Work Under the COVID-19 Pandemic

A Concise Introduction to Spectral Sequences

Xiong Rui

Preface

In this book, I would like to give an acceptable, clear, and concise introduction to spectral sequences. The preliminary is basic homological algebra and basic algebraic topology. Two mini-dictionaries are included for last two chapters.

The first chapter is the most original part. It contains the short proof by the author, and with detailed check. It is not painful and only the elementary stuff are left to reader.

The second chapter is about topology. I introduced the spectral sequences which are easy to introduce. It is a pity that I do not mention enough examples in topology. The interested reader could read [4] and [6] for examples and deeper topics.

The last chapter is about algebra. Here, we used only the spectral sequences for double complex. They are almost all the spectral sequence as I know in algebra which can be introduced shortly. The curious reader is encouraged to ask [8] for more examples.

Last but no mean least, enjoy spectral sequences !

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Comments and criticisms are welcome!

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Chapter 1

Basics of Spectral Sequences

1.1 Definitions

(1.1) Definition (Spectral Sequence) In an abelian category $\mathcal{C}$, a **spectral sequence (of homology type)** is the following

- a family of objects of $\mathcal{C}$

$$\mathsf{E} = \{\mathsf{E}^r_{pq} : p, q \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\};$$

- a family of differentials $\mathrm{d} : \mathsf{E} \to \mathsf{E}$ with d^r of degree $(-r, r-1)$ for each r, that is

$$\mathrm{d} = \{\mathrm{d}^r_{pq} : \mathsf{E}^r_{pq} \to \mathsf{E}^2_{p-r,q+r-1}\}, \quad \mathrm{d}^r \circ \mathrm{d}^r = 0;$$

- a family of isomorphisms of

$$\mathsf{E}^{r+1}_{pq} \cong \mathsf{H}_{pq}(\mathsf{E}^r) = \frac{\ker[\mathsf{E}^r_{pq} \xrightarrow{\mathrm{d}} \cdots]}{\operatorname{im}[\cdots \xrightarrow{\mathrm{d}} \mathsf{E}^r_{pq}]} = \ker \mathrm{d}^r_{pq} \,/\, \operatorname{im} \mathrm{d}^r_{p+r,q-r+1}.$$

(1.2) Definition (Spectral Sequence) In an abelian category $\mathcal{C}$, a **spectral sequence (of cohomology type)** is the following

- a family of objects of $\mathcal{C}$

$$\mathsf{E} = \{\mathsf{E}_r^{pq} : p, q \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\};$$

- a family of differentials $\mathrm{d} : \mathsf{E} \to \mathsf{E}$ with d^r of degree $(r, -r+1)$ for each r, that is

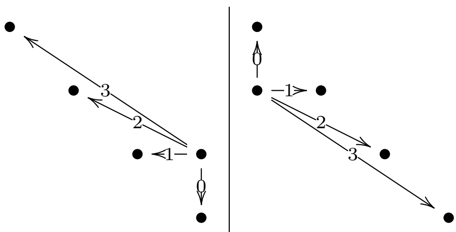
$$\mathrm{d} = \{\mathrm{d}_r^{pq} : \mathsf{E}^r_{pq} \to \mathsf{E}^2_{p+r,q-r+1}\}, \quad \mathrm{d}^r \circ \mathrm{d}^r = 0;$$

- a family of isomorphisms of

$$\mathsf{E}^{pq}_{r+1} \cong \mathsf{H}^{pq}(\mathsf{E}_r) = \frac{\ker[\mathsf{E}_r^{pq} \xrightarrow{\mathrm{d}} \cdots]}{\mathrm{im}[\cdots \xrightarrow{\mathrm{d}} \mathsf{E}_r^{pq}]} = \ker \mathrm{d}_r^{pq} \,/\, \mathrm{im}\, \mathrm{d}_r^{p-r,q+r-1}.$$

(1.3) !! Conventions— A common abuse, we will call directly E a spectral sequence rather than all above data. We will call E^r_{pq} or E_r^{pq} the object in r-th **page** or r-th **stage** of **position** (p, q). We will also call $p + q$ the **(total) order** or **(total) degree** of E^r_{pq}.

(1.4) Hint In one word, spectral sequence is a book of complexes diagrams. The object at $(r+1)$-th page of position (p, q) is the homology group at r-th page of the same position. The morphism moves ↖ or ↘ (depends on homology or cohomology type).



homology type | cohomology type

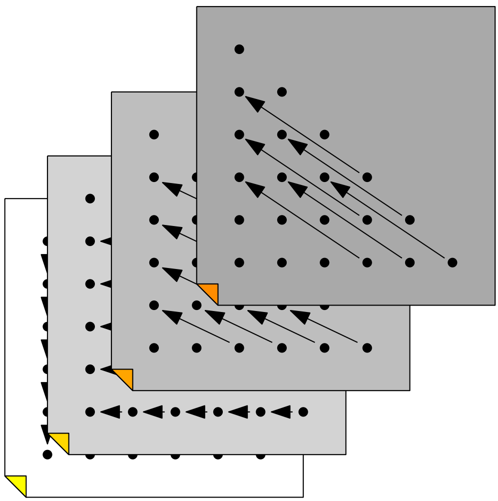
Note that E^r_{pq} is recognized as a subquotient object of E^1_{pq}, thus it defines

$$0 = \mathsf{B}^1_{pq} \subseteq \mathsf{B}^2_{pq} \subseteq \ldots \subseteq \mathsf{B}^r_{pq} \subseteq \quad \ldots \quad \subseteq \mathsf{Z}^r_{pq} \subseteq \ldots \subseteq \mathsf{Z}^2_{pq} \subseteq \mathsf{Z}^1_{pq} = \mathsf{E}^1_{pq}$$

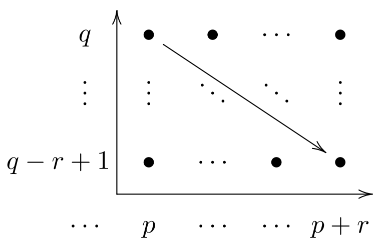
such that $\mathsf{E}^r_{pq} = \mathsf{Z}^r_{pq}/\mathsf{B}^r_{pq}$. Write

$$\mathsf{Z}^\infty_{pq} = \bigcap_{r=1}^{\infty} \mathsf{Z}^r_{pq} \qquad \mathsf{B}^\infty_{pq} = \bigcup_{r=1}^{\infty} \mathsf{B}^r_{pq} \qquad \mathsf{E}^\infty_{pq} = \mathsf{Z}^\infty_{pq}/\mathsf{B}^\infty_{pq}$$

(1.5) Remark Here is a picture of homological type.



(1.6) Remark Here is a picture of cohomological type in r-th stage.



(1.7) Definition (Boundary and cycle) Let E be a spectral sequence of cohomology type. Since the homology group is a subquotient object, E_r^{pq} is recognized as a subquotient object of E_0^{pq}. It defines

$$\begin{array}{cccccccccccc} & 0 \subseteq & \mathsf{B}_0^{pq} \subseteq \mathsf{B}_1^{pq} \subseteq \mathsf{B}_2^{pq} \subseteq \cdots \subseteq \mathsf{B}_r^{pq} \subseteq & \cdots \\ & & & \cap \\ & \mathsf{E}_0^{pq} \supseteq & \mathsf{Z}_0^{pq} \supseteq \mathsf{Z}_1^{pq} \supseteq \mathsf{Z}_2^{pq} \supseteq \cdots \supseteq \mathsf{Z}_r^{pq} \subseteq & \cdots \end{array}$$

such that $\mathsf{E}_{r+1}^{pq} \cong \mathsf{Z}_r^{pq}/\mathsf{B}_r^{pq}$. We call B the **boundary**, Z the **cycle**. The similar notations are also defined for homology type.

(1.8) Remark Consider the variation of pq, we have the following short exact sequence

$$0 \to \underbrace{\mathsf{Z}_{r+1}/\mathsf{B}_r}_{=\ker \mathrm{d}_{r+1}} \xrightarrow{\subseteq} \underbrace{\mathsf{Z}_r/\mathsf{B}_r}_{=\mathsf{E}_{r+1}} \xrightarrow{\mathrm{d}_{r+1}} \underbrace{\mathsf{B}_{r+1}/\mathsf{B}_r}_{=\operatorname{im} \mathrm{d}_{r+1}} \to 0.$$

So we have the following isomorphism

$$@\ (p,q) \qquad \mathsf{Z}_{r+1}/\mathsf{Z}_r \overset{\mathrm{d}_{r+1}}{\cong} \mathsf{B}_{r+1}/\mathsf{B}_r \qquad @\ (p-r, q+r-1).$$

Give this is equivalent to give the family of isomorphisms of $\mathsf{E}_{r+1}^{pq} \cong \ker(\cdots)/\operatorname{im}(\cdots)$.

(1.9) Definition (Infinite terms) Given a spectral sequence E of cohomology type, we can define the **infinite objects**, **infinite boundary** and **infinite cycle**

$$\mathsf{Z}_{pq}^{\infty} = \bigcap_{r=1}^{\infty} \mathsf{Z}_{pq}^{r}, \qquad \mathsf{B}_{pq}^{\infty} = \bigcup_{r=1}^{\infty} \mathsf{B}_{pq}^{r}, \qquad \mathsf{E}_{pq}^{\infty} = \mathsf{Z}_{pq}^{\infty}/\mathsf{B}_{pq}^{\infty}.$$

The similar notations are also defined for homology type.

(1.10) Definition (Convergence) Let E be a spectral sequence of homology type. Given a family of objects $\{\mathsf{H}_n : n \in \mathbb{Z}\}$. We say that $\{\mathsf{E}_{pq}^r\}$ **converges to** $\{\mathsf{H}_n\}$, if there exists a filtration of each H_n

$$0 = \mathcal{F}_s\mathsf{H}_n \subseteq \ldots \subseteq \mathcal{F}_{p-1}\mathsf{H}_n \subseteq \mathcal{F}_p\mathsf{H}_n \subseteq \ldots$$

such that $\bigcup_p F_p\mathsf{H}_n = \mathsf{H}_n$ and $\mathsf{E}_{pq}^{\infty} \cong \mathcal{F}_p\mathsf{H}_{p+q}/\mathcal{F}_{p-1}\mathsf{H}_{p+q}$. We write

$$\mathsf{E}_{pq}^r \Longrightarrow \mathsf{H}_{p+q}.$$

(1.11) Definition (Convergence) Let E be a spectral sequence of cohomology type. Given a family of objects $\{\mathsf{H}^n : n \in \mathbb{Z}\}$. We say that $\{\mathsf{E}_r^{pq}\}$ **converges to** $\{\mathsf{H}^n\}$, if there exists a filtration of each H^n

$$0 = \mathcal{F}^s\mathsf{H}^n \subseteq \ldots \subseteq \mathcal{F}^{p+1}\mathsf{H}^n \subseteq \mathcal{F}^p\mathsf{H}^n \subseteq \ldots$$

such that $\bigcup_p F_p\mathsf{H}^n = \mathsf{H}_n$ and $\mathsf{E}_\infty^{pq} \cong \mathcal{F}^p\mathsf{H}^{p+q}/\mathcal{F}^{p+1}\mathsf{H}^{p+q}$. We write

$$\mathsf{E}_r^{pq} \Longrightarrow \mathsf{H}^{p+q}.$$

(1.12) Hint That is, we can read the factors of each H from the book E. Here is some direction problem. We call the nonzero factor of H which is also a sub-object the lowest factor, i.e. minimal in the filtration.

- For homology type, the lowest factor lies in the topmost position of
$$\{(p,q) : p+q = n, \mathsf{E}_{pq} \neq 0\}$$
i.e. the position (p,q) with $q = n - q$ as large as possible.

- For cohomology type, the lowest factor lies in the lowest position in
$$\{(p,q) : \mathsf{E}^{pq} \neq 0, p+q = n\}$$
i.e. the position (p,q) with $q = n - q$ as small as possible.

(1.13) Definition (Morphism) Given two spectral sequences, E and $\overline{\mathsf{E}}$, we can define the **morphism** between them to be a family of morphisms $f : \mathsf{E} \to \mathsf{E}$ or degree 0, or $\{f_{pq}^r : \mathsf{E}_{pq}^r \to \overline{\mathsf{E}}_{pq}^r\}$ more precisely, such that

$$f \text{ commutes with d;} \qquad f^{r+1} \text{ is induced by } f^r.$$

That is,

$$f_{r+1} = \left[\mathsf{E}_{r+1} = \mathsf{H}(\mathsf{E}_r) \xrightarrow{\mathsf{H}(f_r)} \mathsf{H}(\overline{\mathsf{E}}_r) = \overline{\mathsf{E}}_{r+1}\right].$$

As the end of this section, we introduce the algebra structure. For this, we should fix some tensor product theory for complexes.

(1.14) Koszul convention Assume we have an abelian bifuntor (like tensor product) between abelian categories

$$\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}.$$

For two complexes C and D of $\mathcal{C}$ and $\mathcal{D}$ respectively, one can define

$$C \otimes D = \begin{cases} (C \otimes D)_n & = \bigoplus_{p+q=n} C_p \otimes D_q, \\ \downarrow d \\ (C\otimes D)_{n-1} & : x \otimes y \mapsto dx \otimes y + (-1)^{\deg x} x \otimes dy. \end{cases}$$

It is easy to check $C \otimes D$ is a complex of $\mathcal{E}$. Similar complex can be defined for contravariant, co-contra variant or contra-covariant, for example the functor Hom. This induces a functor

$$\otimes : \mathcal{C}\text{-Complex} \times \mathcal{D}\text{-Complex} \longrightarrow \mathcal{E}\text{-Complex}.$$

The morphisms are defined as usual (no sign).

For an abelian tri-functor, say $- \otimes - \otimes -$, we can also define as

$$d : x \otimes y \otimes z \longmapsto dx \otimes y \otimes z + (-1)^{\deg x} x \otimes dy \otimes z + (-1)^{\deg x + \deg y} x \otimes y \otimes dz.$$

Generally, we can define for arbitrary finite number of variables. It is easy to check that each natural transform between them induces one for complexes (without adjustment in signs).

One most important difference is that, if we denote

$$\times : \mathcal{D} \times \mathcal{C} \longrightarrow \mathcal{E} \qquad (Y, X) \longmapsto X \otimes Y,$$

then for two complexes C and D

$$\tau : D \times C \longrightarrow C \otimes D \qquad y \otimes x \longmapsto (-1)^{\deg x \deg y} x \otimes y$$

is a natural isomorphism.

Another is the shifting. For any complex C, we denote $C[r]_\bullet = C_{\bullet+r}$. Then

$$(C \otimes D)[1] = C \otimes D[1] \cong C[1] \otimes D$$

where the $\cong$ is given by

$$\cong: C \otimes D[1] \longrightarrow C[1] \otimes D \qquad x \otimes y \longmapsto (-1)^{\deg y} x \otimes y.$$

So

$$\begin{array}{ccc} C[1] \otimes D[1] & \xrightarrow{\cong} & (C \otimes D[1])[1] \\ \| \downarrow & \text{anti-comm.} & \downarrow \| \\ (C[1] \otimes D)[1] & \xrightarrow[\cong]{} & (C \otimes D)[2]. \end{array}$$

In one word, using Koszul convention, we only need to be careful when we exchange the order of variables.

For spectral sequence E, we can regard each page E_r disjoint union of complexes with degree by their total degrees. We view E_r as a cochain complex by $(\mathsf{E}_r)^n = \bigoplus_{p+q=n} \mathsf{E}_r^{pq}$, with differential induced by all d's.

(1.15) Definition (Multiplication structure) For three spectral sequence $\mathsf{E}, \overline{\mathsf{E}}, \hat{\mathsf{E}}$ of cohomology type, a **multiplication structure** is family of morphisms

$$\mu_r : \mathsf{E}_r \otimes \overline{\mathsf{E}}_r \to \hat{\mathsf{E}}_r,$$

preserving degrees with μ_{r+1} induced by μ_r. More precisely,

$$\mu_{r+1} : \left[\mathsf{E}_{r+1} \otimes \overline{\mathsf{E}}_{r+1} \cong \mathsf{H}(\mathsf{E}_r) \otimes \mathsf{H}(\overline{\mathsf{E}}_r) \xrightarrow{*} \mathsf{H}(\mathsf{E}_r \otimes \mathsf{E}_r) \xrightarrow{\mathsf{H}(\mu_r)} \mathsf{H}(\hat{\mathsf{E}}_r) \cong \hat{\mathsf{E}}_{r+1} \right].$$

The $*$ above is induced by identity. It induces

$$\mu_\infty : \mathsf{E}_\infty \otimes \overline{\mathsf{E}}_\infty \to \hat{\mathsf{E}}_\infty.$$

Similarly, we can define for mixed type (some of them with homology type), but for convention, the degree of $y \in \overline{E}^r$ is $-\deg y$ for homology type. But since $y \equiv -y \bmod 2$, there is no problem of signs in all.

1.2 Filtered Complexes

(1.16) Definition (Filtered Complexes) A chain complex $C_\bullet$ is called a **filtered (chain) complex** if it is equipped with a filtration

$$\mathcal{F}: \quad \ldots \subseteq \mathcal{F}_{p-1}C_n \subseteq \mathcal{F}_pC_n \subseteq \mathcal{F}_{p+1}C_n \subseteq \ldots \subseteq C_n$$

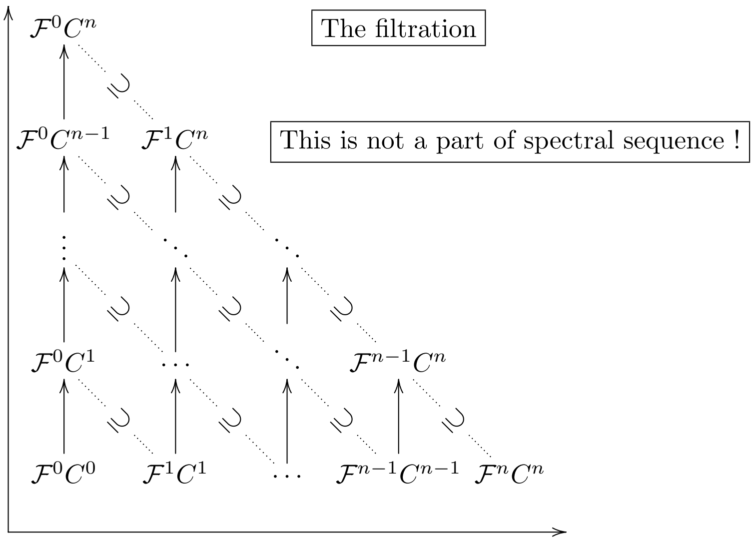
such that $d(\mathcal{F}_pC_n) \subseteq \mathcal{F}_pC_{n-1}$, i.e. $\mathcal{F}_pC_\bullet$ is a subcomplex of $C_\bullet$.

A cochian complex $C^\bullet$ is called a **filtered (cochain) complex** if it is equipped with a filtration

$$\mathcal{F}: \quad \ldots \subseteq \mathcal{F}^{p+1}C^n \subseteq \mathcal{F}^pC^n \subseteq \mathcal{F}^{p+1}C^n \subseteq \ldots \subseteq C_n$$

such that $d(\mathcal{F}^pC^n) \subseteq \mathcal{F}^pC^{n+1}$, i.e. $\mathcal{F}^pC^\bullet$ is a subcomplex of $C^\bullet$.

(1.17) Remark Here is a picture for cohomology type.



(1.18) Definition We say the filtration $\mathcal{F}$ over complex C is

- **exhaustive** if $C_n = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_pC_n$ or $C^n = \bigcup_{p\in\mathbb{Z}} \mathcal{F}^pC^n$;

- **lower bounded** or **bounded below** if for each n, exists p such that $\mathcal{F}_pC_n = 0$ or $\mathcal{F}^pC^n = 0$.

- **upper bounded** or **bounded above** if for each n, exists p such that $\mathcal{F}_pC_n = C_n$ or $\mathcal{F}^pC^n = C^n$.

(1.19) !! Assumption— To protect us to use element-picking method, in this section, we will take an abelian category which can be embedded into some R-Mod for some ring R preserving the colimit (at least the countable-infinite-filtered union appearing). If not, we should change "exhaustive" by "upper bounded" to make the colimit finite. See (1.50).

(1.20) Theorem *Each filtered cochain complex $(C, \mathcal{F})$ determines a spectral sequence* E *of cohomology type with*

$$\begin{aligned} \mathsf{E}_0^{pq} &= \mathcal{F}^pC^{p+q}/\mathcal{F}^{p+1}C^{p+q} \\ \mathsf{E}_1^{pq} &= \mathsf{H}^{p+q}(\mathcal{F}^pC/\mathcal{F}^{p+1}C). \end{aligned}$$

If the filtration $\mathcal{F}$ over A is lower bounded and upper exhaustive then E *converges to* $\mathsf{H}^\bullet(A, d)$*. More exactly,*

$$\mathsf{E}_\infty^{pq} \cong \mathcal{F}^pH^{p+q}(A,d)/\mathcal{F}^{p+1}H^{p+q}(A,d),$$

where $\mathcal{F}$ is lower bounded and exhaustive filtration over $H^\bullet(A,d)$.

The proof is due to Xiong myself. Firstly, it is several elementary exercises in abstract algebra.

(1.21) Λemma (Modular property) *Let A, B, C be three subgroups of some bigger abelian group, if $A \subseteq C$, then*

$$(A + B) \cap C = A + (B \cap C).$$

As a result, it makes no doubt to write $A + B \cap C$.

