underline{\mathbb{C}}_X$-modules, i.e. the category of sheaves of $\mathbb{C}$-vector spaces over X^{an}. Denote the derived category of bounded complexes in $\mathcal{M}(\underline{\mathbb{C}}_X)$ by $\mathcal{D}(X^{\text{an}})$.

We call a sheaf $\mathcal{F} \in \mathcal{M}(\underline{\mathbb{C}}_X)$ **constructible** if there is some stratification $X = \bigcup X_i$ with each X_i locally closed algebraic subvarieties such that $\mathcal{F}|_{X_i^{\text{an}}}$ is finite dimensional and locally constant. Denote the full subcategory of complexes with cohomology group constructible in $\mathcal{D}(X^{\text{an}})$ by $\mathcal{D}_{con}(X^{\text{an}})$. It turns out $f_!, f_*, f^!, f^*, \mathbf{D}$ preserve the constructibility. Furthermore $\mathbf{D}^2(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet$ and $f^* = \mathbf{D}f^!\mathbf{D}$ and $f_! = \mathbf{D}f_*\mathbf{D}$.

A complex $\mathcal{F}^\bullet \in \mathcal{D}(X^{\text{an}})$ is called **perverse sheaf** if $\dim \text{supp}\, H^i(\mathcal{F}) \leq -i$, and $\dim \text{supp}\, H^i(\mathcal{F}) \leq -i$. The full subcategory of perverse sheaves forms an abelian category.

For a smooth locally closed subvariety $Y \subseteq X$, and a local system $\mathcal{E}$ (a locally constant sheaf), the intersection homology complex of Deligne–Goresky–MacPherson, $\mathbf{IC}^\bullet(Y, \mathcal{L})$ is a perverse sheaf with $H^{-\dim Y}(\mathbf{IC}^\bullet(Y, \mathcal{E}))|_Y = \mathcal{E}$ and $H^i(\mathbf{IC}^\bullet(Y, \mathcal{E})) = 0$ if $i < -\dim Y$.

Riemann–Hilbert Correspondence. Denote the **de Rham functor** $\text{dR} : \mathcal{D}(\mathcal{D}_X^{\text{an}}) \to \mathcal{D}(X^{\text{an}})$ by $\text{dR}(\mathcal{F}^\bullet) = \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{F}^\bullet$. It coincides $R\mathcal{H}\text{om}(\mathcal{O}_X, \mathcal{F}^\bullet)[\dim X]$, the derived version of horizontal sections. Also denote $\text{Sol} : \mathcal{D}(\mathcal{D}_X^{\text{an}})^{op} \to \mathcal{D}(X^{\text{an}})$ by $\text{Sol}(\mathcal{F}^\bullet) = R\mathcal{H}\text{om}_{\mathcal{D}_X^{\text{an}}}(\mathcal{F}^\bullet, \mathcal{O}_X^{\text{an}})$. Note that $\text{Sol}(\mathcal{F}^\bullet) \cong \text{dR}(\mathbf{D}\mathcal{F}^\bullet)[-\dim X]$.

The **Riemann–Hilbert correspondence** claims that

$$\text{dR} : \mathcal{D}_h(\mathcal{D}_X) \to \mathcal{D}_{cos}(X^{\text{an}})$$

is an equivalence of categories commuting with $f^!, f^*, f_!, f_*$ and $\mathbf{D}$.

Furthermore, if we recognize a sheaf by a complex centralized at zero position, then regular holomonic sheaves correspond to perverse sheaves; the irreducible regular holomonic $\mathcal{D}_X$-module $\mathcal{L}(Y, \mathcal{E})$ corresponds to the intersection homology complex $\mathbf{IC}^\bullet(Y, \mathcal{E})$.

References

[1] Bernstein. Algebraic Theory of $\mathcal{D}$-modules (unpublished note).

[2] Hotta, Takeuchi, Tanisaki. $\mathcal{D}$-Modules, Perverse Sheaves, and Representation Theory.

[3] Kirwan, Woolf, An introduction to intersection homology.

[4] Etingof. Introduction to algebraic $\mathcal{D}$-modules (unpublished note).

[5] Beilinson, Bernstein. Deligne. Faisceux pervers.

