Combinatorics of Crystals

Xiong Rui

March 11, 2021

1 General Crystal.

Let Λ be the weight lattice. A *crystal structure* over a set $\mathfrak{B}$ is

$$e_i, f_i : \mathfrak{B} \to \mathfrak{B} \sqcup \{0\}$$
$$\epsilon_i, \phi_i : \mathfrak{B} \to \mathbb{Z} \sqcup \{-\infty\}$$
$$\mathsf{wt} : \mathfrak{B} \to \Lambda$$

such that firstly for $x, y \in \mathfrak{B}$,

$$e_i(x) = y \iff f_i(y) = e \qquad \text{in this case} \begin{cases} \epsilon(x) - 1 = \epsilon(y), \\ \phi(x) + 1 = \phi(y), \\ \mathsf{wt}(x) + \alpha_i = \mathsf{wt}(y); \end{cases}$$

secondly, for any $x \in \mathfrak{B}$, $\phi(x) - \epsilon(x) = \langle \mathsf{wt}(x), \alpha_i^\vee \rangle$. We say $\mathfrak{B}$ is a crystal.

For two crystals $\mathfrak{B}$ and $\mathfrak{C}$, we can define their direct sum in the obvious way. We can define their tensor product by setting $\mathfrak{B} \otimes \mathfrak{C}$ the set of formal symbol $b \otimes c$ for $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$;

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y, & \phi_i(x) > \epsilon_i(y), \\ x \otimes f_i(y), & \phi_i(x) \le \epsilon_i(y), \end{cases}$$

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y, & \phi_i(x) \ge \epsilon_i(y), \\ x \otimes e_i(y), & \phi_i(x) < \epsilon_i(y), \end{cases}$$

$$\phi_i(x \otimes y) = \max(\phi_i(y), \phi_i(x) + \langle \mathsf{wt}(y), \alpha_i^\vee \rangle)$$

$$\epsilon_i(x \otimes y) = \max(\epsilon_i(x), \epsilon_i(y) - \langle \mathsf{wt}(x), \alpha_i^\vee \rangle).$$

$$\mathsf{wt}(x \otimes y) = \mathsf{wt}(x) + \mathsf{wt}(y).$$

It is purely combinatorial to show that $(\mathfrak{B} \otimes \mathfrak{C}) \otimes \mathfrak{D} \cong \mathfrak{B} \otimes (\mathfrak{C} \otimes \mathfrak{D})$.

In general, for $x_1, \ldots, x_k$ from some crystal,

$$e_i(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes f_i(x_j) \otimes \cdots \otimes x_k.$$

where i the first value taking the maximal value in

$$\epsilon_i(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \max_{j=1}^{k} \left(\epsilon_i(x_j) - \sum_{h=1}^{j-1} \langle \mathrm{wt}(x_h), \alpha_i^\vee \rangle \right).$$

Alternatively,

$$f_i(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1) = x_k \otimes \cdots \otimes f_i(x_j) \otimes \cdots \otimes x_1.$$

where i the first value taking the maximal value in

$$\phi_i(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1) = \max_{j=1}^{k} \left(\phi_i(x_j) + \sum_{h=1}^{j-1} \langle \mathrm{wt}(x_h), \alpha_i^\vee \rangle \right).$$

A crystal is said to be *of finite type* if ϵ_i and ϕ_i never takes $-\infty$. A crystal is said to be *seminormal* if $\epsilon_i(x) = \max\{k : e_i^k x \neq 0\}$ and $\phi_i(x) = \max\{k : f_i^k x \neq 0\}$. If $\mathfrak{B}$ and $\mathfrak{C}$ are seminormal, so is $\mathfrak{B} \otimes \mathfrak{C}$.