(1.22) Λemma (Exchange Limit) *Let $C_\bullet$ be a directed family of submodules of some bigger abelian group, that is, each pair of C_i, C_j are submodules of some C_k. Assume $A \subseteq B$, then we have*

$$\bigcup \left(A + C_\bullet \cap B\right) = A + \left(\bigcup C_\bullet\right) \cap B.$$

If furthermore, $C_\bullet$ is bounded below, that is, some C_i equals to $\bigcap C_\bullet$, then,

$$\bigcap \left(A + C_\bullet \cap B\right) = A + \left(\bigcap C_\bullet\right) \cap B.$$

(1.23) Λemma (Zassenhaus' Butterfly Lemma) *For four subgroups A, B, C, D of some bigger abelian group, if $A \subseteq B$ and $C \subseteq D$, then*

$$\frac{A + D \cap B}{A + C \cap B} \cong \frac{B \cap D}{(B \cap C) + (A \cap D)} \cong \frac{(A + D) \cap (B + C)}{A + C} \cong \frac{C + B \cap D}{C + A \cap D}.$$

(1.24) Λemma (Adjunct formula) *Let $A \xrightarrow{f} B$ be a homomorphism, X, Y be submodules of A and B respectively.*

$$f(f^{-1}(Y) \cap X) = Y \cap f(X), \qquad f^{-1}(f(X) + Y) = X + f^{-1}(Y).$$

(1.25) Λemma (Adjointness) *If $A = f^{-1}(A')$, $B = f^{-1}(B')$ and $f(C) = C'$, $f(D) = D'$, then*

$$f : \frac{A + D \cap B}{A + C \cap B} \longrightarrow \frac{A' + D' \cap B'}{A' + C' \cap B'}$$

is an isomorphism.

Proof of (1.20), due to Xiong [10].]Define

$$\begin{cases} K_r^{pq} = \mathcal{F}^{p+1}C^{p+q} + d^{-1}(\mathcal{F}^{p+r}C^{p+q+1}) \cap \mathcal{F}^p C^{p+q}, \\ I_r^{pq} = \mathcal{F}^{p+1}C^{p+q} + d(\mathcal{F}^{p-r+1}C^{p+q-1}) \cap \mathcal{F}^p C^{p+q}. \end{cases}$$

Then we get a filtration

$$\mathcal{F}^{p+1}C^{p+q} = I_0^{pq} \subseteq I_1^{pq} \subseteq \cdots \subseteq K_1^{pq} \subseteq K_0^{pq} \subseteq \mathcal{F}^p C^{p+q}.$$

Then

$$\begin{aligned}
\frac{K_r^{pq}}{K_{r+1}^{pq}} &= \frac{\mathcal{F}^{p+1}C^{p+q} + d^{-1}(\mathcal{F}^{p+r}C^{p+q+1}) \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q} + d^{-1}(\mathcal{F}^{p+r+1}C^{p+q+1}) \cap \mathcal{F}^pC^{p+q}} \\
&= \frac{d^{-1}(\mathcal{F}^{p+r+1}C^{p+q+1}) + \mathcal{F}^pC^{p+q} \cap d^{-1}(\mathcal{F}^{p+r}C^{p+q+1})}{d^{-1}(\mathcal{F}^{p+r+1}C^{p+q+1}) + \mathcal{F}^{p+1}C^{p+q} \cap d^{-1}(\mathcal{F}^{p+r}C^{p+q+1})} && \because (1.23) \\
&\xrightarrow[\sim]{d} \frac{\mathcal{F}^{p+r+1}C^{p+q+1} + d(\mathcal{F}^pC^{p+q}) \cap \mathcal{F}^{p+r}C^{p+q+1}}{\mathcal{F}^{p+r+1}C^{p+q+1} + d(\mathcal{F}^{p+1}C^{p+q}) \cap \mathcal{F}^{p+r}C^{p+q+1}} && \because (1.25) \\
&= \frac{I_{r+1}^{p+r,q-r+1}}{I_r^{p+r,q-r+1}}.
\end{aligned}$$

For each stage, define $\mathsf{E}_r^{pq} = \frac{K_r^{pq}}{I_r^{pq}}$, and

$$\mathrm{d} = \left[\mathsf{E}_r^{pq} = \frac{K_r^{pq}}{I_r^{pq}} \twoheadrightarrow \frac{K_r^{pq}}{K_{r+1}^{pq}} \xrightarrow[\sim]{d} \frac{I_{r+1}^{p+r,q-r+1}}{I_r^{p+r,q-r+1}} \hookrightarrow \frac{K_{r+1}^{p+r,q-r+1}}{I_r^{p+r,q-r+1}} = \mathsf{E}_r^{p+r,q-r+1}\right].$$

Then definitely, at the place (p, q),

$$\ker d = \frac{K_{r+1}^{pq}}{I_r^{pq}}, \quad \operatorname{im} d = \frac{I_{r+1}^{pq}}{I_r^{pq}}, \quad \Rightarrow \frac{\ker d}{\operatorname{im} d} = \frac{K_{r+1}^{pq}}{I_{r+1}^{pq}},$$

as desired. So we define a spectral sequence.

To show the convergence, we need to compute the

$$\begin{aligned}
\mathsf{E}_\infty^{pq} = \frac{K_\infty^{pq}}{I_\infty^{pq}} &= \frac{\bigcap_r \mathcal{F}^{p+1}C^{p+q} + d^{-1}(\mathcal{F}^{p+r}C^{p+q+1}) \cap \mathcal{F}^pC^{p+q}}{\bigcup_r \mathcal{F}^{p+1}C^{p+q} + d(\mathcal{F}^{p-r+1}C^{p+q-1}) \cap \mathcal{F}^pC^{p+q}} \\
&= \frac{\mathcal{F}^{p+1}C^{p+q} + \bigcap_r d^{-1}(\mathcal{F}^{p+r}C^{p+q+1}) \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q} + \bigcup_r d(\mathcal{F}^{p-r+1}C^{p+q-1}) \cap \mathcal{F}^pC^{p+q}} && \because (1.22) \\
&= \frac{\mathcal{F}^{p+1}C^{p+q} + d^{-1}\left(\bigcap_r \mathcal{F}^{p+r}C^{p+q+1}\right) \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q} + d\left(\bigcup_r \mathcal{F}^{p-r+1}C^{p+q-1}\right) \cap \mathcal{F}^pC^{p+q}} \\
&= \frac{\mathcal{F}^{p+1}C^{p+q} + \ker d \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q} + \operatorname{im} d \cap \mathcal{F}^pC^{p+q}} \\
&= \frac{\operatorname{im} d + \mathcal{F}^pC^{p+q} \cap \ker d}{\operatorname{im} d + \mathcal{F}^{p+1}C^{p+q} \cap \ker d}
\end{aligned}$$

Hence the nominator and denominator defines a filtration between $\operatorname{im} d$ to $\ker d$. So it induces a filtration on $\mathcal{F}^pH^{p+q}(C, d)$. This is easy to be checked to be lower bounded and exhaustive filtration by (1.22).

(1.26) Corollary *In the theorem (1.20).*

$$\mathsf{Z}_r^{pq} = \frac{K_{r+1}^{pq}}{\mathcal{F}^{p+1}C^{p+q}} = \frac{\mathcal{F}^{p+1}C^{p+q} + d^{-1}(\mathcal{F}^{p+r+1}C^{p+q+1}) \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q}} \subseteq \frac{\mathcal{F}^pC^{p+q}}{\mathcal{F}^{p+1}C^{p+q}};$$

$$\mathsf{B}_r^{pq} = \frac{I_{r+1}^{pq}}{\mathcal{F}^{p+1}C^{p+q}} = \frac{\mathcal{F}^{p+1}C^{p+q} + d(\mathcal{F}^{p-r}C^{p+q-1}) \cap \mathcal{F}^pC^{p+q}}{\mathcal{F}_{p+1}C^{p+q}} \subseteq \frac{\mathcal{F}_pC^{p+q}}{\mathcal{F}_{p+1}C^{p+q}};$$

$$\mathcal{F}^p\mathsf{H}^{p+q} = \frac{\operatorname{im} d + \mathcal{F}^pC^{p+q} \cap \ker d}{\operatorname{im} d} \subseteq \frac{\ker d}{\operatorname{im} d}.$$

The differential in E *is induced by the differential of the cochain complex.*

We will only prove the version of cohomology type, but we state the homology version.

(1.27) Theorem *Each filtered chain complex* $(C, \mathcal{F})$ *determines a spectral sequence* E *of homology type with*

$$\begin{aligned} \mathsf{E}_{pq}^0 &= \mathcal{F}_pC_{p+q}/\mathcal{F}_{p-1}C_{p+q} \\ \mathsf{E}_{pq}^1 &= \mathsf{H}_{p+q}(\mathcal{F}^pC/\mathcal{F}^{p-1}C). \end{aligned}$$

If the filtration $\mathcal{F}$ *over* A *is lower bounded and exhaustive then* E *converges to* $\mathsf{H}_\bullet(A, d)$*. More exactly,*

$$\mathsf{E}_\infty^{pq} \cong \mathcal{F}_pH_{p+q}(A, d)/\mathcal{F}_{p-1}H_{p+q}(A, d),$$

where $\mathcal{F}$ *is lower bounded and exhaustive filtration over* $H_\bullet(A, d)$*.*

(1.28) Corollary *In the theorem (1.27).*

$$\mathsf{Z}_{pq}^r = \frac{\mathcal{F}_{p-1}C_{p+q} + \mathrm{d}^{-1}(\mathcal{F}_{p-r+1}C_{p+q-1}) \cap \mathcal{F}_pC_{p+q}}{\mathcal{F}_{p-1}C_{p+q}} \subseteq \frac{\mathcal{F}_pC_{p+q}}{\mathcal{F}_{p-1}C_{p+q}}$$

$$\mathsf{B}_{pq}^r = \frac{\mathcal{F}_{p-1}C_{p+q} + \mathrm{d}(\mathcal{F}_{p+r}C_{p+q+1}) \cap \mathcal{F}_pC_{p+q}}{\mathcal{F}_{p-1}C_{p+q}} \subseteq \frac{\mathcal{F}_pC_{p+q}}{\mathcal{F}_{p-1}C_{p+q}}$$

$$\mathcal{F}_p\mathsf{H}_{p+q} = \frac{\operatorname{im} d + \mathcal{F}_pC_{p+q} \cap \ker d}{\operatorname{im} d} \subseteq \frac{\ker d}{\operatorname{im} d}.$$

The differential in E *is induced by the differential of the complex.*

(1.29) Definition (Filtered multiplication structure) If we have a multiplication structure over filtered complexes (or objects)

$$C \otimes D \xrightarrow{\mu} E$$

such that $\mu(\mathcal{F}^p C \otimes \mathcal{F}^q C) \subseteq \mathcal{F}^{p+q} C$, we say μ is a **filtered multiplication structure**.

(1.30) Theorem *If we have a filtered multiplication structure over three lower bounded and exhaustive filtered complexes*

$$C \otimes \overline{C} \xrightarrow{\mu} \hat{C}.$$

Then, in the theorem (1.20),

- *the spectral sequence* E, $\overline{\mathsf{E}}$ *and* $\hat{\mathsf{E}}$ *has a multiplication structure.*
- *the filtration over* $\mathsf{H}^\bullet(C)$, $\mathsf{H}^\bullet(\overline{C})$ *and* $\mathsf{H}^\bullet(\hat{C})$ *makes it a filtered multiplication structure.*
- *the isomorphism is an isomorphism between multiplication structure.*

Proof. We only need to check the first two assertions, i.e. the well-definedness, the rest is by our construction. For the first assertion, it suffices to check

$$\mu(\mathsf{Z}_r^{pq} \otimes \overline{\mathsf{Z}}_r^{p'q'}) \subseteq \hat{\mathsf{Z}}_r^{p'+p,q'+q}, \quad \mu(\mathsf{B}_r^{pq} \otimes \overline{\mathsf{Z}}_r^{p'q'} + \mathsf{Z}_r^{pq} \otimes \overline{\mathsf{B}}_r^{p'q'}) \subseteq \hat{\mathsf{B}}_r^{p'+p,q'+q}.$$

$$\text{If } \begin{cases} x \in d^{-1}(\mathcal{F}^{p+r} C^{p+q+1}) \cap \mathcal{F}^p C^{p+q}, \\ y \in d^{-1}(\mathcal{F}^{p'+r} \overline{C}^{p'+q'+1}) \cap \mathcal{F}^{p'} \overline{C}^{p'+q'} \end{cases}, \text{ then}$$

$$d(\mu(x \otimes y)) = \mu(dx \otimes y) \pm \mu(x \otimes dy) \in \mathcal{F}^{p+r} C^{p+p'+q+q'+1}$$

and no problem $\mu(x \otimes y) \in \mathcal{F}^{p+p'} C^{p+p'+q+q'}$.

$$\text{If } \begin{cases} x \in d(\mathcal{F}^{p-r+1} C^{p+q-1}) \cap \mathcal{F}^p C^{p+q} \\ y \in d^{-1}(\mathcal{F}^{p'+r} C^{p'+q'+1}) \cap \mathcal{F}^{p'} C^{p'+q'} \end{cases}, \text{ say } x = dz, \text{ then}$$

$$\mu(dz \otimes y) = \underbrace{\mathrm{d}(\mu(z \otimes y))}_{\in \mathrm{d}(\mathcal{F}^{p+p'-r+1} C^{p+p'+q+q'-1})} - \underbrace{\mu(z \otimes \mathrm{d}y)}_{\in \mathcal{F}^{p+p'+1} C^{p+p'+q+q'}}$$

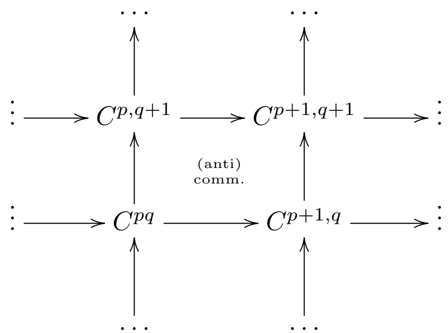
and no problem $\mu(x \otimes y) \in \mathcal{F}^{p+p'}C^{p+p'+q+q'}$.

The checking of the second is easy by construction (1.26) — everything commutes with multiplication.

(1.31) Remark The same multiplication structure theorem (1.30) holds for mixed type. But, still, by a convention, the degree of $y \in \overline{E}^r$ is $-\deg y$ for homology type.

1.3 Double Complexes

Assume we have a **double complex** (or a **bi-complex**) of cohomology type $(C^{\bullet\bullet}, \mathrm{d}^{\rightarrow}, \mathrm{d}^{\uparrow})$,



Here anti-commutativity is just for convention, since the sign change do not effect on homology groups.

We can consider its **total**

$$(\mathrm{Tot}\, C)^n = \bigoplus_{p+q=n} C^{pq}, \quad d = \mathrm{d}^{\uparrow} + \mathrm{d}^{\rightarrow}.$$

Then it admits two filtrations by columns and rows, say ${}^{\mathrm{I}}\mathcal{F}(C) = \mathrm{Tot}(\mathrm{I}_{\leq n} C)$ and ${}^{\mathrm{II}}\mathcal{F}(C) = \mathrm{Tot}(\mathrm{II}_{\leq n} C)$, where

$$(\mathrm{I}_{\leq n} C)^{pq} = \begin{cases} C^{pq}, & p \leq n, \\ 0, & p > n. \end{cases}, \quad (\mathrm{II}_{\leq n} C)^{pq} = \begin{cases} C^{pq}, & q \leq n, \\ 0, & p > n. \end{cases}$$

Let ${}^{\mathrm{I}}\mathsf{E}_*^{**}$ and be ${}^{\mathrm{II}}\mathsf{E}_*^{**}$ their spectral sequences. Note that ${}^{\mathrm{I}}\mathcal{F}^n C/{}^{\mathrm{II}}\mathcal{F}^{n-1}C$ is exactly the n-th column, and we know from the proof of (1.20) that the differential on E_1 is induced by d, so

$${}^{\mathrm{I}}\mathsf{E}_1 = \mathsf{H}(C, \mathrm{d}^{\uparrow}), \qquad {}^{\mathrm{I}}\mathsf{E}_2 = \mathsf{H}(\mathsf{H}(C, d^{\uparrow}), \mathrm{d}^{\rightarrow}).$$

Similarly,

$${}^{\mathrm{II}}\mathsf{E}_1 = \mathsf{H}(C, \mathrm{d}^{\rightarrow}), \qquad {}^{\mathrm{II}}\mathsf{E}_2 = \mathsf{H}(\mathsf{H}(C, d^{\rightarrow}), \mathrm{d}^{\uparrow}).$$

There are several common cases under which ${}^{\mathrm{I}}\mathcal{F}$ and ${}^{\mathrm{II}}\mathcal{F}$ are both exhaustive and lower bounded. For example, when C lies in the first quadrant, or the third quadrant.

(1.32) Theorem *Each double complex $(C^{\bullet\bullet}, \mathrm{d}^{\rightarrow}, \mathrm{d}^{\uparrow})$ of cohomology type, determines two spectral sequences ${}^{\mathrm{I}}\mathsf{E}$ and ${}^{\mathrm{II}}\mathsf{E}$ of cohomology type with*

$$\begin{aligned} {}^{\mathrm{I}}\mathsf{E}^1_{pq} &= \mathsf{H}^q(C^{p\bullet}, \mathrm{d}^{\uparrow}) & \quad {}^{\mathrm{II}}\mathsf{E}^1_{qp} &= \mathsf{H}^p(C^{\bullet q}, \mathrm{d}^{\rightarrow}) \\ {}^{\mathrm{I}}\mathsf{E}^2_{pq} &= \mathsf{H}^p(\mathsf{H}^q(C, \mathrm{d}^{\uparrow}), \mathrm{d}^{\rightarrow}). & \quad {}^{\mathrm{II}}\mathsf{E}^2_{qp} &= \mathsf{H}^q(\mathsf{H}^p(C, \mathrm{d}^{\rightarrow}), \mathrm{d}^{\uparrow}). \end{aligned}$$

If the double complex C lies in the first quadrant, or the third quadrant, then ${}^{\mathrm{I}}\mathsf{E}$ and ${}^{\mathrm{II}}\mathsf{E}$ converge to $\mathsf{H}^{\bullet}(\mathrm{Tot}(C))$.

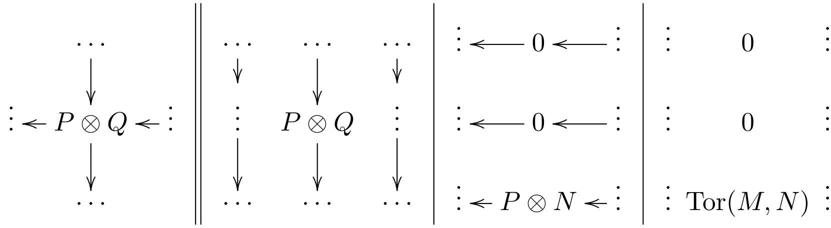
(1.33) Remark Here is not ${}^{\mathrm{II}}\mathsf{E}_2^{pq}$ is to suit our convention (1.4).

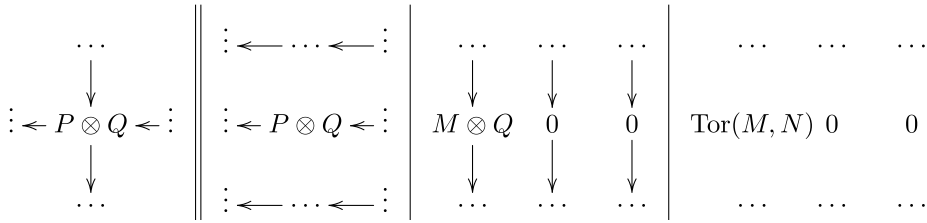
(1.34) Theorem *Each double complex $(C_{\bullet\bullet}, \mathrm{d}^{\leftarrow}, \mathrm{d}^{\downarrow})$ of homology type, determines two spectral sequences ${}^{\mathrm{I}}\mathsf{E}$ and ${}^{\mathrm{II}}\mathsf{E}$ of homology type with*

$$\begin{aligned} {}^{\mathrm{I}}\mathsf{E}_1^{pq} &= \mathsf{H}_q(C_{p\bullet}, \mathrm{d}^{\downarrow}) & \quad {}^{\mathrm{II}}\mathsf{E}_1^{qp} &= \mathsf{H}_p(C_{\bullet q}, \mathrm{d}^{\leftarrow}) \\ {}^{\mathrm{I}}\mathsf{E}_2^{pq} &= \mathsf{H}_p(\mathsf{H}_q(C, \mathrm{d}^{\downarrow}), \mathrm{d}^{\leftarrow}). & \quad {}^{\mathrm{II}}\mathsf{E}_2^{qp} &= \mathsf{H}_q(\mathsf{H}_p(C, \mathrm{d}^{\leftarrow}), \mathrm{d}^{\downarrow}). \end{aligned}$$

If the double complex C lies in the first quadrant, or the third quadrant, then ${}^{\mathrm{I}}\mathsf{E}$ and ${}^{\mathrm{II}}\mathsf{E}$ converge to $\mathsf{H}_{\bullet}(\mathrm{Tot}(C))$.