Appendix: Notations Table

Notations	Explanation
$\mathcal{O}_X$	sheaf of regular functions over X
$\mathcal{O}(U)$	regular functions over U
$\mathcal{O}_p$	stalk of $\mathcal{O}_X$ at $p \in X$
$\mathcal{D}_X$	sheaf of differential operators over X
$\mathcal{D}(U)$	differential operators over U
Θ_X	the sheaf of vector fields
Ω_X^k	the sheaf of k-forms
Ω_X	the sheaf of highest differential forms
$\mathcal{L}(Y, \mathcal{F})$	minimal extension; it is denoted by $i_{*!}\mathcal{F}$ in [1].
$\text{SS}(\mathcal{F})$	the singular support of coherent $\mathcal{D}_X$-module $\mathcal{F}$; it is called by characteristic variety and denoted by $\text{Ch}(\mathcal{F})$ in [2].
$f_*\mathcal{F}$	push forward of coherent sheaf $\mathcal{F}$
$f^*\mathcal{G}$	pull back of coherent sheaf $\mathcal{G}$
$f^{-1}\mathcal{G}$	inverse image of sheaf $\mathcal{G}$
$f^\triangle\mathcal{F}$	push forward of $\mathcal{D}_X$-module $\mathcal{F}$
$f_+\mathcal{F}$	pull back of $\mathcal{D}_X$-module $\mathcal{F}$
$f_*\mathcal{F}^\bullet$	push forward of $\mathcal{D}_X$-module complex $= Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{F}^\bullet)$, it is denoted by $\int_f$ in [2].
$f_!\mathcal{F}^\bullet$	shriek-pull forward of $\mathcal{D}_X$-module complex $= \mathbf{D}f^!\mathbf{D}\mathcal{F}^\bullet$, it is denoted by $\int_{f!}$ in [2]
$f^*\mathcal{G}^\bullet$	pull back of $\mathcal{D}_X$-module complex $= \mathbf{D}f^!\mathbf{D}\mathcal{G}^\bullet$, it is denoted by $f^\bigstar$ in [2]
$f^!\mathcal{G}^\bullet$	shriek–pull back of $\mathcal{D}_X$-module complex $= Lf^\triangle\mathcal{G}[\dim X - \dim Y]$, it is denoted by $f^\dagger$ in [2]
$\mathbf{D}\mathcal{F}^\bullet$	duality functor $= R\mathcal{H}\text{om}^\bullet_{\mathcal{D}_X}(\mathcal{F}^\bullet, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{-1})[\dim X]$
$\mathcal{M}_{qc}(X)$	the category of quasi-coherent sheaves
$\mathcal{M}_c(X)$	the category of coherent sheaves
$\mathcal{D}_{qc}(X)$	derived category of bounded complexes of quasicoherent sheaves
$\mathcal{D}_c(X)$	derived category in $\mathcal{D}_{qc}(X)$ with cohomology coherent
$\mathcal{M}_{qc}(\mathcal{D}_X)$	the category of quasi-coherent left $\mathcal{D}_X$-module
$\mathcal{M}_c(\mathcal{D}_X)$	the category of coherent left $\mathcal{D}_X$-module
$\mathcal{M}_{qc}(\mathcal{D}_X^{op})$	the category of quasi-coherent right $\mathcal{D}_X$-module
$\mathcal{M}_c(\mathcal{D}_X^{op})$	the category of coherent right $\mathcal{D}_X$-module
$\mathcal{D}_{qc}(\mathcal{D}_X)$	derived category of bounded complexes of quasicoherent $\mathcal{D}_X$-sheaves
$\mathcal{D}_c(\mathcal{D}_X)$	derived category in $\mathcal{D}_{qc}(\mathcal{D}_X)$ with cohomology coherent
$\mathcal{D}_h(\mathcal{D}_X)$	derived category in $\mathcal{D}_{qc}(\mathcal{D}_X)$ with cohomology holomonic
$\mathcal{D}_{rh}(\mathcal{D}_X)$	derived category in $\mathcal{D}_{qc}(\mathcal{D}_X)$ with cohomology regular
$\mathcal{M}(\underline{\mathbb{C}}_X)$	the category of sheaves of $\underline{\mathbb{C}}_X$-modules
$\mathcal{D}(X^{\text{an}})$	the derived category of bounded complexes in $\mathcal{M}(\underline{\mathbb{C}}_X)$
$\mathcal{D}_{con}(X^{\text{an}})$	the derived category in $\mathcal{D}(X^{\text{an}})$ with cohomology construcible