Let $L(\lambda)$ be an irreducible finite dimensional $U_q(\mathfrak{sl}_2)$-module with highest weight λ. For a nonzero weight vector $x \in V$, we define the *Kashiwara operators*

$$\tilde{F}x = \frac{F^{n+1}}{[\![n+1]\!]!}\hat{x} \qquad \tilde{E}x = \frac{F^{n-1}}{[\![n-1]\!]!}\hat{x}$$

where $\hat{x} \in V_\lambda$ satisfies $\frac{F^{n+1}}{[\![n]\!]!}\hat{x} = x$. For any finite dimensional $U_q(\mathfrak{sl}_2)$-module (not necessarily irreducible), we can define the Kashiwara operators by a choice of decomposition. It turns out that it does not depend on the choice.

Let V be a finite dimensional $U_q(\mathfrak{g})$-module, where $\mathfrak{g}$ is a semisimple Lie algebra. We define the Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ by embedding $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$. An admissible lattice M of V is a free graded-$\mathbb{C}[q]$-submodule stable under Kashiwara operators. Note that Kashiwara operators will then defined over M/qM. A *crystal basis* $\mathfrak{B}$ of V is a graded-basis for M/qM for an admission lattice M, and it is stable under Kashiwara operators in the following sense

$$\tilde{E}_i(\mathfrak{B}) \subseteq \mathfrak{B} \cup \{0\} \qquad \tilde{F}_i(\mathfrak{B}) \subseteq \mathfrak{B} \cup \{0\}.$$

$$x, y \in \mathfrak{B}, \qquad \tilde{E}_i(x) = y \iff \tilde{F}_i(y) = x.$$

It is proved that crystal basis exists for all finite dimensional representations, and it is unique up to an automorphism.

It is clear that a crystal basis is a finite type seminormal crystal with

$$e_i = \tilde{E}_i, \qquad f_i = \tilde{F}_i, \qquad \mathsf{wt} = \text{the weight}.$$

Let us denote $\mathfrak{B}(V)$ the crystal basis for V. One can show that the tensor product of crystals $\mathfrak{B}(V) \otimes \mathfrak{B}(U)$ is a crystal basis for $V \otimes U$. See

- Jantzen, Lectures on Quantum Groups.
- Lusztig, Canonical Bases Arising from Quantized Enveloping Algebras. II.
- Hong, Kang, Introduction to Quantum Groups and Crystal Bases.

Denote $\chi(\mathfrak{B}) = \sum_{b \in \mathfrak{B}} e^{\mathsf{wt}(b)}$. If $\mathfrak{B}$ is the crystal for V, it is clear $\chi(V) = \chi(\mathfrak{B})$. If $V = U \oplus W$, then $\mathfrak{B}(V) \cong \mathfrak{B}(U) \oplus \mathfrak{B}(W)$. It is clear non-isomorphic representations corresponds to non-isomorphic crystals.

For a crystal $\mathfrak{B}$, let us consider the crystal graph with vertices elements of $\mathfrak{B}$ labeled by whose weight, there is a arrow from x to y labelled by i if and only if $f(x) = y$. It is clear that the crystal graph determines a seminormal crystal. Then the crystal of irreducible representation is connected.

2 Tableaux.

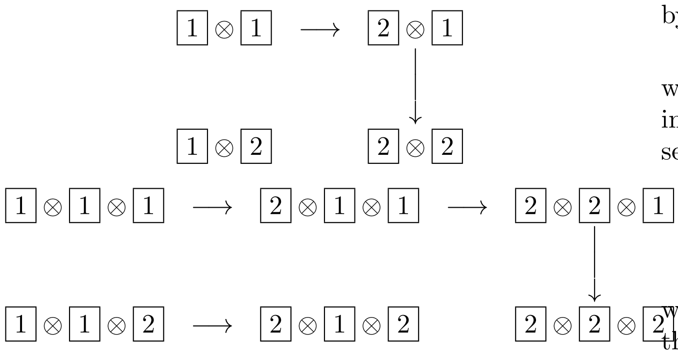
Consider the natural representation V of $U_q(\mathfrak{sl}_2)$. It is easy to deduce its crystal graph to be