(1.35) Application (Balancing Tor and Ext) Let $P_\bullet \to M$ and $Q_\bullet \to N$, the (flat) resolutions as (right and left respectively) modules. Consider the double complex $P \otimes Q$, and compute ${}^{\mathrm{I}}\mathsf{E}$ and ${}^{\mathrm{II}}\mathsf{E}$





We find

$$\mathrm{Tor}_n(M,N) = \mathsf{H}_n(P_\bullet \otimes N) = \mathsf{H}_n(M \otimes Q_\bullet) = \mathsf{H}_n(\mathrm{Tot}(P \otimes Q)).$$

Similarly, for two modules M, N. Let $P_\bullet \to M$ and $N \to\to I^\bullet$ the (projective and injective respectively) resolution. It defines a double (cochain) complex $\mathrm{Hom}(P, I)$, then

$$\mathrm{Ext}^n(M,N) = \mathsf{H}^n(\mathrm{Hom}(P_\bullet, B) = \mathsf{H}^n(\mathrm{Hom}(A, I_\bullet)) = \mathsf{H}^n(\mathrm{Tot}(\mathrm{Hom}(P,I))).$$

(1.36) Application (Mayer-Vietoris Spectral Sequence) Let X be a topological space, and $\mathfrak{U} = \{U_i : i \in I\}$ an open covering of X with I a totally ordered set. We will write $\mathrm{Sing}_\bullet$ for the chain complex of singular homology. Denote

$$U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}, \quad \check{\mathrm{C}}_{\bullet q}(\mathfrak{U}) = \bigoplus_{i_0 < \ldots < i_q \in I} \mathrm{Sing}_\bullet(U_{i_0,\ldots,i_q}).$$

We can define

$$\partial : \check{\mathrm{C}}_{pq}(\mathfrak{U}) \longrightarrow \check{\mathrm{C}}_{p,q-1}(\mathfrak{U}) \qquad \alpha \longmapsto \sum_{j=0}^{q}(-1)^j \alpha_{\hat{j}}$$

where $\alpha \in \mathrm{Sing}_q(U_{i_0,\dots,i_q})$, and $\alpha_{\hat{j}}$ denote the image of α under the map induced by inclusion

$$\mathrm{Sing}_q(U_{i_0,\dots,i_q}) \to \mathrm{Sing}_q(U_{i_0,\dots,\widehat{i_j}\dots,i_q}).$$

It is easy to see $\check{\mathrm{C}}_{pq}(\mathfrak{U})$ forms a double complex.

By a direct computation, $(\check{\mathrm{C}}_{p\bullet}(\mathfrak{U}), \partial)$ is acylic, and

$$\begin{aligned} \mathsf{H}_0(C_{p\bullet}(\mathfrak{U})) &= \textstyle\sum_{U\in\mathfrak{U}} \mathrm{Sing}_p(U) \qquad\qquad (\subseteq \mathrm{Sing}_p(X)) \\ &= \{\sigma \in \mathrm{Sing}_p(X) : \text{the image of } \sigma \text{ lies in some } U \in \mathfrak{U}\} \\ &\simeq \mathrm{Sing}_p(X). \end{aligned}$$

As a result, $\mathsf{H}(\mathrm{Tot}(\check{\mathrm{C}})) \cong \mathsf{H}(X)$. But if we consider the other direction, we will find

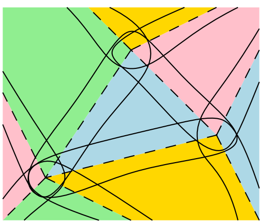
$$\mathsf{E}^1_{pq} = \bigoplus_{i_0,\dots,i_q\in I} \mathsf{H}_p(U_{i_0,\cdots,i_q}).$$

We define the homology groups of $\mathsf{E}^1_{0\bullet}$ the **Čech homology** $\check{\mathsf{H}}_{\mathfrak{U}}(X)$. Of course, the chain complex enjoys a combinatorial description (left to readers).

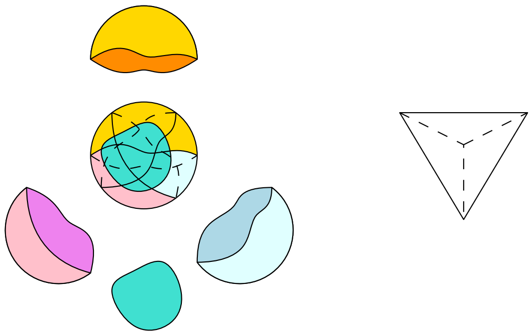
In particular, when all nonempty $U_{i_0,\dots,i_p}$ are acyclic, then

$$\check{\mathsf{H}}_{\mathfrak{U}}(X) = \mathsf{H}(X).$$

In some sense, Čech homology is a kind of "fat" simplicial homology, tell us how to glue acyclic space into general space.



All above can be carried to cohomology of sheaves, known as **Čech cohomology**.



1.4 Exact couples

(1.37) Remark In the definition of spectral sequence, we assume that every spectral sequence starts from 0-th stage ($r \in \mathbb{Z}_{\geq 0}$). But we can also define it after removing this assumption. We will call such a spectral sequence **starting from ℓ-th page** if r is assumed to be in $\mathbb{Z}_{\geq r}$.

(1.38) Definition (Exact couple) Let D, E be two objects in some abelian category, and $i : D \to D$, $j : D \to E$ and $k : E \to D$ be morphisms. We present these data as in the diagram:

$$\begin{array}{ccc} D & \xrightarrow{\quad i \quad} & D \\ & {}_{k}\nwarrow \qquad \swarrow_{j} & \\ & E & \end{array}$$

We say (D, E, i, j, k) is an **exact couple** if this diagram is exact at each position, that is, $\operatorname{im} i = \ker j$, $\operatorname{im} j = \ker k$ and $\operatorname{im} k = \ker i$.

(1.39) Definition (limit) For an exact couple (D, E, i, j, k), we define the **limit** of it to be

$$\mathsf{H} = \varinjlim \left[\cdots \xrightarrow{i} D \xrightarrow{i} \cdots \right].$$

(1.40) !! Assumption— Similarly, to protect us to use element-picking method, in this section, we will take an abelian category which can be embedded into some R-Mod for some ring R preserving the colimit (at least the $\{0 \to 1 \to 2 \to \cdots\}$-colimit appearing). See (1.50).

(1.41) Remark We will use the case when $D = D^{pq}, E = E^{pq}$ are bigraded, and

$$\deg i = (-1, 1), \quad \deg j = (0, 0), \quad \deg k = (1, 0).$$

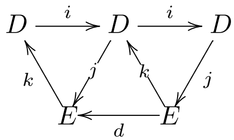
$$\mathsf{H}^n = \varinjlim_{p+q=n} D^{pq} = \varinjlim \left[\cdots \xrightarrow{i} D^{pq} \xrightarrow{i} \cdots \right].$$

(1.42) Theorem *Each exact couple of bigraded modules* (D, E, i, j, k) *determines a spectral sequence* E *starting from the first page with*

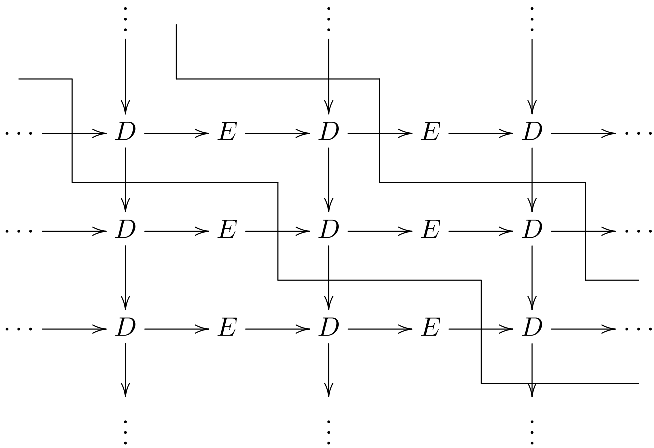
$$\begin{aligned} \mathsf{E}_1^{pq} &= E^{pq} \\ \mathsf{E}_2^{pq} &= H(E^{pq}, j \circ k). \end{aligned}$$

If the exact couple is lower bounded, i.e. $\left[\cdots \xrightarrow{i} D \xrightarrow{i} \cdots \right]$ *is started from* 0*. then* E *converges to* H^**.*

(1.43) Remark It is easy to see $\mathrm{d} = j \circ k$ is a differential, i.e. $\mathrm{d} \circ \mathrm{d} = 0$.



(1.44) Remark Someone prefer the following big diagram

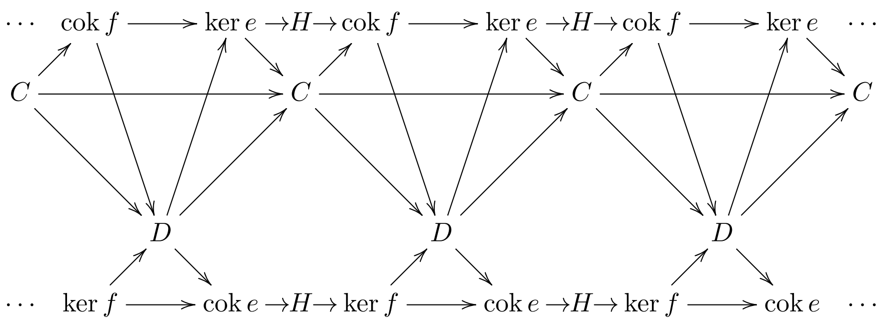


with sequence in ▭ exact, each row the first stage of spectral sequence, and the inductive limit of column the limit.

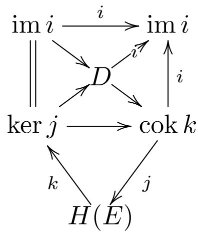
(1.45) Λemma[9] Theorem 8 *Let C be a differential object, assume d is decomposed into $C \xrightarrow{e} D \xrightarrow{f} C$, with $D \xrightarrow{f} C \xrightarrow{e} D$ exact. Then we have the following long two exact sequences*

$$\cdots \to H(C,d) \to \ker f \to \operatorname{cok} e \to H(C,d) \to \cdots$$
$$\cdots \to H(C,d) \to \operatorname{cok} f \to \ker e \to H(C,d) \to \cdots$$

As some readers like, we have the following commutative diagram

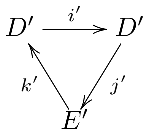


So for any exact couple (D, E, i, j, k), we have the following triangle

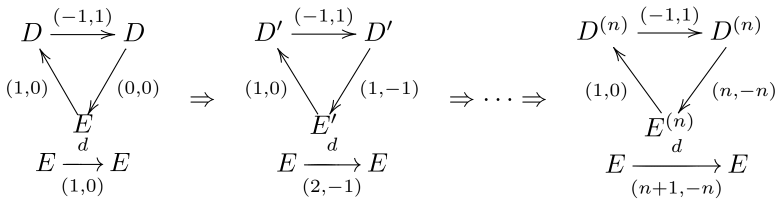


(1.46) Definition (Derived couple) Let (D, E, i, j, k) be an exact couple, we define the **derived couple** to be

$D' = \operatorname{im} i$, $E' = H(E, d)$, and i' induced by i, j' induced by ji^{-1}, and k' induced by k.



We will denote $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$ the n-th derived couple.



As a result, $D^{(n)} = \operatorname{im} i^n$, and $i^{(n)}$ induced by i, $j^{(n)}$ induced by ji^{-n}, and $k^{(n)}$ induced by k.

PROOF OF (1.42). Denote $E_r^{pq} = (E^{(r-1)})^{pq}$. Let us do some computation

$$\begin{aligned} \mathsf{B}_r^{pq} &= \operatorname{im} d^{(r-1)} = j^{(r-1)}(\operatorname{im} k^{(r-1)}) \\ &= j(i^{-(r-1)}(\ker i^{(r-1)})) = j(i^{-r}(0)), \end{aligned}$$

and

$$\begin{aligned} \mathsf{Z}_r^{pq} &= \ker d^{(r-1)} = (k^{(r-1)})^{-1}(\ker j^{(r-1)}) \\ &= (k^{(r-1)})^{-1}(\operatorname{im} i^{(r-1)}) = k^{-1}(i^r(D)) \cap \mathsf{Z}_{r-1}^{pq} \\ &= k^{-1}(i^r(D)) \cap k^{-1}(i^{r-1}(D)) \cap \mathsf{Z}_{r-2}^{pq} = k^{-1}(i^r(D)) \cap \mathsf{Z}_{r-2}^{pq} \\ &= \cdots = k^{-1}(i^r(D)). \end{aligned}$$

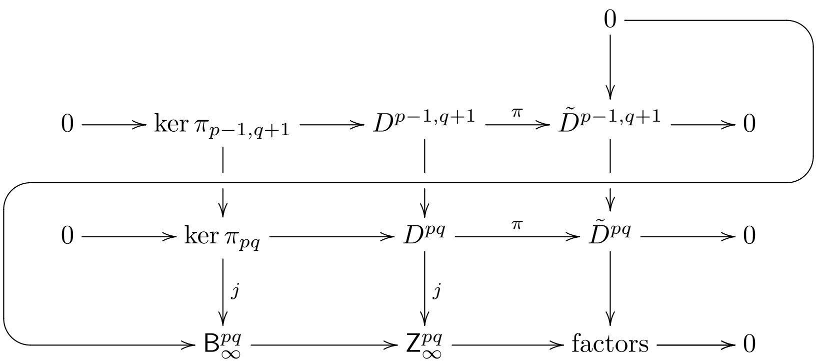
Then let us see the convergence

$$\begin{aligned} \mathsf{B}_\infty^{pq} &= \bigcup_r \mathsf{B}_r^{pq} = \bigcup_r j(i^{-r}(0)) & \qquad \mathsf{Z}_\infty^{pq} &= \bigcap_r \mathsf{Z}_r^{pq} = \bigcap_r k^{-1}(i^r(D)) \\ &= j\big(\bigcup_r i^{-r}(0)\big), & &= k^{-1}\big(\bigcap_r i^r(D)\big). \end{aligned}$$

Denote the image of D^{pq} in the H^{p+q} by $\tilde{D}^{pq}$. Now we assume the exact couple is lower bounded, then

$$\bigcap_r i^r(D) = 0, \qquad \bigcup_r i^{-r}(0) = \ker[D \to \tilde{D}].$$

Thus $\mathsf{Z}_\infty^{pq} = k^{-1}(0) = \ker k = \operatorname{im} j$. Now, consider the diagram

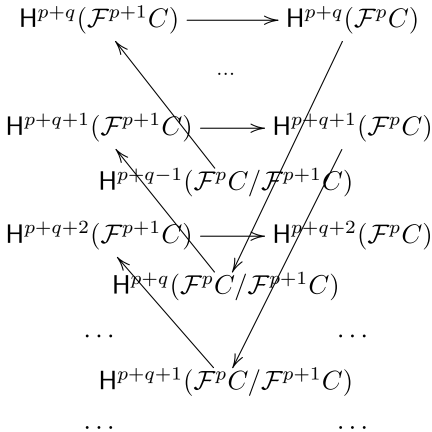


The exactness of leftmost row. If $x \in \ker \pi_{pq}$, with $j(x) = 0$, then $x = i(y)$ for some $y \in D^{p-1,q+1}$. But $x \in \ker \pi_{pq} = \bigcup_r i^{-r}(0)$, so $x \in \bigcup_r i^{-r}(0) = \ker \pi_{p-1,q+1}$. The rest exactness is clear.

(1.47) EXAMPLE Let C be a filtered complex of modules. Consider the short exact sequence

$$0 \to \mathcal{F}^{p+1}C \xrightarrow{i} \mathcal{F}^p C \to \mathcal{F}^p C/\mathcal{F}^{p+1}C \to 0$$

which gives rise to



It is an exact couple.

(1.48) Theorem *By above, we can define*

$$\mathsf{E}^{pq} = \mathsf{H}^{p+q}(\mathcal{F}^{p}C/\mathcal{F}^{p+1}C).$$

This coincides the spectral sequence of filtered complex (from the first page).

Proof. The differential maps are all induced by d, so it suffices to check the case $r = 1$. Let us see E_r^{pq}.

$$\mathsf{E}_{r+1}^{pq} = \frac{\mathsf{Z}_r^{pq}}{\mathsf{B}_r^{pq}} = \frac{k^{-1}(i^r(D))}{j(i^{-r}(0))}$$

Pick $x \bmod (\cdots) \in \mathsf{E}_1^{pq} = \mathsf{H}^p(\mathcal{F}^pC/\mathcal{F}^{p+1}C)$, with $x \in \mathcal{F}^pC$ with $dx \in \mathcal{F}^{p+1}C$,

$$\begin{aligned} x \bmod (\cdots) \in k^{-1}(i^r(D)) &\iff dx \in \operatorname{im} i^r + \operatorname{im}[\mathcal{F}^{p+1}C \xrightarrow{d} \mathcal{F}^{p+1}C] \qquad (*) \\ &\iff dx \in \mathcal{F}^{p+r+1}C + \operatorname{im}[\mathcal{F}^{p+1}C \xrightarrow{d} \mathcal{F}^{p+1}C] \\ &\iff x \in d^{-1}(\mathcal{F}^{p+r+1}C) + \mathcal{F}^{p+1}C. \end{aligned}$$

where the i in $(*)$ is by the definition of connected morphism that $k(x \bmod (\cdots)) = dx \in \mathsf{H}(\mathcal{F}^{p+1}C)$.

$$
\begin{aligned}
x \bmod (\cdots) \in j(i^{-r}(0)) &\iff \exists y \begin{cases} y \in \ker[\mathcal{F}^p C \xrightarrow{d} \mathcal{F}^p C], \\ i^r(y) \in \operatorname{im}[\mathcal{F}^{p-r} C \xrightarrow{d} \mathcal{F}^{p-r} C], \\ x \equiv y \bmod \mathcal{F}^{p+1} C. \end{cases} \\
&\iff \exists y \begin{cases} i^r(y) \in \operatorname{im}[\mathcal{F}^{p-r} C \xrightarrow{d} \mathcal{F}^{p-r} C], \\ x \equiv y \bmod \mathcal{F}^{p+1} C. \end{cases} \\
&\iff \exists y \begin{cases} y \in d(\mathcal{F}^{p-r} C), \\ x \equiv y \bmod \mathcal{F}^{p+1} C. \end{cases} \\
&\iff x \in d(\mathcal{F}^{p-r} C) + \mathcal{F}^{p+1} C.
\end{aligned}
$$

So

$$
\begin{cases} \mathsf{Z}_r^{pq} = \dfrac{\mathcal{F}^{p+1} C + d^{-1}(\mathcal{F}^{p+r+1} C) \cap \mathcal{F}^p C}{d^{-1}(\mathcal{F}^{p+1} C)} \\ \mathsf{B}_r^{pq} = \dfrac{\mathcal{F}^{p+1} C + d(\mathcal{F}^{p-r} C) \cap \mathcal{F}^p C}{d^{-1}(\mathcal{F}^{p+1} C)} \end{cases}
$$

exactly coincides what we defined for filtered complex (where it starts from 0-th page).

(1.49) Remarks on limit and colimit In the case of modules, we may use the fact that

$$
\varinjlim : \mathcal{C}^{\{0 \to 1 \to 2 \to \cdots\}} \longrightarrow \mathcal{C} \qquad \left(M_0 \xrightarrow{\rho_0} M_1 \xrightarrow{\rho_1} \cdots\right) \longmapsto \varinjlim_i M_i
$$

is exact. But in the case of modules, its dual

$$
\varprojlim : \mathcal{C}^{\{\cdots \to 2 \to 1 \to 0\}} \longrightarrow \mathcal{C} \qquad \left(\cdots \xrightarrow{\rho_2} M_1 \xrightarrow{\rho_1} M_0\right) \longmapsto \varprojlim_i M_i
$$

is not generally exact (but left exact).

As a result, $\varinjlim$ commutes with homology groups of complex. In particular, if C is some filtered complex,

$$
\varinjlim_p \mathsf{H}^n(\mathcal{F}^p C) = \mathsf{H}^n\Big(\bigcup \mathcal{F}^p C\Big).
$$

(1.50) Nothing general In these two sections, we made very narrow assumption on our underlying abelian category. Because we have infinite union and even the concrete construction of filtered limit. The point is, even though we have **Freyd-Mitchell embedding theorem** that

> any small abelian category admits an exact fully faithful embedding to some category of modules.

It only claims the embedding preserves kernel and cokernel. It also perserves finite sum since it can be defined by equations, and thus finite limit and colimit, but may fail for infinite (co)limit.

In conclusion, the interesting theory of "general spectral sequences" is not known, or, the theory of spectral sequences known to be interesting is only for modules. Where "interesting" means that we can construct spectral sequences from interesting objects. For usage of other case, we should make further assumption to reduce to modules, for example

- the colimit appearing are all finite, then everything follows from Freyd-Mitchell embedding theorem;

- the abelian category which can be embedded into R-Mod for some ring R preserving the colimit as we assumed.

Another fact is that unfortunately the dual category of R-Mod is not the case as remarked in (1.49).

1.5 Calculations

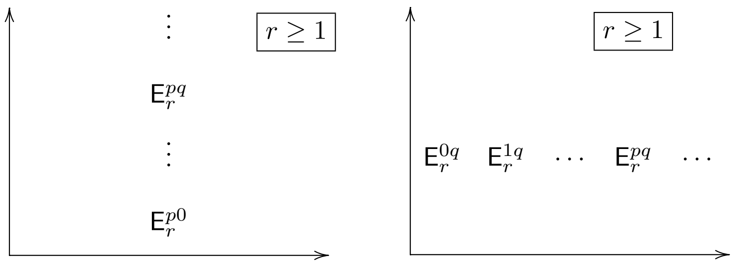
Before the applications, we firstly show some calculation of special type of spectral sequences.

For fixed n, let us call the line $\{(p,q) : p+q=n\}$ **codiagonal**.

(1.51) Trivial case *If* E *all vanish in* r*-th stage, then* $\mathsf{H}=0$.

(1.52) Easy cases *If E has at most one nonzero module in each codiagonal in r-th stage, then $E_r^{pq} = H^{p+q}$.*

(1.53) EXAMPLE (One column/row type) When there rests only one column or row, $\mathsf{H}^{p+q} = \mathsf{E}_r^{p+q}$.



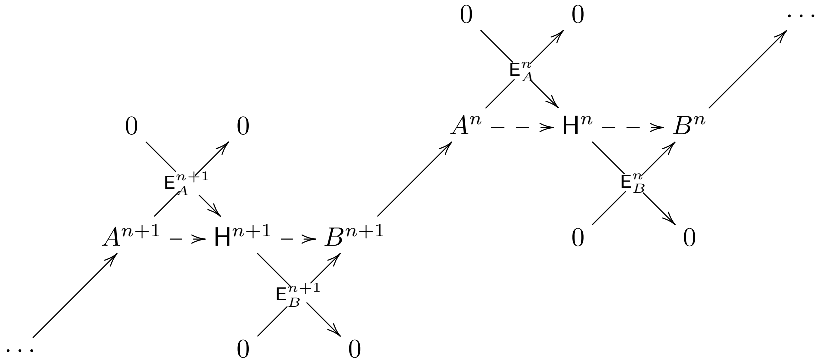
(1.54) Computable cases *If E_*^{**} is of homological type, and has at most two nonzero modules at each codiagonal in r-th stage, then we have only two terms in infinity stage. Assume that*

- *A^n, B^n are the only two modules on the codiagonal $\{(p,q) : p+q = n\}$ in some stage such that the map between A and B are all zero except $B^{n+1} \to A^n$ if some of them are not.*
- *the infinite term at A^n and B^n is E_A^n and E_B^n respectively;*
- *the submodule $\mathsf{H}_\dagger^n \subseteq \mathsf{H}^n$ is the filtration of H^*.*

Then there exist the following exact sequences

$$0 \to \mathsf{E}_B^{n+1} \to B^{n+1} \xrightarrow{d} A^n \to \mathsf{E}_A^n \to 0,$$
$$0 \to 0 \to \mathsf{H}_\dagger^n \to \mathsf{E}_A^n \to 0,$$
$$0 \to \mathsf{H}_\dagger^n \to \mathsf{H}^n \to \mathsf{E}_B^n \to 0.$$

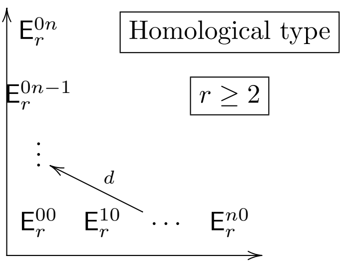
So we have



Now, we get a long exact sequence

$$\cdots \to B^{n+1} \xrightarrow{d} A^n \to \mathsf{H}^n \to B^n \xrightarrow{d} A^{n-1} \to \cdots .$$

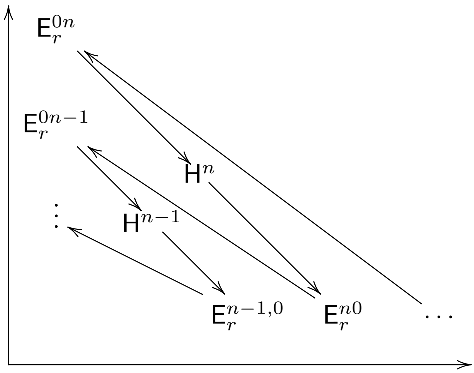
(1.55) EXAMPLE (Resolution Type) Especially these cases



where we have the following long exact sequence

$$\cdots \longrightarrow \mathsf{E}_r^{0n} \longrightarrow \mathsf{H}^n \longrightarrow \mathsf{E}_r^{n0} \xrightarrow{d} \mathsf{E}_r^{0,n-1} \longrightarrow \mathsf{H}^{n-1} \longrightarrow \mathsf{E}_r^{n-1,0} \cdots \cdots \mathsf{E}_r^{0,r-1} \longrightarrow \mathsf{H}^{r-1} \longrightarrow \mathsf{E}_r^{r-1,0} \xrightarrow{d=0} 0$$

Or we can illustrate it as

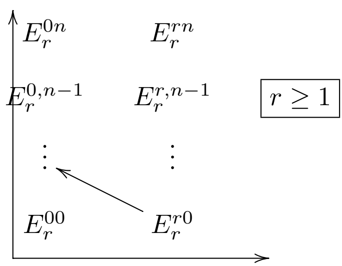


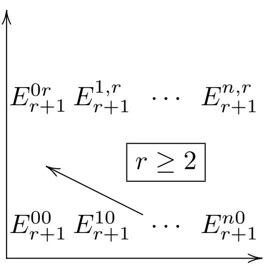
(1.56) EXAMPLE Dually, If E_*^{**} is of cohomological type, and has at most two nonzero modules at codiagonal in r-th stage, then we have only two terms in infinity stage. We have a long exact sequence

$$\cdots \to B^{n-1} \xrightarrow{d} A^n \to \mathsf{H}^n \to B^n \xrightarrow{d} A^{n+1} \to \cdots ,$$

with now B higher, and similar assumptions.

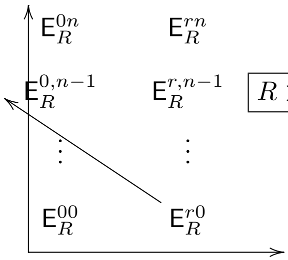
(1.57) EXAMPLE (Spherical type) So the following two cases are also useful.

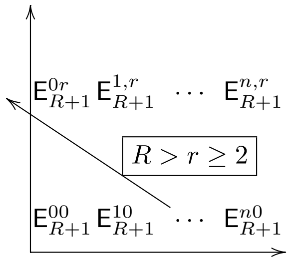




Actually, due to topological reason, they are called **of spherical base** and **of spherical fibre** respectively.

(1.58) EXAMPLE (Trivial Spherical type) Of course, if the gap between the rest two columns or rows are more narrow, then the long exact sequence splits, since $d = 0$ now.





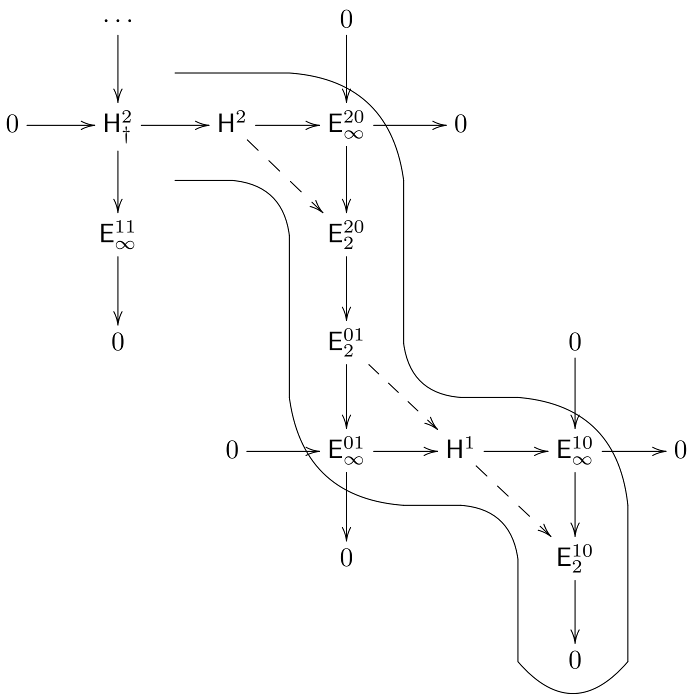
(1.59) Small terms *If E_*^{**} is a homological type spectral sequence in the first quadrant. Then, when $p - r < 0$, i.e. $p < r$, the arrow from E_r^{pq} is zero; when $q - r + 1 < 0$, i.e. $q + 1 < r$, the arrow to E_r^{pq} is zero. As a result, when $r = \max(p+1, q+2)$, $\mathsf{E}_r^{pq} = \mathsf{E}_\infty^{pq}$. Similarly, for cohomological type, when $r = \max(p+1, q+2)$, $\mathsf{E}_r^{pq} = \mathsf{E}_\infty^{pq}$.*

(1.60) EXAMPLE (First five terms) So it is useful to get the first several terms if we know E_2. In the homological type, we have

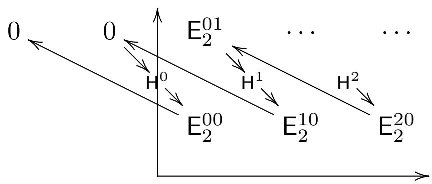
- For $n = 0$, then no problem, $\mathsf{E}_2^{00} = \mathsf{H}^0$.
- For $n = 1$, $\begin{cases} 0 \to \mathsf{E}_\infty^{20} \to \mathsf{E}_2^{20} \to \mathsf{E}_2^{01} \to \mathsf{E}_\infty^{10} \to 0, \\ 0 \to \mathsf{E}_\infty^{01} \to \mathsf{H}^1 \to \mathsf{E}_2^{10} \to 0. \end{cases}$
- For $n = 2$, we know that $\mathsf{E}_\infty^{02} = \mathsf{E}_2^{02}$ forms a topmost factor of H^2.
- So we have the following exact sequence

$$\mathsf{H}^2 \to \mathsf{E}_2^{20} \to \mathsf{E}_2^{01} \to \mathsf{H}^1 \to \mathsf{E}_2^{10} \to 0.$$

The diagram is

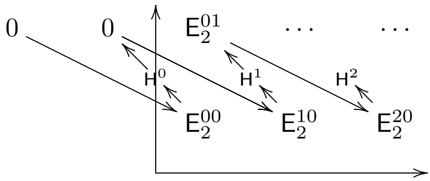


Or, in summary,



(1.61) EXAMPLE (First five terms) Dually, in the cohomological type, besides $E^{00}_2 = H^0$, we have the following exact sequence

$$0 \to E^{10}_2 \to H^1 \to E^{01}_2 \to E^{20}_2 \to H^2$$



Exercises

▶ **(1.62)** EXERCISE. Prove all the lemmate we do not prove (1.21), (1.22), (1.23), (1.25) and (1.45).

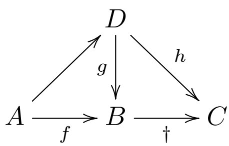
▶ **(1.63)** EXERCISE. Prove the derived exact couple (1.46) is exact couple by element-picking.

▶ **(1.64)** EXERCISE. Check the exactness claimed in (1.49).

▶ **(1.65)** EXERCISE. If we have the following diagram with the row exact,

show that it induces a short exact sequence

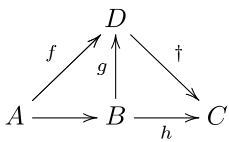
$$0 \to \operatorname{im} f \to \operatorname{im} g \xrightarrow{\dagger} \operatorname{im} h \to 0.$$



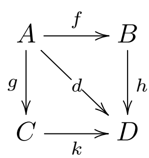
▶ **(1.66)** EXERCISE. If we have the following diagram with the row exact

show that it induces a short exact sequence

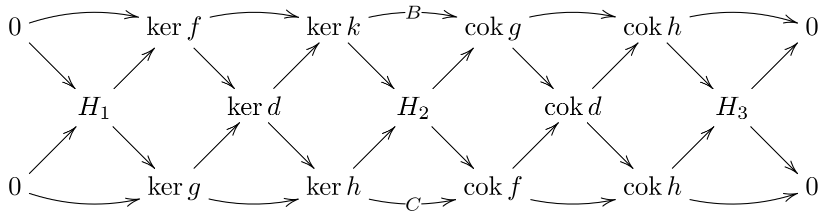
$$0 \to \operatorname{im} f \to \operatorname{im} g \xrightarrow{\dagger} \operatorname{im} h \to 0.$$



▶ **(1.67)** EXERCISE *([9], Theorem 3).* For



show that we have the following diagram



with each braid exact. Where H_1, H_2, H_3 are exactly the homology groups of

$$0 \to \overset{1}{A} \to \overset{2}{B \oplus C} \to \overset{3}{D} \to 0.$$

`Hint: View it as a double complex, then two of them follows from spectral sequence of double complex. The rest two is kernel-coke sequence for composition.`

▶ **(1.68)** Exercise *(Transgression).* Given a chain map between two chain complexes $C_\bullet \xrightarrow{f} D_\bullet$ which are both exact, show that

$$\mathsf{H}_{\bullet-1}(\ker f) \cong \mathsf{H}_{\bullet+1}(\operatorname{cok} f).$$

This isomorphism is called **transgression**. Show this reprove the snake lemma and five lemma. `Hint: View it as a bicomplex with only two rows.`

▶ **(1.69)** Exercise. Given a finite length sequence of exact complexes

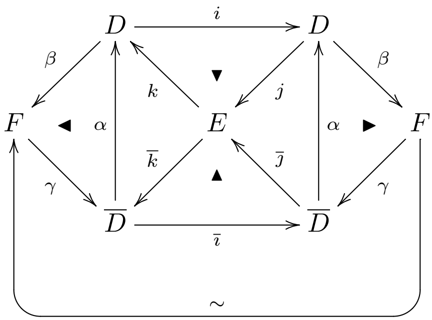
$$0 \to A^1_\bullet \to \cdots \to A^n_\bullet \to 0$$

then we get

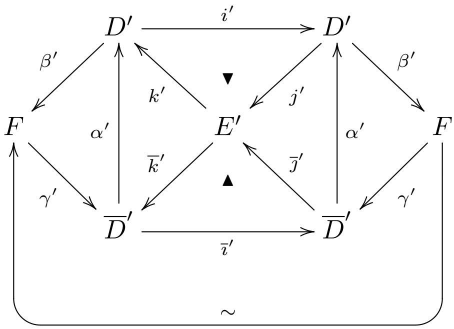
$$0 \to \mathsf{H}_\bullet(A^1) \to \cdots \to \mathsf{H}_\bullet(A^n) \to 0$$

if for all positions other than $\mathsf{H}_\bullet(A^i)$ is exact, show that it is exact.

▶ **(1.70)** Problem *(Rees system).* Historically, the following commutative diagram



where ▲'s all exact at each position, is called a **Rees system**. It is easy to see $j \circ k = \bar{\jmath} \circ k$. Show that the following diagram is also a Rees system



with the two ▲'s the derived couples, and α' induced by α, β' by βi^{-1}, γ' by γ.

▶ **(1.71)** PROBLEM *(Comparison)*. If we have a morphism $C \xrightarrow{f} \overline{C}$ between filtered complexes (f compatible with the filtration) where C and $\overline{C}$ are both lower bounded and exhaustive. If the induced morphism $\mathsf{E}_\bullet \xrightarrow{f} \overline{\mathsf{E}}_\bullet$ is isomorphic when $\bullet \gg 0$, show that $\mathsf{H}(C) \overset{f}{\cong} \mathsf{H}(\overline{C})$.

`Hint: By five or snake lemma,` $\mathcal{F}^p\mathsf{H}(C) \overset{f}{\cong} \mathcal{F}^p\mathsf{H}(\overline{C})$`, then take the union.`

▶ **(1.72)** PROBLEM. Here is some further remarks on the exactness of limit (1.49).

(1) Give an example where it fails to be right exact. `Hint: Consider` $0 \to p^n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$.

(2) If $(\cdots \to A_2 \to A_1 \to A_0)$ satisfies the following **Mittag-Leffler condition**

$$\forall j \gg 0, \quad \exists i > j, \quad \forall k > i, \quad \mathrm{im}[A_k \to A_j] = \mathrm{im}[A_i \to A_j].$$

That is, descending condition holds on the image at each position. Then $\varprojlim$ is exact at $0 \to A \to B \to C \to 0$.

(3) If we have a filtered complex C, show that

$$\varprojlim_p \mathsf{H}^p(C/\mathcal{F}^pC) = \mathsf{H}^p(\,\widehat{C}\,)$$

where $\widehat{C} = \varprojlim_p C/\mathcal{F}^pC$ the **completion**. `Hint:` $\varprojlim$ `commutes with kernel by left exactness. On image, in our case, the map between them are all surjective, thus of course they satisfy Mittag-Leffler condition. By (2),` $\varprojlim$ `commutes with` $\mathsf{H} = \ker / \mathrm{im}$.

▶ **(1.73)** Exercise. Consider the complex and the filtration

$$0 \to \mathbb{Z} \xrightarrow{n \mapsto 3n} \mathbb{Z} \to 0, \quad \cdots \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$$

Compute that $\mathsf{E}_1 = 0$, but $\mathsf{H}(C) \neq 0$.

Chapter 2

Applications in Topology

2.1 Mini-dictionary of Topology

Local coefficient homology For a topological space X, denote its fundamental groupoid by $\Pi_1(X)$. A local coefficient system $\mathcal{R}$ is an assignment of abelian group R_x to each point of x, with $\Pi_1(X)$ acting on R_x.

§ We can define

$$\mathrm{Sing}_n(X;\mathcal{R}) = \bigoplus_{x\in X} F_x^n \otimes \mathcal{R}_x,$$

where F_x^n is the free abelian group generated by the continuous map $\triangle^n \xrightarrow{f} X$, with the centre of $\triangle^n$ mapped to x. Then we can define the differential

$$[\triangle^n \xrightarrow{f} X] \otimes r_x \mapsto \sum_{i=0}^{n} (-1)^n [\triangle^{n-1} \xrightarrow{i\text{-th boundary}} \triangle^n \xrightarrow{f} X] \otimes \gamma_i(r_x)$$

where γ_i is the path $\big[[0,1] \xrightarrow{*} \triangle^n \xrightarrow{f} X\big]$, with $*$ mapping the centre of $\triangle^n$ to centre of i-th boundary. The homology group of $\mathrm{Sing}_n(X,\mathcal{R})$ is called **local coefficient homology**.

§ We can also define **local coefficient cohomology** by

$$\mathrm{Sing}_n(X;\mathcal{R}) = \bigoplus_{x\in X} \mathrm{Hom}(F_x^n;\mathcal{R}_x).$$

We can also define **relative local coefficient (co)homology**. One can prove homotopy invariance, excision etc.

§ If X admits a cellular structure, say

$$X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots$$

then we can define the cellular homology by

$$\mathrm{Cell}_n = \mathsf{H}_n(X^n, X^{n-1};\mathcal{R})$$

with the differential the connection morphism. An easy exercise is that $\mathsf{H}_n(\mathrm{Cell};\mathcal{R}) = \mathsf{H}_n(X;\mathcal{R})$, more precisely, $\mathsf{H}_\bullet(X^n, X^{n-1};\mathcal{R})$ vanishes only except n.

Let

$$\begin{cases} C^n & = \text{small open disks around centres of all } n\text{-cells} \\ D^n & = \overline{C^n} = \text{small closed disks around centres of all } n\text{-cells} \\ |B|^{n-1} & = B^n \setminus C^n \\ \|B\|^{n-1} & = B^n \setminus D^n \end{cases}$$

The standard computation

$$\begin{aligned} \mathsf{H}_n(X^n, X^{n-1};\mathcal{R}) &= \mathsf{H}_n\big(B^n,\, |B|^{n-1};\mathcal{R}\big) \\ &= \mathsf{H}_n\big(B^n \setminus \|B\|^{n-1},\, |B|^{n-1} \setminus \|B\|^{n-1};\mathcal{R}\big) \\ &= \mathsf{H}_n\big(D^n,\, C^n;\mathcal{R}\big) \\ &= \textstyle\bigoplus_{\text{all } n\text{-cells } c} \mathsf{H}_n\big(\mathbb{D}^n,\, \mathbb{S}^{n-1};\mathcal{R}_c\big) \\ &= \textstyle\bigoplus_{\text{all } n\text{-cells } c} \mathsf{H}_p(\mathbb{D}^p, \mathbb{S}^p) \otimes \mathsf{H}_q\big(F_c\big) \end{aligned}$$

where $\mathcal{R}_c$ stands for abelian group of the centre of c. Then by chasing through the isomorphisms, the differential is given by $\partial \otimes \gamma_i$, where γ_i the the path from the center of n-cell to the target $(n-1)$-cell.

§ When the action of Π_1 is trivial, for example, the space is simply connected, we say $\mathcal{R}_x$ is constant. In particular, when X is path connected then the local coefficient (co)homology coincide with the (co)homology of coefficient of any $\mathcal{R}_{x_0}$ for $x_0 \in X$.

Fibre Generally, a **bundle** is a surjective between topological space $E \xrightarrow{\xi} B$. We call E the **total space**, B the **base space**. For any $x \in B$, $\{x \in E : \xi(x) = B\}$ is called the **fibre** at x. For convention, we will write $E = E(\xi)$ and $B = B(\xi)$, and $E(\xi)_x$ for the fibre at x. For any subset $U \subseteq B$, we denote $E|_U = \xi^{-1}(U)$.

§ In topology, we say a bundle $\xi : E \to B$ is a **fibration** if it satisfy **homotopy lifting property (HLP)** for any space X, more precisely,

For any α and λ makes the square commutes, there exists Λ making two triangle commute.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & E \\ {\scriptstyle =X\times\{0\}}\downarrow & \overset{\Lambda}{\nearrow} & \downarrow{\scriptstyle \xi} \\ X \times [0,1] & \xrightarrow[\lambda]{} & B \end{array}$$

For any $x \in B$, denote F the fibre at x. Pick some $y \in F_x \subseteq E$, then it induces long exact sequence for homotopy group

$$\cdots \to \pi_1(B, x) \to \pi_0(F, y) \to \pi_0(E, y) \to \pi_0(B, x) \to 1.$$

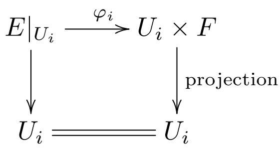
Clearly, covering is a standard example of fibration.

If B is path connected and $x, y \in B$ are two points, by the HLP, for any path $x \xrightarrow{\gamma} y$, there is a transformation between fibres $F_x \to F_y$ at x and y, called **change of fibre** through γ. If γ_1 and γ_2 are vertices-fixing-homotopic, then the resulting transformation is homotopic. In particular, in this case, all fibres are of the same homotopy type, we will write

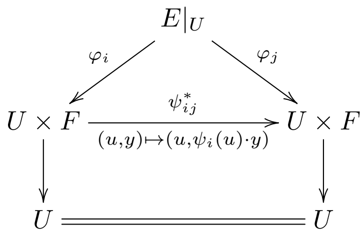
$$F \to E \to B.$$

§ In geometry, we starts from some topological group G. We say $E \xrightarrow{\xi} B$ is a **G-bundle** of fibre F, if F is a G-set, and there is an open covering $\mathfrak{U} = \{U_i\}$ such that

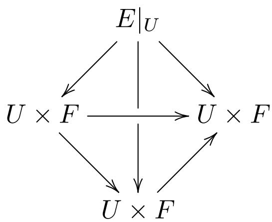
For each i, there is a homeomorphism φ_i make the square commute.