$$\mathfrak{B}_\square : \boxed{1} \xrightarrow{1} \boxed{2}.$$

where $\boxed{i} = x_i \in \Lambda = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2/(x_1 + x_2)$. For $\mathfrak{B}_\square^{\otimes d}$, f_i acts on

$$\boxed{n_1} \otimes \cdots \otimes \boxed{n_d}$$

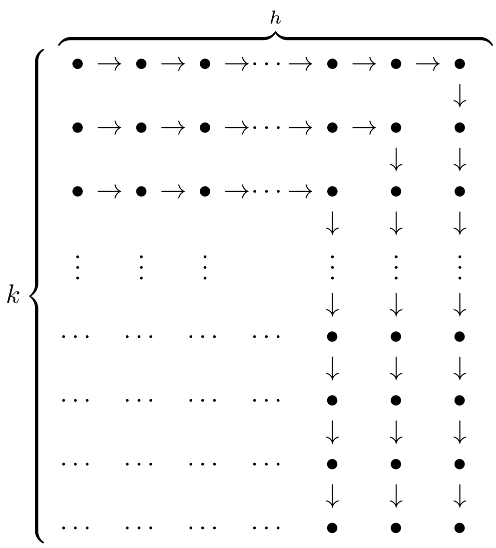
as following— after canceling adjacent $\boxed{1} \otimes \boxed{2}$, we exchange the first $\boxed{1}$ to $\boxed{2}$ and 0 if impossible. Similarly, e is to exchange the first $\boxed{2}$ to $\boxed{1}$ and 0 if impossible after cancelation. For example,



Let $\mathfrak{B}(d)$ be the crystal of unique d-dimensional representation. Its crystal graph is

$$\bullet \to \cdots \to \bullet.$$

Then $\mathfrak{B}(k) \otimes \mathfrak{B}(h)$ decomposition as the following



the classic *ClebschGordan formula*. Note that $\mathfrak{B}(k) \otimes \mathfrak{B}(h) \cong \mathfrak{B}(h) \otimes \mathfrak{B}(k)$ but not by $x \otimes y \mapsto y \otimes x$.

Now consider the natural representation V of $U_q(\mathfrak{sl}_n)$. It is easy to deduce its crystal graph to be

$$\mathfrak{B}_\square : \boxed{1} \xrightarrow{1} \boxed{2} \to \cdots \xrightarrow{n-1} \boxed{n}. \qquad \boxed{i} = x_i \in \Lambda$$

Consider $\mathfrak{B}_\square^{\otimes d}$. Then f_i acts similarly as $\mathfrak{sl}_2$ case on

$$\boxed{n_1} \otimes \cdots \otimes \boxed{n_d}$$

as following— after canceling adjacent $\boxed{i} \otimes \boxed{i+1}$, we exchange the first $\boxed{i}$ to $\boxed{i+1}$ and 0 if impossible. Similarly, e_i is to exchange the first $\boxed{i+1}$ to $\boxed{i}$ and 0 if impossible after cancelation.

For a partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_r$, denote $\mathfrak{B}(\lambda)$ the component of

$$\boxed{1} \otimes \overset{\lambda_1}{\cdots} \otimes \boxed{1} \otimes \cdots\cdots\cdots\cdots \otimes \boxed{r} \otimes \overset{\lambda_r}{\cdots} \otimes \boxed{r}$$

in $\mathfrak{B}^{\otimes|\lambda|}$. Then e_i acts trivially on it, so it is of highest weight. Denote

$$\boxed{x_{1\lambda_1}} \otimes \overset{\lambda_1}{\cdots} \otimes \boxed{x_{11}} \otimes \cdots\cdots\cdots\cdots \otimes \boxed{x_{r\lambda_r}} \otimes \overset{\lambda_r}{\cdots} \otimes \boxed{x_{r1}}$$

by a tableaux $\begin{array}{|c|c|c} \hline x_{11} & x_{12} & \cdots \\ \hline x_{21} & x_{22} & \cdots \\ \hline \vdots & \vdots & \ddots \end{array}$ of sharp λ. The tableaux which is weakly increasing in row and strictly increasing in column is called *semistandard*. It is not hard to see the set of semistandard tableaux is exactly $\mathfrak{B}(\lambda)$.