For each i, j, denote $U = U_i \cap U_j$, there is a continuous map $U \xrightarrow{\psi_{ji}} G$ such that the square commute.



For each i, j, k, then $\psi_{kj}\psi_{ji} = \psi_{ki}$. In particular, the diagram commutes where $U = U_i \cap U_j \cap U_k$.



We will call G the **structure group**. It is easy to define the morphisms, isomorphisms between fibre bundles. For example, when $G = \mathrm{GL}_n$, F is an n dimensional space, then it is nothing but **vector bundle**.

§ If $F = G$, then we call ξ a **G-principal bundle**. For any topological group G, there exists (called **Milnor construction**) a G-principal bundle $EG \to BG$ with EG contractible. Then for any CW-complex X, the equivalence class of G-principal bundle over X is classified by the homotopy class from X to BG, say

$$[X, BG] \longrightarrow G\text{-}\mathsf{Prin}_X \qquad [X \xrightarrow{f} BG] \longmapsto f^*\xi.$$

The fibre bundle $EG \xrightarrow{\xi} BG$ is called **classifying bundle**.

§ For a bundle $E \xrightarrow{\xi} B$, and a map $B' \xrightarrow{f} B$, then we can define

$$E' = \{(x, y) \in B' \times E : f(x) = \xi(y)\} \xrightarrow{\text{projection}} B'$$

called **pull back** by f, and denoted by $f^*\xi$. Note that the pull back shares the same fibre. An exercise is to prove when ξ is fibration, then so is its pull back.

2.2 The Leray–Serre Spectral sequence

The purpose is to calculate the (co)homology group $\mathsf{H}(E; R)$ for some commutative ring R with E in the fibration

$$F \hookrightarrow E \to B.$$

We assume B to be path-connected. The fibre change defines a system of local coefficient $\mathcal{H}(F)_x = \mathsf{H}(F)_x$.

(2.1) Leray–Serre *Assume we have a fibration $E \xrightarrow{\pi} B$ with fibre F, then there is a spectral sequence E with*

$$\mathsf{E}^2_{pq} = \mathsf{H}_p(B; \mathcal{H}_q(F; R))$$

converging to $\mathsf{H}_\bullet(B; R)$.

Proof. We will omit R for simplicity.

We use the technique of Cw-approximation. Let $B' \to B$ be a Cw-approximation, i.e. B' is Cw complex, and it induces isomorphism between homotopy groups. Let $E' \to B'$ be the pull back of $E \to B$ (of the same fibre). By long exact sequence of fibration, $E' \to E$ is weakly equivalence, thus induces the isomorphism between homology.

So it suffices to deal with when B is a Cw complex, say filtered by

$$B^0 \subseteq B^1 \subseteq \cdots \subseteq B^p \subseteq \cdots \qquad \subseteq B$$

Denote $E^i = E|_{B^i} = \pi^{-1}(B_i)$. Now, $\mathrm{Sing}_\bullet(E)$ is filered by this lower bounded and exhaustive filtration

$$\mathrm{Sing}_\bullet(X^0) \subseteq \mathrm{Sing}_\bullet(X^1) \subseteq \cdots \subseteq \mathrm{Sing}_\bullet(X^p) \subseteq \cdots$$

So it induces a spectral sequence E with

$$\mathsf{E}^0_{pq} = \operatorname{Sing}_{p+q}(X^p)/\operatorname{Sing}_{p+q}(X^{p-1}).$$

Then,

$$\mathsf{E}^1_{pq} = \mathsf{H}_{p+q}(X^p, X^{p-1}).$$

By our construction d inside E^1 is given by

$$\mathrm{d} : \mathsf{H}_{p+q}(X^p, X^{p-1}) \xrightarrow{\text{connecting map}} \mathsf{H}_{p+q-1}(X^{p-1}, X^{p-2}).$$

Let

$$\begin{cases} C^p & = \text{small open disks around centres of all } p\text{-cells} \\ D^p & = \overline{C^p} = \text{small closed disks around centres of all } p\text{-cells} \\ |B|^{p-1} & = B^p \setminus C^p \\ \|B\|^{p-1} & = B^p \setminus D^p \end{cases}$$

Then

$$\begin{aligned} \mathsf{H}_p(B^p, B^{p-1}) &= \mathsf{H}_p\big(B^p,\, |B|^{p-q}\big) \\ &= \mathsf{H}_p\big(B^p \setminus \|B\|^{p-1},\, |B|^{p-q} \setminus \|B\|^{p-1}\big) \\ &= \mathsf{H}_{p+q}\big(D^p,\, C^p\big) \\ \mathsf{H}_{p+q}(X^p, X^{p-1}) &= \mathsf{H}_{p+q}\big(\pi^{-1}(B^p),\, \pi^{-1}(B^{p-1})\big) \\ &= \mathsf{H}_{p+q}\big(\pi^{-1}(B^p),\, \pi^{-1}(|B|^{p-1})\big) \\ &= \mathsf{H}_{p+q}\big(\pi^{-1}(B^p) \setminus \pi^{-1}(\|B\|^{p-1}),\, \pi^{-1}(|B|^{p-1}) \setminus \pi^{-1}(\|B\|^{p-1})\big) \\ &= \mathsf{H}_{p+q}\big(\pi^{-1}(D^p),\, \pi^{-1}(C^p)\big) \end{aligned}$$

Now every isomorphism commutes with connecting map.

$$\begin{aligned} \mathsf{H}_p(B^p, B^{p-q}) &= \textstyle\bigoplus_{\text{all } n\text{-cells } c} \mathsf{H}_{p+q}\big(\mathbb{D}^p, \mathbb{S}^p\big) \\ \mathsf{H}_{p+q}(X^p, X^{p-1}) &= \textstyle\bigoplus_{\text{all } n\text{-cells } c} \mathsf{H}_{p+q}\big(\mathbb{D}^p \times F_c, \mathbb{S}^p \times F_c\big) \\ &= \textstyle\bigoplus_{\text{all } n\text{-cells } c} \mathsf{H}_p(\mathbb{D}^p, \mathbb{S}^p) \otimes \mathsf{H}_q\big(F_c\big) \end{aligned}$$

Here we use some homotopy induced by HLP. So under the isomorphism, d is given exactly by the local coefficient homology complex. As a result,

$$\mathsf{E}^2_{pq} = \mathsf{H}^p(B, \mathcal{H}^q(F, R)).$$

The proof is complete.

(2.2) Leray–Serre *Assume we have a fibration $E \xrightarrow{\pi} B$ with fibre F, then there a spectral sequence E with*

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(B; \mathcal{H}^q(F; R))$$

which converges to $\mathsf{H}^\bullet(B; R)$.

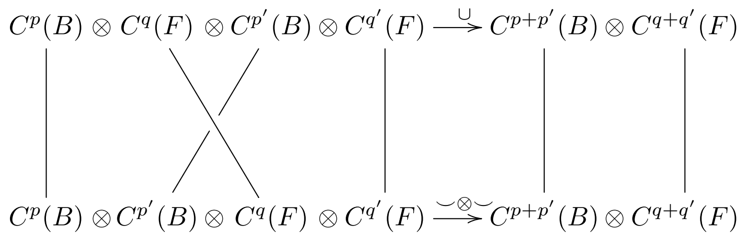
The next problem is about the cup product and cap product.

(2.3) Corollary *Assume we have a fibration $E \xrightarrow{\pi} B$ with fibre F, then the cup product $\cup$ induced from X on E_2^{pq} is given by*

$$\cup : \mathsf{E}_2^{pq} \times \mathsf{E}_2^{p'q'} \longrightarrow \mathsf{E}_2^{p+p',q+q'} \qquad (x, y) \longmapsto (-1)^{p'q} x \smile y$$

where $\smile$ is the cup product for $\mathsf{H}^p(B; \mathcal{H}^q(F; R))$.

PROOF. Note that $\mathsf{H}^p(B; \mathcal{H}^q(F; R))$ are from $C^p(B) \otimes C^q(F)$, so



one need $* = (-1)^{p'q}$ to adjust the sign by Koszul convention (1.14).

(2.4) Corollary *Assume we have a fibration $E \xrightarrow{\pi} B$ with fibre F, then the cap product $\cap$ induced from X on E^2_{pq} is given by*

$$\cap : \mathsf{E}_2^{pq} \times \mathsf{E}^2_{p'q'} \longrightarrow \mathsf{E}^2_{p'-p,q'-q} \qquad (x, y) \longmapsto (-1)^{p'q} x \frown y$$

where $\frown$ is the cap product for $\mathsf{H}^p(B; \mathcal{H}^q(F; R))$.

$\mathsf{H}^0(B, \mathsf{H}^2(F; R))$	$\mathsf{H}^1(B, \mathsf{H}^2(F; R))$	$\mathsf{H}^2(B, \mathsf{H}^2(F; R))$
$\mathsf{H}^0(B, \mathsf{H}^1(F; R))$	$\mathsf{H}^1(B, \mathsf{H}^1(F; R))$	$\mathsf{H}^2(B, \mathsf{H}^1(F; R))$
$\mathsf{H}^0(B, \mathsf{H}^0(F; R))$	$\mathsf{H}^1(B, \mathsf{H}^0(F; R))$	$\mathsf{H}^2(B, \mathsf{H}^0(F; R))$

(2.5) EXAMPLE (Gysin sequence) Let $S^n \to E \to B$ be a sphere bundle. If B is orientable, since automorphism of $\mathsf{H}^\bullet(S^n)$ is $\{\pm 1\}$, then $\mathcal{H}^\bullet(S^n)$ is constant.

$$
\begin{array}{l|ccccc}
 & \mathsf{H}^0(B;\mathsf{H}^n(S^n)) & \mathsf{H}^1(B,\mathsf{H}^n(S^n)) & \mathsf{H}^2(B;\mathsf{H}^n(S^n)) & & \\
 & \vdots & \ddots & \vdots & & \\
 & 0 & 0 & \ddots & & \\
\searrow & \vdots & \vdots & \vdots & \ddots & \\
 & \mathsf{H}^0(B) & \mathsf{H}^1(B) & \mathsf{H}^2(B) & \cdots & \mathsf{H}^n(B) \\
\hline
\end{array}
$$

So we get $\mathsf{H}^i(B) = \mathsf{H}^i(E)$ for $0 \le i \le n-1$, and exact sequence

$$0 \to \mathsf{H}^n(B) \to \mathsf{H}^n(E) \to \mathsf{H}^0(B) \xrightarrow{*} \mathsf{H}^{n+1}(B) \to \mathsf{H}^{n+1}(E) \to \mathsf{H}^1(B) \xrightarrow{*} \mathsf{H}^{n+2}(B) \to$$

The map $\mathsf{H}^i(B) \xrightarrow{*} \mathsf{H}^{i+n+1}(B)$ is actually given by a cup product. Let γ be the image of $1 \in \mathsf{H}^0(B)$ in $\mathsf{H}^{n+1}(B)$. Let $[x] \in \mathsf{H}^i(B;\mathsf{H}^n(S^n))$, say presented by $x \otimes 1_n$, then

$$
\begin{aligned}
d(x \otimes 1_n) &= d(x \otimes 1_0 \smile 1 \otimes 1_n) \\
&= d(x \otimes 1_0) \smile 1 \otimes 1_n \pm x \otimes 1_0 \smile d(1 \otimes 1_n) \\
&= \pm x \otimes 1_0 \smile \gamma
\end{aligned}
$$

its image in $\mathsf{H}^{i+n+1}(B;\mathbb{Z})$ is $\pm x \smile \gamma$. The γ is known as **Euler class**.

(2.6) EXAMPLE (Leray-Hirsch) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a filtration with B path-connected. Assume $\mathsf{H}^\bullet(F;R)$ is a finitely generated free R-module for each $\bullet$, and there is a subset $\{c_i\} \subseteq \mathsf{H}^\bullet(E)$ such that $i^*(c_i)$ forms an R-basis for $\mathsf{H}^\bullet(F)$. Then by **Leray-Hirsch theorem**,

$$L : \mathsf{H}^\bullet(B;R) \otimes \mathsf{H}^\bullet(F;R) \longrightarrow \mathsf{H}^\bullet(E;R) \qquad b \otimes i^*(c_i) \longmapsto p^*(b) \smile c_j$$

is an isomorphism.

Now, by the HLP, the property holds for any fibre, and the local system is constant. So

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(B;\mathcal{H}(F;R)) = \mathsf{H}^p(B;R) \otimes_R \mathsf{H}^q(F;R).$$

By our proof, locally, the isomorphism $\mathsf{E}_2^{pq} = \mathsf{H}^p(B; \mathcal{H}(F; R))$ is given by product, so $\mathsf{E}_\infty^{pq} \Longrightarrow \mathsf{H}^\bullet(E)$ is induced by L above. So $\mathsf{E}_2^{pq} = \mathsf{E}_\infty^{pq}$ yet.

2.3 The Eilenburg–Moore Spectral sequence

The purpose is to calculate the cohomology group of pull back

Where $E \xrightarrow{\pi} B$ is a fibration, $X \xrightarrow{f} B$ a map, and E_f is the pull back.

$$\begin{array}{ccc} E_f & \overset{\tilde{f}}{\dashrightarrow} & E \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \underset{f}{\longrightarrow} & B \end{array}$$

We will use some convention of differential algebra.

(2.7) Definition (Cross product) Consider

$$\times_f : \mathrm{Sing}^\bullet(X) \otimes_{\mathrm{Sing}^\bullet(B)} \mathrm{Sing}^\bullet(E) \longrightarrow \mathrm{Sing}^\bullet(E_f) \qquad x \otimes y \longmapsto \xi^* x \smile \tilde{f}^* y$$

it induces **cross product**

$$\mathsf{H}^\bullet(X) \otimes_{\mathsf{H}^\bullet(B)} \mathsf{H}^\bullet(X) \xrightarrow{\times_f} \mathsf{H}^\bullet(E).$$

(2.8) Eilenburg–Moore *Let $E \to B$ be a fibration over simply connected space B, and $X \xrightarrow{f} B$ be some map. Denote the pull back of ξ through f by E_f. Then there exists a spectral sequence E such that*

$$\mathsf{E}_2^{pq} = \textit{degree } p \textit{ part of } \mathrm{Tor}_q^{\mathsf{H}^\bullet(B;R)}(\mathsf{H}^\bullet(X;R), \mathsf{H}^\bullet(E;R))$$

and E converges to $\mathsf{H}^\bullet(E_f; R)$.

Proof. By Leray-Serre spectral sequence, we have a filtration $\mathcal{F}$ over $\mathrm{Sing}^\bullet(E_f)$ and $\mathrm{Sing}^\bullet(X)$. Note that they are also $S^\bullet(E)$ and $S^\bullet(B)$-module, so one can take resolution with compatible with filtration

$$\begin{array}{ccc} \mathcal{P}_*^\bullet(E_f) & \xrightarrow{\sim} & \mathrm{Sing}^\bullet(E_f) \\ \uparrow & & \uparrow \\ \mathcal{P}_*^\bullet(X) & \xrightarrow[\sim]{} & \mathrm{Sing}^\bullet(X) \end{array}$$

So it induces

$$\begin{array}{ccc} \mathcal{P}^{\bullet}_{*}(E_f) & \xrightarrow{\sim} & \mathrm{Sing}^{\bullet}(E_f) \\ \uparrow * & & \uparrow \times \\ \mathcal{P}^{\bullet}_{*}(X) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E) & \longrightarrow & \mathrm{Sing}^{\bullet}(X) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E) \end{array}$$

It is funny that over the total of $\mathcal{P}^{\bullet}_{*}(X) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E)$, there are three filtrations, the column, the row, and the mixed filtration induced by $\mathcal{F} \otimes \mathcal{F}$. Consider the spectral sequence induced by mixed filtration on

$$\left(\mathrm{Tot}_{\bullet *}\ \mathcal{P}^{\bullet}_{*}(X)\right) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E) \xrightarrow{*} \mathrm{Tot}_{\bullet *}\ \mathcal{P}^{\bullet}_{*}(E_f) \xrightarrow{\sim} \mathrm{Sing}^{\bullet}(E_f).$$

Since $\mathcal{P}^{\bullet}_{*}(X)$ is free $\mathrm{Sing}^{\bullet}(B)$-module, so the mixed filtration of left hand side can be computed. Also note that $\mathrm{Tot}\,\mathcal{P}^{\bullet}_{*}(X)$ is weakly equivalence to $\mathrm{Sing}^{\bullet}(X)$, On E_1-level, it is

$$\left(\mathsf{H}^{\bullet}(X) \underset{\mathsf{H}^{\bullet}(B)}{\otimes} \mathsf{H}^{\bullet}(B^{\bullet}, B^{\bullet-1}) \otimes H^{q}(F)\right)_{\bullet=p} \to \mathsf{H}^{p}(X^{p}, X^{p+1}) \otimes \mathsf{H}^{q}(F)$$

On E_2-level, it is

$$\mathsf{H}^{\bullet}(X) \underset{\mathsf{H}^{\bullet}(B)}{\otimes} \mathsf{H}^{\bullet}(B, H^{q}(F)) \xrightarrow{\times} \mathsf{H}^{\bullet}(X, \mathsf{H}^{q}(F))$$

an isomorphism. As a result, the homology group of

$$\mathrm{Tot}\ \mathcal{P}^{\bullet}_{*}(X) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E)$$

is exactly $\mathsf{H}(E_f)$. It is also a double complex, so

$$\begin{array}{l} \left(\mathrm{Tot}\ \mathcal{P}^{\bullet}_{*}(X) \underset{\mathrm{Sing}^{\bullet}(B)}{\otimes} \mathrm{Sing}^{\bullet}(E)\right)_{*=p,\bullet=q} \\ \quad \Big|^{q} \left(\mathrm{Tot}\ \mathcal{P}^{\bullet}_{*}(X) \underset{\mathsf{H}^{\bullet}(B)}{\otimes} \mathsf{H}^{\bullet}(E)\right)_{*=p,\bullet=q} \\ \qquad \Big|^{p} \left(\mathrm{Tor}_{q}^{\mathsf{H}^{\bullet}(B)}\left(\mathsf{H}^{\bullet}_{*}(X), \mathsf{H}^{\bullet}(E)\right)\right)_{*=p}. \end{array}$$

The proof is complete.

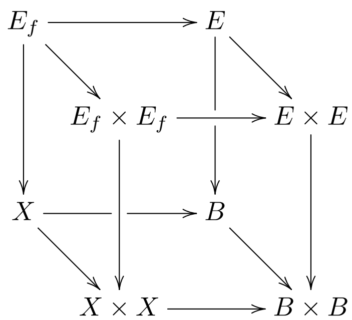
(2.9) Remark Actually, we have

$$\mathrm{Tor}^{\mathrm{Sing}^\bullet(B)}(\mathrm{Sing}^\bullet(X), \mathrm{Sing}^\bullet(E)) = \mathsf{H}^\bullet(E_f; R)$$

by computing $\left(\mathrm{Tot}_{\bullet *}\, \mathcal{P}^\bullet_*(X)\right) \otimes_{\mathrm{Sing}^\bullet(B)} \mathrm{Sing}^\bullet(E)$ directly.

(2.10) Corollary *The natural product of* Tor *on* E_2^{pq} *coincides the cup product.*

PROOF. Consider



and we can take the resolution of $\mathsf{H}^\bullet(B) \otimes \mathsf{H}^\bullet(B)$ to be $\mathcal{P} \otimes \mathcal{P}$ in the proof.

(2.11) Remark By a similar argument, if X and Y are G-spaces with X a principal bundle, then

$$\mathrm{Tor}^{\mathrm{Sing}_\bullet(G)}(\mathrm{Sing}_\bullet(X), \mathrm{Sing}_\bullet(Y)) = \mathrm{Sing}_\bullet(X \times_G Y).$$

and there exists spectral sequence E with

$$\mathsf{E}^2 = \mathrm{Tor}^{\mathsf{H}(G)}(\mathsf{H}(X), \mathsf{H}(Y))$$

converges to $\mathsf{H}(X \times_G Y)$.

2.4 The Bockstein Spectral Sequences

Let X be a topological space. Then the short exact sequence

$$0 \to \mathrm{Sing}_\bullet(X) \xrightarrow{\cdot n} \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(X;\mathbb{Z}/n) \to 0$$

gives rise to an exact couple

$$\begin{array}{ccc} \mathsf{H}_\bullet(X) & \xrightarrow{\ i\ } & \mathsf{H}_\bullet(X) \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & \mathsf{H}_\bullet(X;\mathbb{Z}/n) & \end{array}$$

To make it a bigraded exact couple, we consider

$$D_{pq} = \mathsf{H}_{p+q}(X), \text{ and } E_{pq} = \mathsf{H}_{p+q}(X;\mathbb{Z}/n).$$

(2.12) Definition (Bockstein operator) The first page differential is given by $j \circ k$, or

$$\div n : \mathsf{H}_\bullet(X,\mathbb{Z}/n) \longrightarrow \mathsf{H}_\bullet(X,\mathbb{Z}/n) \qquad [x] \longmapsto [y]$$

where $x \in \mathrm{Sing}_\bullet(X)$ with $dx = ny$ for some $y \in \mathrm{Sing}_\bullet(Y)$. This is called **Bockstein operator**.

(2.13) Proposition *The Bockstein operator is the connected morphism induced by*

$$0 \to \mathrm{Sing}_\bullet(X;\mathbb{Z}/n) \xrightarrow{\cdot n} \mathrm{Sing}_\bullet(X;\mathbb{Z}/n^2) \to \mathrm{Sing}_\bullet(X;\mathbb{Z}/n) \to 0$$

PROOF. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & \mathbb{Z}/n^2 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \end{array}$$

(2.14) EXAMPLE The Bockstein operator for $n = 2$ is exactly the first steenrod operator Sq^1.

(2.15) Bockstein Spectral Sequences *For a topological space X with $\mathsf{H}_\bullet(X)$ a finite generated abelian group for each $\bullet$, there is a spectral sequence E with*

$$\begin{cases} \mathsf{E}^1_{pq} & = \mathsf{H}_{p+q}(X; \mathbb{Z}/n), \\ \mathsf{E}^2_{pq} & = \mathsf{H}_{p+q}(\mathsf{H}_\bullet(X; \mathbb{Z}/n), \div n) \\ \mathsf{E}^\infty_{pq} & = \big(\text{free part of } \mathsf{H}^{p+q}(X)\big) \otimes \mathbb{Z}/n \end{cases}$$

"converging" to $S^{-1}\mathsf{H}_\bullet(X)$, the localization with respect to $S = \{1, n, n^2, \ldots\}$. Here "convergence" means there is a filtration $\mathcal{F}$ on $H_\bullet = \mathsf{H}_\bullet(X)_n$, with $\bigcap \mathcal{F}H = 0$, and $\bigcup \mathcal{F}H = H$, and $\mathcal{F}_pC_{p+q}/\mathcal{F}_{p-1}C_{p+q} \cong E^\bullet_{pq}$.

PROOF. Recall the proof of (1.42). The crucial step is

$$\bigcap_r i^r(D) = 0, \qquad \bigcup_r i^{-r}(0) = \ker[D \to \tilde{D}],$$

which holds when the topological space is of finite type, i.e. $\mathsf{H}_\bullet(X)$ is a finite generated abelian group. So the proof still holds, if we loose the assumption of convergence.

The rest is some computation,

$$\begin{aligned} \varprojlim \left[\cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots\right] &= \varprojlim \left[\cdots \xrightarrow{\subseteq} \mathbb{Z} \xrightarrow{\subseteq} \tfrac{1}{n}\mathbb{Z} \xrightarrow{\subseteq} \cdots\right] \\ &= \textstyle\bigcup_k \dfrac{1}{n^k}\mathbb{Z} = S^{-1}\mathbb{Z} \end{aligned}$$

Then filtered limit commutes tensor product, so

$$\varprojlim \left[\cdots \xrightarrow{i} D \xrightarrow{i} \cdots\right] = S^{-1}\mathsf{H}_\bullet(X).$$

Also, in the proof,

$$\begin{aligned} \mathsf{E}^\infty_{pq} &= \operatorname{im}\big[\mathsf{H}_\bullet(X) \to \mathsf{H}_\bullet(X)_n\big] \otimes \mathbb{Z}/n \\ &= \big(\text{free part of } \mathsf{H}^{p+q}(X)\big) \otimes \mathbb{Z}/n. \end{aligned}$$

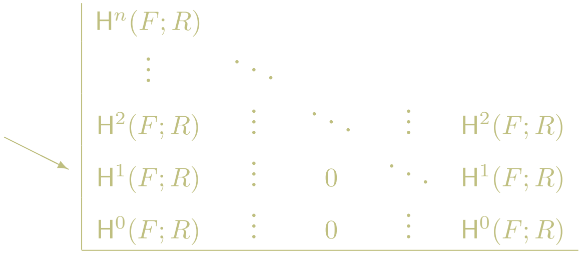
The proof is complete.

Exercises

▶ **(2.16)** PROBLEM *(Wang sequence).* Let $F \to E \to S^n$ be bundle over sphere with $n \neq 0, 1$. Show that there is the following **Wang sequence**

$$0 \to \mathsf{H}^{n-1}(F; R) \xrightarrow{*} \mathsf{H}^0(F; R) \to \mathsf{H}^n(E; R) \to \mathsf{H}^n(F; R) \xrightarrow{*} \mathsf{H}^1(F; R) \to \mathsf{H}^{n+1}(E; R$$

and $\mathsf{H}^i(F; R) = \mathsf{H}^i(E; R)$ for $0 \leq i \leq n-2$. `Hint: Consider`



`Note that the` $\Theta = *$ `satisfies` $\Theta(x \smile y) = \Theta(x) \smile y + (-1)^{|x|(n-1)} x \smile \Theta y$.

Chapter 3

Applications in Algebra

3.1 Mini-dictionary of Algebra

Hyper resolutions Let $\mathcal{A}$ be an abelian category of enough projectives. Let $C_\bullet$ be a complex of $\mathcal{A}$.

§ A double complex P_{**} is said to be a **Hyper resolution** if for each column i,

$$\begin{array}{ccc} P_{i\bullet} & \mathrm{im}[P_{i-1,\bullet} \to P_{i\bullet}] & \mathsf{H}[P_{i-1,\bullet} \to P_{i\bullet} \to P_{i+1,\bullet}] \\ \downarrow & \downarrow & \downarrow \\ C_i & \mathrm{im}[C_{i-1} \to C_i] & \mathsf{H}[C_{i-1} \to C_i \to C_{i+1}] \end{array}$$

are all projective resolutions.

§ As a result, the following are automatically projective resolution.

$$\begin{array}{cc} \ker[P_{i,\bullet} \to P_{i+1,\bullet}] & \mathrm{cok}[P_{i-1,\bullet} \to P_{i\bullet}] \\ \downarrow & \downarrow \\ \ker[C_i \to C_{i+1}] & \mathrm{cok}[C_{i-1} \to C_r] \end{array}$$

For each row,

$$\cdots \to P_{*,q+1} \to P_{*,q} \to P_{*,q-1} \to \cdots$$

is direct sum of $0 \to \bullet = \bullet \to 0$, and $0 \to \bullet \to 0$.

§ By horseshoe lemma, there always Hyper resolution for all complexes.

Group (co)homology Let G be a discrete group. Consider the category of $G\text{-}\mathsf{Mod} = \mathbb{Z}[G]\text{-}\mathsf{Mod}$. For any left G-module M, we can view it as a right module by $x \cdot g = g^{-1}x$. So we will not distinguish left and right modules, but distinguish two-side and one-side.

§ For a G-module M, we define the **group cohomology** and **homology** by

$$\mathsf{H}^i(G; M) = \mathrm{Ext}^i_G(\mathbb{Z}, M), \quad \mathsf{H}_i(G; M) = \mathrm{Tor}^G_i(\mathbb{Z}, M),$$

where $\mathbb{Z}$ is the trivial module. Note that, there is an augment map $\mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z}$, whose kernel is $\mathbb{Z}$-free generated by $\{(1-g) : g \in G \setminus 1\}$. It is easy to see that

$$\begin{cases} \mathsf{H}^0(G; M) = \mathrm{Hom}_G(\mathbb{Z}, M) = \{x \in M : \forall g \in G, gx = x\} = M^G \\ \mathsf{H}_0(G; M) = \mathbb{Z} \otimes_G M = M/\left\langle (1-g)x : g \in G \right\rangle = M/\mathbb{I}_G M = M_G \end{cases}$$

Conversely, $\mathsf{H}^\bullet(G; -)$ is the right derived functor of $M \mapsto M^G$ and $\mathsf{H}_\bullet(G; -)$ is the left derived functor of $M \mapsto M_G$.

§ For two left G-modules M, N,

$$\mathrm{Hom}_{\mathbb{Z}}(M, N), \quad M \otimes_{\mathbb{Z}} N$$

has a natural G-structure by $g \cdot f : x \mapsto gf(g^{-1}x)$, and $g \cdot (x \otimes y) = gx \otimes gy$. Under this, we have the following isomorphisms of G-module

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(N, L)) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(N, L)) = \mathrm{Hom}_{\mathbb{Z}}(N; \mathrm{Hom}_{\mathbb{Z}}(M, L)).$$

This is known as the structure of **Hopf module**.

We have

$$\mathrm{Hom}_{\mathbb{Z}}(M, N)^G = \mathrm{Hom}_G(M, N), \quad \left(M \otimes_{\mathbb{Z}} N\right)_G = M \otimes_G N.$$

Generally, if we take a free resolution $P_\bullet \to \mathbb{Z}$, then $P_\bullet \otimes_{\mathbb{Z}} M \to M$ has two structure $g(x \otimes y) = gx \otimes y$ and $gx \otimes gy$, but they are isomorphic. Actually, over $\mathbb{Z}[G] \otimes_G M$,
$$\left[\begin{array}{ccc} h \otimes x & \longrightarrow & h \otimes hx \\ {\scriptstyle g\otimes 1}\downarrow & & \downarrow{\scriptstyle g\otimes g} \\ gh \otimes x & \longrightarrow & gh \otimes ghx \end{array}\right]$$
, so is all free modules. Then

$$\begin{aligned} \operatorname{Ext}^i_G(M,N) &= \mathsf{H}^i(\operatorname{Hom}_G(P_\bullet \otimes_{\mathbb{Z}} M, N)) \\ &= \mathsf{H}^i(\operatorname{Hom}_G(P_\bullet, \operatorname{Hom}_{\mathbb{Z}}(M,N))) = \mathsf{H}^i(G; \operatorname{Hom}_{\mathbb{Z}}(M,N)), \\ \operatorname{Tor}_i^G(M,N) &= \mathsf{H}_i(P_\bullet \otimes_{\mathbb{Z}} M \otimes_G N) = \mathsf{H}^i(G; M \otimes_G N). \end{aligned}$$

§ For a subgroup $H \le G$, any G-module M is naturally an H-module, but we write $M \downarrow^G_H = M$ if necessary to differ. For any H-module N, we can define the induced module

$$\begin{cases} N \Uparrow^G_H = \operatorname{Hom}_H(\mathbb{Z}[G], N), & g \cdot f : x \mapsto f(xg). \\ N \uparrow^G_H = \mathbb{Z}[G] \otimes_H N, & g \cdot (z \otimes y) = gz \otimes y \end{cases}$$

Then,

$$\begin{cases} \operatorname{Hom}_G(M, N \Uparrow^G_H) = \operatorname{Hom}_H(M \downarrow^G_H, N), \\ \operatorname{Hom}_G(N \uparrow^G_H, M) = \operatorname{Hom}_H(N, M \downarrow^G_H). \end{cases}$$

Note that a G-resolution $F_\bullet \to \mathbb{Z}$ is also an H-resolution, so

$$\mathsf{H}^\bullet(G; M \Uparrow^G_H) = \mathsf{H}^\bullet(\operatorname{Hom}_G(F_\bullet, N \Uparrow^G_H)) = \mathsf{H}^\bullet(\operatorname{Hom}_H(F_\bullet, N)) = \mathsf{H}^\bullet(H; M).$$

$$\mathsf{H}_\bullet(G; M \uparrow^G_H) = \mathsf{H}_\bullet(F_\bullet \otimes_G \mathbb{Z}[G] \otimes_H N) = \mathsf{H}_\bullet(F_\bullet \otimes_H N) = \mathsf{H}_\bullet(H; M).$$

These facts are known as **Shapiro lemma**.

If a G-module is of the form $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) = A \Uparrow^G$ for some abelian group A, it will be called **coinduced G-module**. Dually, a G-module is of the form $\mathbb{Z}[G] \otimes A = A \uparrow^G$ for some abelian group A, is called **induced G-module** Note that,

$$\mathsf{H}^\bullet(G; A \Uparrow^G) = \mathsf{H}^\bullet(1, A) = \begin{cases} A, & \bullet = 0, \\ 0, & \bullet \ge 1. \end{cases}$$

$$\mathsf{H}_\bullet(G; A \uparrow^G) = \mathsf{H}_\bullet(1, A) = \begin{cases} A, & \bullet = 0, \\ 0, & \bullet \ge 1. \end{cases}$$

If G is finite, then $\uparrow$ coincides with $\Uparrow$ (as G-module) by $f \mapsto \sum_{g\in G} g \otimes f(g^{-1})$. Since $gf \mapsto \sum_{x\in G} x \otimes f((x)^{-1}g) = \sum_{x\in G} gx \otimes f(x^{-1})$.

§ For a subgroup H, G-module M, we have maps by universal property of derived functor

$$\mathrm{res} : \mathsf{H}^i(G; M) \to \mathsf{H}^i(H; M), \quad \mathrm{cores} : \mathsf{H}_i(H; M) \to \mathsf{H}_i(G; M),$$

called **restriction** and **corestriction** induced by

$$\mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Hom}_H(P_\bullet, M), \quad P_\bullet \otimes_G M \to P_\bullet \otimes_H M.$$

When we have a normal subgroup N, G-module M, we also have

$$\mathsf{H}^i(G/N; M) \xrightarrow{\mathrm{inf}} \mathsf{H}^i(G; M^N), \quad \mathsf{H}_i(G; M) \xrightarrow{\mathrm{coinf}} \mathsf{H}_i(G/N; M_N),$$

called **inflaction** and **coinflation** induced by

$$\mathrm{Hom}_{G/N}(P_\bullet^N, M^N) \to \mathrm{Hom}_G(P_\bullet, M), \quad P_\bullet \otimes_G M \to (P_\bullet)_N \otimes_N M_N.$$

§ When $H \subseteq G$ is of finite index, and G-module H, one can define **transfer** and **cotransfer**

$$\mathsf{H}^i(H; M) \xrightarrow{\mathrm{tr}} \mathsf{H}^i(G; M), \quad \mathsf{H}_i(G; M) \xrightarrow{\mathrm{cotr}} \mathsf{H}_i(H; M).$$

induced by $A^H \xrightarrow{x \mapsto Nx} A^G$ and $A_G \xrightarrow{x \mapsto xN} A_H$, with N the sum of representatives of coset in G/H. Then by a direct computation,

$$\begin{cases} \mathsf{H}^i(G; M) \xrightarrow{\mathrm{tr}} \mathsf{H}^i(H; M) \xrightarrow{\mathrm{res}} \mathsf{H}^i(G; M), \\ \mathsf{H}_i(G; M) \xrightarrow{\mathrm{cores}} \mathsf{H}_i(H; M) \xrightarrow{\mathrm{cotr}} \mathsf{H}_i(G; M), \end{cases}$$

are just scalar product by the index $[G : H]$.

§ When G is finite, we can define **Tate cohomology** by

$$\tilde{\mathsf{H}}^i(G; M) = \begin{cases} \mathsf{H}^i(G; M), & i \geq 1, \\ \mathrm{cok}[M_G \xrightarrow{N} M^G] = M^G / N \cdot M, & i = 0, \\ \ker[M_G \xrightarrow{N} M^G] = \{x \in M : Nx = 0\}/\mathbb{I} \cdot M, & i = -1, \\ \mathsf{H}_{-1-i}(G; M) & i \leq -2. \end{cases}$$

It also has two-side long exact sequence for each short exact sequence of G-Mod. By a trick of dimension shift, res, cores, tr and cotr can be extended on Tate cohomology, and res = cotr and cores = tr.

§ If we take resolution $P_\bullet \to \mathbb{Z}$, it is easy to find $P_\bullet \otimes_{\mathbb{Z}} P_\bullet$ is a resolution also for $\mathbb{Z}$. So the map induced **cup product**

$$\mathsf{H}^i(G;M) \otimes \mathsf{H}^j(G;N) \xrightarrow{\smile} \mathsf{H}^i(G;M \otimes N).$$

In particular, when $M = N = \mathbb{Z}$, $\mathsf{H}^\bullet(G;\mathbb{Z})$ forms a graded commutative ring. More precisely, for $\alpha \in \mathsf{H}^i(G;M)$ presented by $P_i \xrightarrow{a} M$ and $\beta \in \mathsf{H}^j(G;N)$ presented by $P_i \xrightarrow{\beta} N$, then $\alpha \smile \beta$ is presented by $P_i \otimes P_j \xrightarrow{a\otimes b} M \otimes N$. When $M = N = \mathbb{Z}$, using Koszul convention, and Eckmann-Hilton argument

$$\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha.$$

§ Geometrically, we can find a contractible G-cellular space EG with free G-action. Denote $BG = EG/G$, then the natural map $EG \to BG$ is a covering, and $\pi_i(BG) = \begin{cases} G, & i = 1 \\ 0 & i \neq 1 \end{cases}$. Such BG is called $K(G;1)$. Then, now, the chain complex of cellular homology of EG, say $C_\bullet(EG)$, forms a G-resolution of $\mathbb{Z}$. Given a G-module M, $EG \times_G M$ forms a local coefficient system over BG, say $\mathcal{M}$, then

$$\mathsf{H}^\bullet(G;M) = \mathsf{H}^\bullet(BG;\mathcal{M}), \quad \mathsf{H}_\bullet(G;M) = \mathsf{H}_\bullet(BG;\mathcal{M}).$$

In particular, if M is trivial, then it is just the (co)homology of BG with coefficient M. More particularly, if $M = \mathbb{Z}$, then two cup products coincide.

3.2 The Künneth Spectral Sequences

It is natural to compare the homology of a complex and after tensoring some module. Let $C_\bullet$ be a complex in R-Mod, and M a right R-module. We will compare $H_\bullet(M \otimes C_\bullet)$ and $M \otimes H_\bullet(C_\bullet)$.

Let $P_\bullet \to M$ be a resolution as R module. It forms a double complex $P_\bullet \otimes C_\bullet$. Then

$$\begin{array}{ccc|ccc|ccc} & \downarrow & & & \cdots & & & \cdots & \\ \vdots & P_p \otimes C_q & \vdots & \leftarrow & P_p \otimes \mathsf{H}_q(C_\bullet) & \leftarrow & \vdots & \mathrm{Tor}_p(M, H_q(C_\bullet)) & \vdots \\ & \downarrow & & & \cdots & & & \cdots & \end{array}$$

To get clear result, we assume $C_\bullet$ is flat for each $\bullet$, then

$$\begin{array}{ccc|ccc|ccc} & \cdots & & \downarrow & 0 & \cdots & \cdots & 0 & \cdots \\ \leftarrow & P_p \otimes C_q & \leftarrow & M \otimes C_q & 0 & \cdots & \mathsf{H}_q(M \otimes C_\bullet) & 0 & \cdots \\ & \cdots & & \downarrow & 0 & \cdots & \cdots & 0 & \cdots \end{array}$$

It goes from $\leftarrow$ to $\downarrow$, so we need to exchange pq to suit our convention (1.4).

(3.1) Theorem *Let $C_\bullet$ be a nonnegative complex in R-*Mod*, and M a right R-module. Then there is a spectral sequence* E *with*

$$\mathsf{E}^2_{pq} = \mathrm{Tor}_p(M, \mathsf{H}_q(C_\bullet))$$

converging to $\mathsf{H}_\bullet(M \otimes C_\bullet)$.

If $C_\bullet$ satisfies

$$\text{for each } n,\ C_n \text{ is flat, and } \mathrm{d}(C_n) \subseteq C_{n-1} \text{ is also flat,} \qquad (*)$$

Then $\ker \mathrm{d}$ is flat by considering the long exact sequence of $0 \to \ker \to C \to \mathrm{im} \to 0$. Then $\mathsf{H}(C)$ has flat dimension at most 1 by considering the long exact sequence of $0 \to \mathrm{im} \to \ker \to H(C) \to 0$.

$$\begin{array}{|cccc} M \otimes \mathsf{H}_q(C_\bullet) & \mathrm{Tor}_1(M, \mathsf{H}_q(C_\bullet)) & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ M \otimes \mathsf{H}_0(C_\bullet) & \mathrm{Tor}_1(M, \mathsf{H}_0(C_\bullet)) & 0 & \cdots \\ \hline \end{array}$$

(3.2) Corollary (Universal coefficient theorem) *If $C_\bullet$ satisfies $(*)$, then we have the exact sequence*

$$0 \to M \otimes_R \mathsf{H}_n(C_\bullet) \to \mathsf{H}_n(M \otimes_R C_\bullet) \to \mathrm{Tor}_1^R(M, \mathsf{H}_{n-1}(C_\bullet)) \to 0.$$

Here we do not assume the $C_\bullet$ to be nonnegative since we can firstly truncate the complex.

(3.3) Remark If furthermore, $\mathrm{d}(C\bullet)$ is projective, then the sequence splits. Consider

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M\otimes \mathrm{im} & \longrightarrow & M\otimes \ker & \longrightarrow & M\otimes H(C) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{im}(M\otimes C) & \longrightarrow & \ker(M\otimes C) & \longrightarrow & H(M\otimes C) & \longrightarrow & 0
\end{array}$$

Now, $\ker \mathrm{d}$ is summand of $C_\bullet$, so $M\otimes \ker \mathrm{d}$ is summand of $M\otimes C$, so is summand of $\ker(M\otimes C)$. Since summand is summand of subgroup containing it. More precisely,

$$\left.\begin{array}{l} A\oplus B = C, \\ A\subseteq D\subseteq C \end{array}\right\} \Longrightarrow A\oplus(B\cap D) = D.$$

Then the right row splits. More categorially, note that a short exact sequence $\mathbf{C}$ splits iff $\mathrm{Hom}(X,\mathbf{C})$ is short exact for any X, then it reduces to diagram chasing over

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X, M\otimes\ker) & \longrightarrow & \mathrm{Hom}(X, \mathrm{Tor}_1^R(M, \mathsf{H}_{n-1}(C_\bullet))) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X, \ker(M\otimes C)) & \longrightarrow & \mathrm{Hom}(X, \mathrm{Tor}_1^R(M, H(M\otimes C))) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X, \mathrm{cok}) & = & \mathrm{Hom}(X, \mathrm{cok}) \\
\downarrow & & \\
0 & &
\end{array}$$

where $\mathrm{cok} = \mathrm{cok}[M\otimes\ker \to \ker(M\times C)]$. Or, equivalently, take $X = \mathrm{cok}$ in the above diagram.