Another choice is

$$\boxed{\mu_l} \otimes \overset{\mu_1}{\cdots} \otimes \boxed{1} \otimes \cdots\cdots\cdots \otimes \boxed{\mu_1} \otimes \overset{\mu_1}{\cdots} \otimes \boxed{1}$$

where $\mu_1 \geq \cdots \geq \mu_l$ is the transposition of λ. Actually, there is an algorithm to determine component of $\mathfrak{B}_0^n$, i.e. Robinson-Schensted-Knuth (RSK) algorithm.

- Notes on Crystals.

Note that by definition

$$\epsilon_i(x \otimes y) \geq \epsilon_i(x) \qquad \phi_i(x \otimes y) \geq \phi_i(y).$$

We say x is *highest* if $e_i(x) = 0$ for all i. Let $\mathfrak{B}(\lambda)$ be a crystal. Then $x \otimes \boxed{i} \in \mathfrak{B}(\lambda) \otimes \mathfrak{B}_\square$ is highest if and only if x is highest and adding a box in i-th row of λ is still a Young diagram. So we get the *Pieri rule*

$$\mathfrak{B}(\lambda) \otimes \mathfrak{B}_\square = \bigoplus_{\mu = \lambda + \square} \mathfrak{B}(\mu).$$

We define $\lambda \leftarrow \boxed{i}$ to be the tableaux obtained by adding a box at i-th row, and 0 if impossible. So we get the classic *Littlewood–Richardson rule*

$$\mathfrak{B}(\lambda) \otimes \mathfrak{B}(\mu) = \bigoplus_{\boxed{i_1} \otimes \cdots \otimes \boxed{i_{|\mu|}} \in \mathfrak{B}(\mu)} \mathfrak{B}((\cdots(\lambda \leftarrow i_1) \leftarrow \cdots \leftarrow i_{|\mu|})).$$

This can be generalized to classical Lie algebras.

- Nakashima, Crystal base and a generalization of the Littlewood-Richardson rule for the classical Lie algebras.
- Hong, Kang, Introduction to Quantum Groups and Crystal Bases.

3 Littelmann Path Model.

A path in Λ is a pairwise linear map $[0,1] \to \Lambda \otimes \mathbb{R}$ with endpoint in Λ, and staring from 0. We recognize two paths if they are different by a re-coordinating. Let π be a path, we define the weight $\mathsf{wt}(\pi)$ to be its end point. We will define e_i and f_i to make them a crystal.

For two paths π_1 and π_2, we define $\pi_1 * \pi_2$ by the path

$$0 \xrightarrow{\pi_1} \mathsf{wt}(\pi_1) \xrightarrow{\mathsf{wt}(\pi_1)+\pi_2} \mathsf{wt}(\pi_1) + \mathsf{wt}(\pi_2).$$

For a path π, we can reflect it $s_i\pi$. Let $[a,b] \subseteq [0,1]$, we define $\pi[a,b]$ to be the path

$$0 \xrightarrow{\pi|_{[a,b]} - \pi(a)} \pi(b) - \pi(a).$$

Define

$$s_i^{[a,b]}\pi = \pi[0,a] * s_i(\pi[a,b]) * \pi[b,1].$$

We can define s_i^I for a union of interval $I \subseteq [0,1]$.

Assume the minimum of $h_i(t) = \langle \pi(t), \alpha_i^\vee \rangle$ is $m \leq -1$. Let I the set of $t \in [0,1]$ with $h_i(t) \leq m-1$ such that $x \leq t \Rightarrow h_i(x) \leq f_i(t)$. That is, the sunshine set of $0 \times [m, m-1]$. Then I cut π into pieces, we define

$$e_i(\pi) = \begin{cases} s_i^I, \pi & m \leq -1, \\ 0, & \text{otherwise.} \end{cases}$$