Generally, we want a theorem to compute the homology group of tensor product of two complexes. For this, we need a resolution to a complex. Let $M_\bullet$ and $C_\bullet$ be two complexes. Let $P_{**} \to M_\bullet$ be a proper resolution. Now, we need to consider the tri-complex

$$P_{\bullet\bullet} \otimes C_\bullet$$

Then

$$\bigoplus_{s+t=q} P_{ps} \otimes C_t \,\Big|^{q}\, \bigoplus_{s+t=q} \mathsf{H}_s(P_{p*}) \otimes \mathsf{H}_t(C) \,\Big|^{p}\, \bigoplus_{s+t=q} \mathrm{Tor}_p(\mathsf{H}_s(M), \mathsf{H}_t(C))$$

The first computation is because $P_{p\bullet}$ splits. To get clear result, we assume $C_\bullet$ is flat for each $\bullet$, then

$$\bigoplus_{s+t=q} P_{ps} \otimes C_t \,\Big|^{p}\, \begin{cases} \bigoplus_{s+t=q} M_s \otimes C_t & p = 0 \\ 0 & p \neq 0 \end{cases} \Longrightarrow \mathsf{H}_\bullet(M \otimes C)$$

(3.4) Künneth spectral sequences *If two complexes $C_\bullet, D_\bullet$ are non-negative, with one of them flat, then we have a spectral sequence* E *with*

$$E^2_{pq} = \bigoplus_{s+t=q} \mathrm{Tor}_p(H_s(M_\bullet), H_t(C_\bullet))$$

which converges to $H_\bullet(C \otimes D)$.

(3.5) Classic Künneth theorem *If two complexes $C_\bullet, D_\bullet$ are non-negative, with one of them satisfying $(*)$, then we have the following spilt short exact sequence*

$$\bigoplus_{p+q=n} \mathsf{H}_p(C) \otimes H_q(D) \hookrightarrow \mathsf{H}^n(C \otimes D) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(\mathsf{H}_s(M_\bullet), \mathsf{H}_t(C_\bullet)).$$

(3.6) Remark If furthermore, when the complex satisfying $(*)$ with $\mathrm{d}(C\bullet)$ projective, the sequence splits. The proof is the same.

3.3 The Hoshschild–Serre Spectral Sequences

Let G be a discrete group, and N be a normal subgroup. Fix an G-module M. We want to study the relation of $\mathsf{H}^i(N;M)$ and $\mathsf{H}^i(G;M)$. Let $P_\bullet \to \mathbb{Z}$ be a free G-resolution.

$$\mathsf{H}^i(G;M) = \mathsf{H}^i(\mathrm{Hom}_G(P_\bullet, M)) = \mathsf{H}^i(\mathrm{Hom}_N(P_\bullet, M)^{G/N})$$

Then we can take a G/N-resolution $Q_\bullet \to \mathbb{Z}$, and consider the double complex

$$C^{\bullet\bullet} = \mathrm{Hom}_{G/N}(Q_\bullet, \mathrm{Hom}_N(P_\bullet, M)).$$

Then, $\mathsf{H}^i(\mathrm{Tor}(C)) = \mathsf{H}^i(G;M)$.

$$\begin{array}{ccc|cccc|ccc} & \uparrow & & \vdots \to & 0 & \to \vdots & & \vdots & 0 & \vdots \\ \vdots & C^{pq} & \vdots & \vdots \to & 0 & \to \vdots & & \vdots & 0 & \vdots \\ & \uparrow & & \vdots \to & \mathrm{Hom}_N(P_p, M)^{G/N} & \to \vdots & & \cdots & \mathsf{H}^p(G;M) & \cdots \end{array}$$

On the other hand,

$$\mathrm{Hom}_{G/N}(Q_q, \mathrm{Hom}_N(P_p, M)) \;\Big|^{p}\; \mathrm{Hom}_{G/N}(Q_q, \mathsf{H}^p(N;M)) \;\Big|^{q}\; \mathsf{H}^q(G/N; \mathsf{H}^p(N;M)).$$

(3.7) Hoshschild–Serre *Let G be a discrete group, and N be a normal subgroup. For any G-module M, there a spectral sequence E with*

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(G/N; \mathsf{H}^q(N;M))$$

converging to $\mathsf{H}^\bullet(G;M)$.

$$\begin{array}{|ccc} \mathsf{H}^2(N;M)^{G/N} & \mathsf{H}^1(G/N;\mathsf{H}^2(N;M)) & \mathsf{H}^2(G/N;\mathsf{H}^2(N;M)) \\ \mathsf{H}^1(N;M)^{G/N} & \mathsf{H}^1(G/N;\mathsf{H}^1(N;M)) & \mathsf{H}^2(G/N;\mathsf{H}^1(N;M)) \\ (M^N)^{G/N} & \mathsf{H}^1(G/N;M^N) & \mathsf{H}^2(G/N;M^N) \\ \hline \end{array}$$

By (1.60), we get first term sequence.

(3.8) Corollary *We have the following exact sequence*

$$0 \longrightarrow \mathsf{H}^1(G/N; M^N) \xrightarrow{\text{inf}} \mathsf{H}^1(G; M) \xrightarrow{\text{res}} \mathsf{H}^1(N; M)^{G/N} \xrightarrow{d} \mathsf{H}^2(G/N; M^N) \xrightarrow{\text{inf}} \mathsf{H}^2(G; M)$$

PROOF. Only rest to prove the map given by spectral sequence coincide the inflation and restriction. The map $\mathsf{H}^i(G/N; M^N) \xrightarrow{\text{inf}} \mathsf{H}^i(G; M)$ is induced by some chain map

$$\mathrm{Hom}_{G/N}(Q_i, \mathrm{Hom}_N(P_0, M)) \to \mathrm{Hom}_N(P_\bullet, M)^{G/N} \to \mathrm{Hom}_G(P_i, M).$$

If we take $Q_i = P_i^N$, this is what we defined. The map $\mathsf{H}^1(G; M) \to \mathsf{H}^1(N; M)^{G/N}$ is induced by

$$\mathrm{Hom}_G(P_i, M) \to \mathrm{Hom}_{G/N}(Q_0, \mathrm{Hom}_N(P_i, M)).$$

So it also coincides the definition of restriction.

Dually, we have the following.

(3.9) Hoshschild-Serre *Let G be a discrete group, and N be a normal subgroup. For any G-module M, there a spectral sequence E with*

$$\mathsf{E}_2^{pq} = \mathsf{H}_p(G/N; \mathsf{H}_q(N; M))$$

which converges to $\mathsf{H}_\bullet(G; M)$.

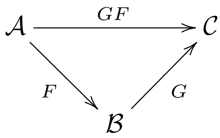
(3.10) Corollary *We have the following exact sequence*

$$\mathsf{H}_2(G; M) \xrightarrow{\text{coinf}} \mathsf{H}_2(G/N; M_N) \xrightarrow{d} \mathsf{H}_1(N; M)_{G/N} \xrightarrow{\text{cores}} \mathsf{H}_1(G; M) \xrightarrow{\text{coinf}} \mathsf{H}_1(G/N; M_N) \longrightarrow 0$$

3.4 The Grothendieck Spectral Sequences

Here is a generalization of (3.7) last section.

Consider the functors between abelian categories with enough projectives



assume F, G are both right exact. Denote $\boldsymbol{L}_i F$ and $\boldsymbol{L}_i G$ the corresponding left derived functors satisfies

> F sends projective objects to acyclic objects of G.
> It means, there are enough projective objects, say
> A, in $\mathcal{A}$, such that $\boldsymbol{L}_{\geq 1} G(A) = 0$.

(3.11) Grothendieck spectral sequences *As above, for any object A, there exists a spectral sequence E with*

$$\mathsf{E}^2_{pq} = (\boldsymbol{L}_p G \circ \boldsymbol{L}_q F)(A)$$

converging to $\boldsymbol{L}_*(G \circ F)(A)$.

PROOF. Take the projective resolution $P_\bullet \to A$, and take the proper resolution $Q_{**} \to F(P_\bullet)$. Now let us compute the homology of $G(Q_{**})$.

$$G(Q_{pq}) \,\Big|^{q}\, \boldsymbol{L}_q G(F(P_p)) = \begin{cases} G(F(P_p)), & q = 0, \\ 0, & q \neq 0. \end{cases} \Longrightarrow \boldsymbol{L}_\bullet (GF)(A).$$

Secondly,

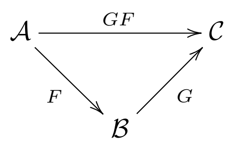
$$G(Q_{pq}) \,\Big|^{p}\, G(\mathsf{H}_p(Q_{\bullet q})) \,\Big|^{q}\, \boldsymbol{L}_p G(\boldsymbol{L}_q F(A)),$$

since $\mathsf{H}_p(Q_{\bullet q})$ is a resolution of the homological group of $F(P_\bullet)$, i.e. $\boldsymbol{L}_q F(A)$.

(3.12) Corollary *We have the following exact sequence of functors*

$$\boldsymbol{L}_2(F\circ G)\to \boldsymbol{L}_2F\circ G\to F\circ \boldsymbol{L}_1(G)\to \boldsymbol{L}_1(F\circ G)\to \boldsymbol{L}_1F\circ G\to 0.$$

Dually, we have the similar result for left exact functors. Consider the functors between abelian categories with enough injectives where F, G are both left exact. Denote $\boldsymbol{R}^iF$ and $\boldsymbol{R}^iG$ the corresponding right derived functors. Assume F sends injective objects to acyclic objects of G.

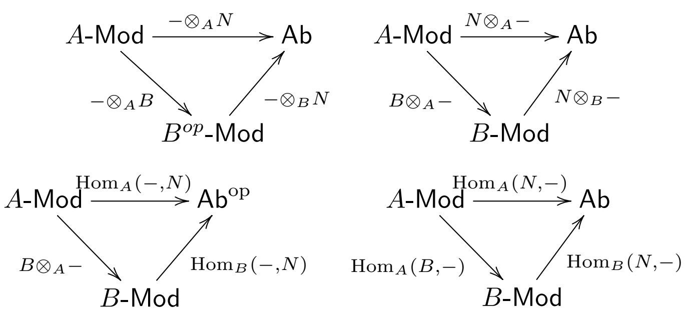


(3.13) Grothendieck spectral sequences *As above, for any object A, there exists a spectral sequence E with*

$$\mathsf{E}_2^{pq} = (\boldsymbol{R}^pG\circ \boldsymbol{R}^qF)(A)$$

which converges to $\boldsymbol{R}^*(GF)(A)$.

(3.14) EXAMPLE Let $A\xrightarrow{\varphi} B$ be a ring homomorphism. Let M be an A module, N be a B module (left or right indicated by notations).



We have the spectral sequences

- E with $\mathsf{E}^2_{pq} = \mathrm{Tor}_p^B(\mathrm{Tor}_q^A(M,B),N)$ converges to $\mathrm{Tor}_*^A(M,N)$.
- E with $\mathsf{E}^2_{pq} = \mathrm{Tor}_p^B(N,\mathrm{Tor}_q^A(B,M))$ converges to $\mathrm{Tor}_*^A(N,M)$.

- E with $\mathsf{E}_2^{qp} = \mathrm{Ext}_B^p(\mathrm{Tor}_q^A(B, M), N)$ converges to $\mathrm{Ext}_A^*(M, N)$.

- E with $\mathsf{E}_2^{pq} = \mathrm{Ext}_B^p(N, \mathrm{Ext}_q^A(B, M))$ converges to $\mathrm{Ext}_A^*(N, M)$.

That the third is not E_2^{pq} is to suit our convention (1.4).

(3.15) EXAMPLE (Leray Sequences) Let $X \to Y$ be map between topological space, then

$$\begin{array}{ccc} \mathsf{Sheaf}(X) & \xrightarrow{\;f_*\;} & \mathsf{Sheaf}(Y) \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\Gamma} & \\ & \mathsf{Ab} & \end{array}$$

gives rise to a spectral sequence E for each sheaf $\mathscr{F}$ over X, with

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(Y; \boldsymbol{R}^q f_* \mathscr{F})$$

converges to $\mathsf{H}^\bullet(X, \mathscr{F})$. For sheaves, since the double complex of first quadratic involves only finite colimit, so there is no problem mentioned (1.49).

Exercises

▶ **(3.16) Exercise.** Under the assumption of (3.7), if further more, $\mathsf{H}^\bullet(N; M) = 0$ for $1 \leq \bullet \leq q-1$, show that we have the following exact sequence

$$0 \longrightarrow \mathsf{H}^q(G/N; M^N) \xrightarrow{\text{inf}} \mathsf{H}^q(G; M) \xrightarrow{\text{res}} \mathsf{H}^q(N; M)^{G/N} \xrightarrow{\;d\;} \mathsf{H}^{q+1}(G/N; M^N) \xrightarrow{\text{inf}} \mathsf{H}^{q+1}(G; M)$$

Hint: Now it becomes

$$
\begin{array}{|lllll}
\mathsf{H}^q(N;M)^{G/N} & \mathsf{H}^1(G/N;\mathsf{H}^q(N;M)) & \mathsf{H}^2(G/N;\mathsf{H}^q(N;M)) & & \\
0 & \ddots & 0 & & \\
\vdots & \vdots & \ddots & & \\
0 & 0 & 0 & \ddots & \\
(M^N)^{G/N} & \mathsf{H}^1(G/N;M^N) & \mathsf{H}^2(G/N;M^N) & \cdots & \mathsf{H}^q(G/N;M^N) \\
\hline
\end{array}
$$

The same trick we used in (1.60).

Appendix A

Cohomology for Topological Groups

The main result of this chapter is by Borel in [2].

A.1 Comparison Theorem

(A.1) Lemma *For a chain of complex $C_\bullet$, we have the following exact sequence*

$$0 \to \ker \mathrm{d} \to C \xrightarrow{\mathrm{d}} C \to \operatorname{cok} \mathrm{d} \to 0$$

$$0 \to \mathsf{H}(C) \to \operatorname{cok} \xrightarrow{\mathrm{d}} \ker \to \mathsf{H}(C) \to 0$$

(A.2) Lemma *If we have the following diagram with rows exact*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & 0 \end{array}$$

Then

$$\beta \text{ is injective} \Rightarrow \alpha \text{ is injective}$$

$$\left.\begin{array}{r}\beta \text{ is surjective}\\ \gamma \text{ is injective}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\alpha \text{ is surjective}\\ \delta \text{ is injective}\end{array}\right.$$

$$\gamma \text{ is surjective} \Rightarrow \delta \text{ is surjective}$$

(A.3) Λemma *Let $f : \mathsf{E} \to \overline{\mathsf{E}}$ be a morphism of spectral sequences of cohomology type. Then if the condition*

$$f \text{ is } \begin{cases} \text{isomorphic on } \mathsf{E}_r^{pq} \text{ for } p \leq k - r \\ \text{injective on } \mathsf{E}_r^{pq} \text{ for } p \leq k \end{cases}$$

holds for some $r = R$, then it holds for $r \geq R$.

(A.4) Λemma *Let $f : \mathsf{E} \to \overline{\mathsf{E}}$ be a morphism of spectral sequences of cohomology type. Then if the condition*

$$f \text{ is } \begin{cases} \text{isomorphic on } \mathsf{E}_r^{pq} \text{ for } q \leq k - r + 1 \\ \text{sujective on } \mathsf{E}_r^{pq} \text{ for } q \leq k \end{cases}$$

holds for some $r = R$, then it holds for $r \geq R$.

(A.5) Zeeman comparison theorem *Assume we have a morphism between two first quadratic spectral sequences from the second page $\mathsf{E} \xrightarrow{f} \overline{\mathsf{E}}$, with the following diagram with rows exact*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{E}_2^{p0} \otimes \mathsf{E}_2^{01} & \longrightarrow & \mathsf{E}_2^{pq} & \longrightarrow & \mathrm{Tor}_1(\mathsf{E}_2^{p+1,0}, \mathsf{E}_2^{0q}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Tor}(f,f)} & & \\ 0 & \longrightarrow & \overline{\mathsf{E}}_2^{p0} \otimes \overline{\mathsf{E}}_2^{01} & \longrightarrow & \overline{\mathsf{E}}^{pq} & \longrightarrow & \mathrm{Tor}_1(\overline{\mathsf{E}}_2^{p+1,0}, \overline{\mathsf{E}}_2^{0q}) & \longrightarrow & 0 \end{array}$$

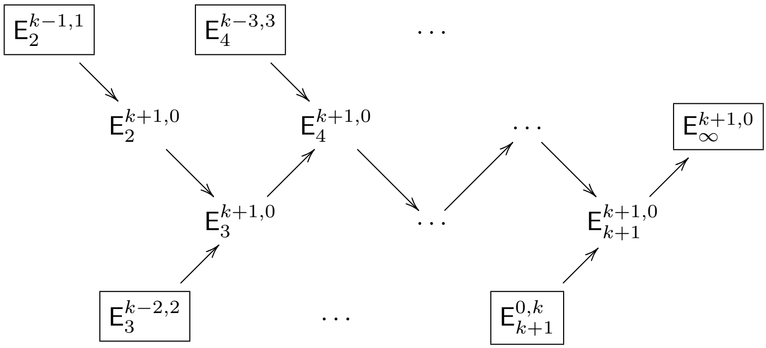
Then any two of the following conditions

- *f is isomorphic on E_2^{p0} for each p.*
- *f is isomorphic on E_2^{0q} for each q.*
- *f is isomorphic on E_∞^{pq} for each p, q.*

imply that f is an isomorphism.

PROOF. The first two imply the f is isomorphism on E_2^{pq}, so f is an isomorphism.

Assume f is isomorphism on E_∞^{pq} and E_2^{0q}. If f is isomorphic on E_2^{pq} for $p \leq k$, then consider



The f on the modules in boxes have been proven to be isomorphic by lemma above. So f is isomorphic on $\mathsf{E}_2^{k+1,0}$. Then by the assumption of big diagram, f is isomorphic on E_2^{pq} for $p \leq k+1$. The proof is complete.

When f is isomorphism on E_∞^{pq} and E_2^{p0}. It follows by the similar induction.

A.2 Transgression

(A.6) Definition (Transgression) If we have the diagram with row exact

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\pi} & C & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\ \cdots & \longrightarrow & A' & \xrightarrow[\iota]{} & B' & \longrightarrow & C' & \longrightarrow & \cdots \end{array}$$

There is a well-defined map called **transgression**

$$\iota^{-1}(\beta(B)) \longrightarrow C/\pi(\beta^{-1}(0)) \qquad x \longmapsto \beta(\pi^{-1}(\iota(x))).$$

(A.7) Definition For a fibration $F \to E \to B$, the **transgression** is defined to be the transgression of the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathsf{H}_n(E) & \longrightarrow & \mathsf{H}_n(E,F) & \longrightarrow & \mathsf{H}_{n-1}(F) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathsf{H}_n(B) & \longrightarrow & \mathsf{H}_n(B,*) & \longrightarrow & \mathsf{H}_{n-1}(*) & \longrightarrow & \cdots \end{array}$$

Note that for first quadratic spectral sequence E, $\mathsf{E}^{\geq 2}_{p0}$ are all subgroup of E^2_{p0} and $\mathsf{E}^{\geq 2}_{0q}$ are all quotient group of E^2_{0q}.

(A.8) Theorem *In Leray-Serre spectral sequence* E *of fibration*

$$[F \xrightarrow{i} E \xrightarrow{p} B]$$

induces to commute diagram

$$\begin{array}{ccccccccccc} & & \mathsf{H}_n(B) & \longrightarrow & \mathsf{H}_n(B,*) & & & & & & \\ & & \uparrow & & \uparrow & & & & & & \\ 0 & \longrightarrow & \mathsf{E}^{\infty}_{n0} & \longrightarrow & \mathsf{E}^{n}_{n0} & \longrightarrow & \mathsf{E}^{n}_{0,n-1} & \longrightarrow & \mathsf{E}^{\infty}_{0,n-1} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \operatorname{im}\mathsf{H}(p_*) & \longrightarrow & \heartsuit & \longrightarrow & \diamondsuit & \longrightarrow & \operatorname{im}\mathsf{H}(i_*) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \downarrow & & \\ & & \mathsf{H}_n(E) & \longrightarrow & \mathsf{H}_n(E,F) & \longrightarrow & \mathsf{H}_{n-1}(F) & \longrightarrow & \mathsf{H}_{n-1}(E) & & \end{array}$$

where $\heartsuit \to \diamondsuit$ *is the transgression.*

PROOF. We argue by the funtoriality of Leray-Serre spectral sequences.

$$\begin{bmatrix} F \\ \downarrow \\ E \\ \downarrow \\ B \end{bmatrix} \to \begin{bmatrix} * \\ \downarrow \\ B \\ \downarrow \\ B \end{bmatrix} \Longrightarrow \qquad \begin{array}{ccccc} \mathsf{E}^2_{n0} & = & \mathsf{H}_n(B) & \rightarrow & \mathsf{H}_n(B) \\ & & \uparrow & & \| \\ & & \mathsf{E}^\infty_{n0} & \longrightarrow & \overline{\mathsf{E}}^\infty_{n0} \\ & & \uparrow & & \| \\ & & \mathsf{H}_n(E) & \rightarrow & \mathsf{H}_n(B) \end{array}$$

shows

$$\mathsf{E}^\infty_{n0} = \mathrm{im}[\mathsf{H}_n(E) \to \mathsf{H}_n(B)] = \mathrm{im}\,\mathsf{H}(p_*)$$

The relative version, since $\mathsf{E}^2_{0q} = \mathsf{H}_0(E, F; \mathsf{H}_q(F)) = 0$,

$$\begin{bmatrix} F \\ \downarrow \\ (E, F) \\ \downarrow \\ (B, *) \end{bmatrix} \to \begin{bmatrix} * \\ \downarrow \\ (B, *) \\ \downarrow \\ (B, *) \end{bmatrix} \Longrightarrow \quad \begin{array}{ccccc} & & \mathsf{H}_n(B, *) & \longrightarrow & \mathsf{H}_n(B, *) \\ & & \uparrow & & \| \\ \mathsf{E}^n_{n0} & = & \mathsf{E}^\infty_{n0} & \longrightarrow & \mathsf{E}^\infty_{0n} \\ & & \uparrow & & \| \\ & & \mathsf{H}_n(E, F) & \longrightarrow & \mathsf{H}_n(B, *) \end{array}$$

for $n \geq 1$, then

$$\mathsf{E}^\infty_{n0} = j^{-1}\,\mathrm{im}[\mathsf{H}_n(E, F) \to \mathsf{H}_n(B, *)]$$

where $j : \mathsf{H}_n(B) \to \mathsf{H}_n(B, *)$.