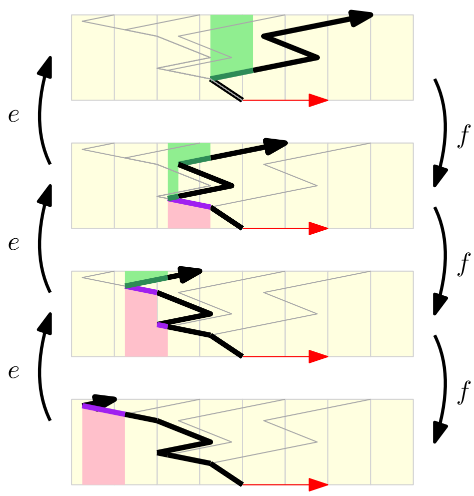
Assume the minimum of $h_i(t) = \langle \pi(t), \alpha_i^\vee \rangle$ is $m \leq h_i(1) - 1$. Let I' the set of $t \in [0,1]$ with $h_i(t) \leq m-1$ such that $x \geq t \Rightarrow h_i(x) \geq f_i(t)$. That is, the sunshine set for $1 \times [m, m-1]$. Then I cut π into pieces, we define

$$f_i(\pi) = \begin{cases} s_i^{I'}, \pi & m \leq h_i(1) - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\epsilon_i = \lfloor 0 - m \rfloor$, and $\phi_i = \lfloor h_i(1) - m \rfloor$.

One can check that the crystal structure over $\{\pi_1 \otimes \pi_2\}$ is the same to $\{\pi_1 \otimes \pi_2\}$.

Let us first do $\mathfrak{sl}_2$.



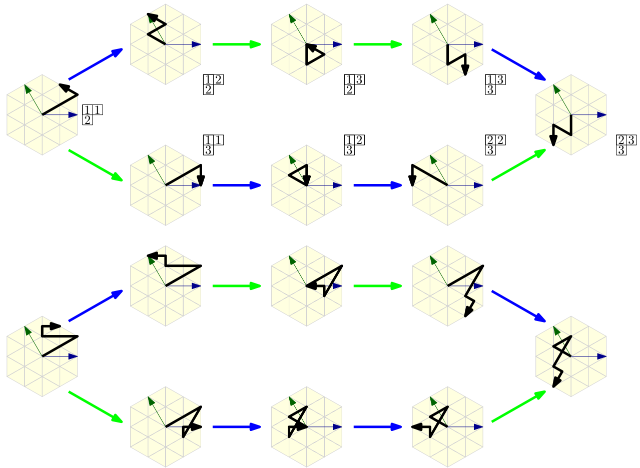
Let $C^+ = \{v \in \Lambda \otimes \mathbb{R} : \langle v, \alpha_i^\vee \rangle > 0\}$ the interior of Weyl chambre. Then π is highest if and only if the $\pi + \rho$ lies in C^+ completely. Equivalently, π in $\{v \in \Lambda \otimes \mathbb{R} : \langle v, \alpha_i^\vee \rangle > -1\}$.

It is not easy to prove that for a path crystal $\mathfrak{B}$,

$$\mathfrak{B} = \bigoplus_{\substack{\pi \in \mathfrak{B} \\ \pi + \rho \in C^+}} \mathfrak{B}(\mathsf{wt}(\pi)).$$

That is, any path crystal is isomorphic to the crystal of some representation.

Actually, Littelmann path model is much general than tableaux before



Then we get a *generalized Littlewood–Richardson* rule

$$\mathfrak{B}(\lambda) \otimes \mathfrak{B}(\mu) = \bigoplus_{\substack{\pi \in \mathfrak{B}(\mu) \\ \rho + \lambda + \pi \text{ lies in } C^+ \text{ completely}}} \mathfrak{B}(\lambda + \mathsf{wt}(\pi)).$$

But to prove it gives the character formula is relatively easier. One can also prove a crystal formula of Demazure character (formula).

- Littelmann, Paths and Root Operators in Representation Theory.
- Littelmann, A Littlewood–Richardson rule for symmetrizable Kac-Moody algebras.