The morphism between fibrations

$$\begin{bmatrix} F \\ \downarrow \\ F \\ \downarrow \\ * \end{bmatrix} \to \begin{bmatrix} F \\ \downarrow \\ E \\ \downarrow \\ B \end{bmatrix} \Longrightarrow \quad \begin{array}{ccc} \mathsf{H}_n(F) & \longrightarrow & \mathsf{H}_n(F) \\ \| & & \downarrow \\ \hat{\mathsf{E}}^\infty_{0n} & \longrightarrow & \mathsf{E}^\infty_{0n} \\ \| & & \downarrow \\ \mathsf{H}_n(F) & \longrightarrow & \mathsf{H}_n(E) \end{array}$$

So

$$\mathsf{E}^\infty_{0n} = \mathrm{im}[\mathsf{H}_n(F) \to \mathsf{H}_n(E)] = \mathrm{im}\,\mathsf{H}(i_*).$$

Last, we have established (the middle □ commutes since they are all induced by d)

$$\begin{array}{ccccccccccc}
 & & \mathsf{H}_n(B) & \longrightarrow & \mathsf{H}_n(B,*) & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
0 & \longrightarrow & \mathsf{E}^{\infty}_{n0} & \longrightarrow & \mathsf{E}^{n}_{n0} & \xrightarrow{\mathrm{d}} & \mathsf{E}^{n}_{0,n-1} & \longrightarrow & \mathsf{E}^{\infty}_{0,n-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & \square & \uparrow & & \uparrow & & \\
 & & \mathsf{H}_n(E) & \longrightarrow & \mathsf{H}_n(E,F) & \xrightarrow[\mathrm{d}]{} & \mathsf{H}_{n-1}(F) & \longrightarrow & \operatorname{im} i_* & &
\end{array}$$

So by an algebraic argument, $\mathsf{E}^{n}_{0,n-1}$ must be the codomain of transgression, say, by diagram chasing, or computation the push out of the □.

(A.9) Definition (transgression) For a fibration $F \to E \to B$, the **transgression** is defined to be the transgression of the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathsf{H}^{n-1}(*) & \longrightarrow & \mathsf{H}^{n}(B,*) & \longrightarrow & \mathsf{H}^{n}(B) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathsf{H}^{n-1}(F) & \longrightarrow & \mathsf{H}^{n}(E,F) & \longrightarrow & \mathsf{H}^{n}(E) & \longrightarrow & \cdots
\end{array}$$

(A.10) Theorem *In Leray-Serre spectral sequence* E *of fibration*

$$[F \xrightarrow{i} E \xrightarrow{p} B]$$

induces to commute diagram

$$
\begin{array}{ccccccccccc}
 & & & & & & \mathsf{H}^n(B,*) & \longrightarrow & \mathsf{H}^n(B) & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{E}_\infty^{0,n-1} & \longrightarrow & \mathsf{E}_n^{0,n-1} & \longrightarrow & \mathsf{E}_n^{n0} & \longrightarrow & \mathsf{E}_\infty^{n0} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \\
0 & \longrightarrow & \operatorname{im}\mathsf{H}(i^*) & \longrightarrow & \heartsuit & \longrightarrow & \diamondsuit & \longrightarrow & \operatorname{im}\mathsf{H}(p^*) & \longrightarrow & 0 \\
 & & \uparrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathsf{H}^{n-1}(E) & \longrightarrow & \mathsf{H}^{n-1}(F) & \longrightarrow & \mathsf{H}^n(E,F) & \longrightarrow & \mathsf{H}^n(E) & &
\end{array}
$$

where $\heartsuit \to \diamondsuit$ is the transgression.

(A.11) Definition We will call the elements in the domain of transgression the **transgressive elements**. It is, by theorem, the element x with $dx = 0$ until the last d.

A.3 The Borel Theorem

(A.12) Borel *Let k be a field. Consider*

a fibration with E contractible and B simply connected. | $F \to E \to B$

If

$$\mathsf{H}^\bullet(F;k) = \Lambda_k(x_1,\dots,x_n)$$

the exterior algebra in $x_1,\dots,x_n$, transgressive elements of odd degrees, then

$$\mathsf{H}^\bullet(B;k) = k[y_1,\dots,y_n]$$

the polynomial ring in $y_1,\dots,y_n$ the image of $x_1,\dots,x_n$ under transgression.

Zeeman, [11]. Since k is a field,

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(B;k)\otimes\mathsf{H}^q(F;k).$$

Consider

$$ {}_i\mathsf{E}_2^{pq} = \big(k[y_i]\big)_p \otimes \big(\Lambda_k(x_i)\big)_q $$

which has only two rows, and define

$$ \mathrm{d} : {}_i\mathsf{E}_r^{pq} \longrightarrow {}_i\mathsf{E}_r^{p+r,q-r+1} \qquad y_i^p \otimes x_i \longmapsto \begin{cases} y_i^{p+1} & |y_i| = r \\ 0 & \text{otherwise} \end{cases} $$

This makes ${}_i\mathsf{E}$ a spectral sequence with multiplication structure. Define

$$ f_2 : {}_i\mathsf{E}_2 \longrightarrow \mathsf{E}_2 \qquad y_i^p \otimes x_i \longmapsto y_i^p \otimes x_i. $$

If f_{r-1} is defined, then $f_r = \mathsf{H}(f_{r-1})$ is a chain map for $r < |y_i|$. Actually, since ${}_i\mathsf{E}$ has only two rows, we need to check

$$ \begin{array}{ccc} 0 & \longrightarrow & \mathsf{E}_r^{p-r,r-1} \\ \downarrow & & \downarrow \\ {}_i\mathsf{E}_r^{p0} & \longrightarrow & \mathsf{E}_r^{p0} \end{array} \qquad \begin{array}{ccc} {}_i\mathsf{E}_r^{0q} & \longrightarrow & \mathsf{E}_r^{0q} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{E}_r^{r,q-r+1} \end{array} $$

The first square always commutes, and the second commutes since x_i is assumed to be transgressive. The rest nonzero entries commute because of multiplication structure.

In the last step, that is, when $r = |y_i|$, f_r is also a chain morphism, since

$$ \begin{array}{ccccc} x_i & \in & {}_i\mathsf{E}_r^{0,r-1} & \longrightarrow & \mathsf{E}_r^{0,r-1} \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \text{transgression} \\ y_i & \in & {}_i\mathsf{E}_r^{r0} & \longrightarrow & \mathsf{E}_r^{r0} \end{array} $$

The rest is also due to multiplication structure.

So we get a well-defined map

$$ f : \overline{\mathsf{E}} = \bigotimes_{i=1}^{n} {}_i\mathsf{E} \xrightarrow{\otimes f} \bigotimes_{i=1}^{n} \mathsf{E} \xrightarrow{\smile} \mathsf{E}. $$

Then $\overline{\mathsf{E}}_\infty^{pq} = \begin{cases} k, & (p,q) = 0, \\ 0, & (p,q) \neq 0. \end{cases}$ since each ${}_iE$ does, and are made by free k-modules. So by Zeeman comparison theorem (A.5), $\mathsf{H}^\bullet(B;k) = k[y_1,\ldots,y_n]$.

(A.13) Borel *Let k be a field. Consider*

a fibration with E contractible and B simply connected.	$F \to E \to B$

If

$$\mathsf{H}^\bullet(F;k) = \Lambda_k(a_1,\ldots,a_n)$$

with $a_1,\ldots,a_n$ of odd degrees, then we can pick transgressive elements $x_1,\ldots,x_n$, such that

$$\mathsf{H}^\bullet(F;k) = \Lambda_k(x_1,\ldots,x_n).$$

PROOF. Since k is a field,

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(B;k) \otimes \mathsf{H}^q(F;k).$$

We will make induction.

Assume we have adjusted $\{x_i\}$ such that $\mathrm{d}_\bullet x_i \neq 0$ only when the target touch E^{p0} for $\bullet < r$.

Assume $\mathrm{d}_r x_\bullet \neq 0$ only when the target touch E^{p0} is proved for $\deg x_\bullet \leq |x_i|$.

Consider the $\mathrm{d}_r x_i \in \mathsf{E}_r^{\rho\varsigma}$. We can assume

$$\mathrm{d}_r x_i \text{ is presented by } \sum \beta_k \otimes \gamma_k \in \mathsf{E}_2^{\rho\varsigma} = \mathsf{H}^\rho(B) \otimes \mathsf{H}^\varsigma(F).$$

with γ_k polynomial in $\{x_\bullet : |x_\bullet| < x_i\}$. Then some transgressive element $\alpha_k \mapsto \beta_k$ through transgression. Fortunately, since $|x_i|$ is of odd degree, so $\mathrm{d}_r\gamma_k$ do not touch E^{*0}, so $\sum \alpha_k \otimes \mathrm{d}_r\gamma_k$ is 0.

$$\mathrm{d}_r\Big(\sum \alpha_k \wedge \gamma_k\Big) = \sum \beta_k \otimes \gamma_k + \underbrace{\sum \alpha_k \otimes \mathrm{d}_r\gamma_k}_{=0}$$

Next, for $\bullet < r$,

$$\mathrm{d}_\bullet\big(\sum \alpha_k \wedge \gamma_k\big) = \underbrace{\sum \mathrm{d}_\bullet \beta_k \wedge \gamma_k}_{0} + \underbrace{\sum \alpha_k \wedge \mathrm{d}_\bullet \gamma_k}_{(*)}.$$

The first term vanishes since α_k is transgressive. Note that $\mathrm{d}_\bullet(\mathrm{d}_r x_i) = \sum \beta_k \otimes \mathrm{d}_\bullet \gamma_k = 0$. So $\mathrm{d}_r(*) = 0$. But there is no nonzero differential to the position of $(*)$ after r-th page, so $(*) = 0$.

The proof above is given by author myself.

A.4 Cohomology Computation

(A.14) !! Assumption— We assume in this section that k is a field of characteristic zero.

(A.15) Borel–Hopf *If a graded commutative Hopf k-algebra H is generated by x of nonzero degree as k-algebra, if x is of odd degree, then $H = \Lambda_k(x)$, the exterior algebra, if x is of even degree, $H = k[x]$, the polynomial ring.*

Proof. If x is of odd degree, it is easy, since $x^2 = -x^2$. When x is of even degree, since x is of least nonzero degree, so one must have

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

Then

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

So $x^k \neq 0 \neq x^h$ implies $x^{k+h} \neq 0$.

(A.16) Corollary *For a path-connected topological group G of finite dimensional* Cw*-complex structure,*

$$\mathsf{H}^\bullet(G; k) = \Lambda_k(x_1, \ldots, x_n)$$

with $x_1, \ldots, x_n$ of odd degrees.

PROOF. Note that any nonzero element, of least nonzero degree is primitive. Let x_1 be such an element in $\mathsf{H}^\bullet(G;k)$. Consider the ideal generated by x_1, and so on. We can find that $\mathsf{H}^\bullet(G;k)$ itself is generated by elements of odd degrees as an ideal, so as an algebra.

(A.17) Corollary *For a path-connected topological group G of finite dimensional* CW*-complex structure,*

$$\mathsf{H}^\bullet(BG;k) = k[y_1, \ldots, y_n]$$

with $y_1, \ldots, y_n$ of even degrees.

PROOF. By Borel's theorem (A.12) and (A.13).

(A.18) EXAMPLE For torus $\mathbb{S}^1$, $B\mathbb{S}^1 = \mathbb{C}P^\infty$,

$$\mathsf{H}^\bullet(\mathbb{S}^1;k) = \Lambda(x), \qquad \deg x = 1.$$

$$\mathsf{H}^\bullet(\mathbb{C}P^\infty;k) = k[y], \qquad \deg y = 2.$$

(A.19) EXAMPLE For unitary group $\mathsf{U}(n)$, $B\mathsf{U}(n) = \mathsf{Grass}^{\mathbb{C}}(n)$ the infinite complex Grassmannian,

$$\mathsf{H}^\bullet(\mathsf{U}(n);k) = \Lambda(x_1, \ldots, x_{2n-1}), \qquad \deg x_\bullet = \bullet$$

$$\mathsf{H}^\bullet(\mathsf{Grass}^{\mathbb{C}}(n);k) = k[c_2, \ldots, c_{2n}], \qquad \deg c_\bullet = \bullet$$

(A.20) EXAMPLE For symplectic group $\mathsf{Sp}(2n)$, $B\mathsf{Sp}(2n) = \mathsf{Grass}^{\mathbb{H}}(n)$ the infinite Hermiterian Grassmannian,

$$\mathsf{H}^\bullet(\mathsf{Sp}(2n);k) = \Lambda(x_3, \ldots, x_{4n-1}), \qquad \deg x_\bullet = \bullet$$

$$\mathsf{H}^\bullet(\mathsf{Grass}^{\mathbb{H}}(n);k) = k[p_4, \ldots, p_{4n}], \qquad \deg \deg p_\bullet = \bullet$$

(A.21) EXAMPLE For unitary group $\mathsf{SO}(n)$, $B\mathsf{SO}(n) = \mathsf{Grass}^{\mathbb{R}}(n)$ the infinite real Grassmannian,

$$\mathsf{H}^\bullet(\mathsf{SO}(2k+1);k) = \Lambda(x_3,\dots,x_{4k-1}), \qquad \deg x_\bullet = \bullet$$

$$\mathsf{H}^\bullet(\mathsf{Grass}^{\mathbb{R}}(2k+1);k) = k[p_4,\dots,p_{4n}], \qquad \deg p_\bullet = \bullet$$

$$\mathsf{H}^\bullet(\mathsf{SO}(2k+2);k) = \Lambda(x_3,\dots,x_{4k-1},x_{2k+1}), \qquad \deg x_\bullet = \bullet$$

$$\mathsf{H}^\bullet(\mathsf{Grass}^{\mathbb{R}}(2k+1);k) = k[p_4,\dots,p_{4n},p_{2k+2}], \qquad \deg p_\bullet = \bullet$$

(A.22) Collapse theorem *Let G be a path-connected topological group of finite dimensional* CW*-complex structure, and H be a closed subgroup of G. There is a spectral sequence E with*

$$\mathsf{E}_2 = \mathrm{Tor}^{\mathsf{H}(BG)}(k,\mathsf{H}(BH))$$

collapsing at $\mathsf{E}_{\geq 2}$ and converging to $\mathsf{H}(G/H)$.

PROOF. The existence is just an application of Eilenburg–Moore spectral sequence (2.8) by considering the pull back square

$$\begin{array}{ccc} G/H & \longrightarrow & BH \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array}$$

By (A.17), $\mathsf{H}(BH)$ and $\mathsf{H}(BG)$ are polynomial ring with only even dimension, so there no differential anymore.

(A.23) Remark When k is not of characteristic zero, the cohomology group is complicated. The calculation over $\mathbb{C}$ can be done by Chern-Weil theory.

Exercises

► **(A.24)** PROBLEM *(Universal transgressive)*. For a topological group G, $x \in \mathsf{H}^\bullet(G)$ is said to be **universal transgressive** if for any G-principal bundle $E \to B$, x is transgressive. Show that, x is universal transgressive if and only if it is transgressive for classifying bundle $EG \to BG$. `Hint: Since one can find` $(E \to B) \to (EG \to BG)$`. Then consider`

$$\begin{bmatrix} G \\ \downarrow \\ EG \\ \downarrow \\ BG \end{bmatrix} \leftarrow \begin{bmatrix} G \\ \downarrow \\ E \\ \downarrow \\ B \end{bmatrix} \Longrightarrow \begin{array}{ccc} \mathsf{H}^\bullet(G) & = & \mathsf{H}^\bullet(G) \\ \downarrow & & \downarrow \\ \mathsf{E} & \longrightarrow & \overline{\mathsf{E}} \end{array}$$

Appendix B

Cohomology for Compact Lie Groups

The topic is mainly from [5].

B.1 The Koszul Complex

Let R be a commutative ring.

(B.1) Definition (Koszul complex) For $x \in R$, and M an R-module, denote

$$K(x): \quad 0 \to R \xrightarrow{x} R \to 0$$

For arbitrary $x_1, \ldots, x_n$, denote

$$K(x_1, \ldots, x_n) = \mathrm{Tot}\left(K(x_1) \otimes_R \cdots \otimes_R K(x_n)\right)$$

the **Koszul complex**.

(B.2) Theorem *The Koszul complex $K(x_1, \ldots, x_n)$ is isomorphic to $\Lambda^\bullet R^{\oplus n}$ with*

$$d : \Lambda^p R^{\oplus n} \longrightarrow \Lambda^{p-1} R^{\oplus n} \qquad e_1 \wedge \cdots \wedge e_p \longmapsto \sum_{i=1}^{p} (-1)^i x_i \cdot e_1 \wedge \cdots \widehat{e_i} \cdots \wedge e_p$$

PROOF. By definition,

$$K(x_1,\dots,x_n)_p = \bigoplus_{1\le i_1<\cdots<i_p\le n} \overbrace{R\otimes\cdots\otimes R}^{\text{the } i_\bullet\text{-th tensormand is of degree 1}}$$

So set the summand for $i_1 < \cdots < i_p$ isomorphic to $e_{i_1}\wedge\cdots\wedge e_{i_p}R$. Then use Koszul convention, we get the expression of differential.

(B.3) Definition Let $x=(x_1,\dots,x_n)$ a sequence of elements in R, and an R-module M, denote

$$\mathsf{H}^n(x,M)=\mathsf{H}^n(\mathrm{Hom}_R(K(x),M)),\qquad \mathsf{H}_n(x,M)=\mathsf{H}_n(K(x)\otimes_R M)$$

(B.4) Proposition *For $x=(x_1,\dots,x_n)$, and an R-module M,*

$$\begin{cases}\mathsf{H}_0(x,M) &= M/\langle x_1,\dots,x_n\rangle M,\\ \mathsf{H}^0(x,M) &= \{a\in M : x_1a=0,\dots,x_na=0\}.\end{cases}$$

(B.5) Künneth theorem *For a complex $C_\bullet$, and $x\in R$*

$$0\to \mathsf{H}_0(x,\mathsf{H}_q(C))\to \mathsf{H}_q(K(x)\otimes C_\bullet)\to \mathsf{H}_1(x,\mathsf{H}_{q-1}(C))\to 0.$$

PROOF. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & 0\\ & & \downarrow & & \downarrow{\scriptstyle x} & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

and tensoring it with C, we get

$$\mathsf{H}_{q+1}(C[-1])\xrightarrow{x}\mathsf{H}_q(C)\to\mathsf{H}_q(K(x)\otimes C_\bullet)\to\mathsf{H}_q(C[-1])\xrightarrow{x}\mathsf{H}_q(C).$$

This is exactly the desired exact sequence.

(B.6) Definition (Regular sequence) Let $x = (x_1, \ldots, x_n)$, and an R-module M, we say x is **regular** on M, if x_i is not a zero divisor over $M/\langle x_1, \ldots, x_{i-1}\rangle M$ for all i.

(B.7) Theorem *If the sequence $x = (x_1, \ldots, x_n)$ is regular on M, then* $\mathsf{H}_\bullet(x, M) = \begin{cases} M/\langle x_1, \ldots, x_n\rangle M, & \bullet = 0, \\ 0 & \bullet \neq 0. \end{cases}$.

PROOF. By induction from (B.5).

(B.8) Theorem *If we have two ideals $\mathfrak{a}$ and $\mathfrak{b}$ generated by regular sequences $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ respectively, then*

$$\mathrm{Tor}_\bullet^R(R/\mathfrak{a}, R/\mathfrak{b}) = \mathsf{H}_\bullet(x, R/\mathfrak{b}) = \mathsf{H}_\bullet(y, R/\mathfrak{a}).$$

PROOF. Because the Koszul complex $R(x)$ and $R(y)$ form a free resolution of $R/\mathfrak{a}$ and $R/\mathfrak{b}$ respectively.

B.2 Some Commutative Algebra

For sake of lack of reference, the proof is presented here. The paper mentioned assume for local ring, but one can move the proof over polynomial ring.

(B.9) Theorem *Let R be a commutative ring, M a module, $x \in R$ not a zero divisor of R and M. Then $p.\dim_{R/xR} M/xM \leq p.\dim_R M$.*

PROOF. Consider a resolution of $P_\bullet \to M$, then $P_\bullet \otimes R/xR \to M/xM$ forms a resolution, since $\mathrm{Tor}(R/x, M) = 0$ by long exact sequence of $0 \to R \xrightarrow{x} R \to R/xR \to 0$.

(B.10) Auslander–Buchsbaum, [1] Proposition 6.2 *Let R be a noetherian ring. If M admits a finite finitely generated projective resolution of equal rank, then the annihilator of M is trivial or contains a nonzero divisor in R.*

PROOF. Denote the annihilator of M by $\mathfrak{a}$. Take $\mathfrak{p} \in \operatorname{ass} R$. Then $M_{\mathfrak{p}}$ itself is free. Actually, consider

$$0 \to R_{\mathfrak{p}}^n \xrightarrow{\varphi} R_{\mathfrak{p}}^m \to M \to 0.$$

If the graded ideal $\mathfrak{b}$ generated by n-mirror of matrix of φ lies in $\mathfrak{p}$. But some $x \in R$ with $\mathfrak{p} = \{y \in R : xy = 0\}$, then $\varphi(x, \dots, x) = 0$, so $zx = 0$ for some $z \notin \mathfrak{p}$, a contradiction. So $\mathfrak{b} = R$, so the φ splits.

But $\mathfrak{a} \otimes R_{\mathfrak{p}}$ kills $M_{\mathfrak{p}}$. So either $\mathfrak{a} \otimes R_{\mathfrak{p}} = 0$ or $M_{\mathfrak{p}} = 0$. Since it is of finite projective dimension, by consider the rank of each module of free resolution, all $M_{\mathfrak{p}} = 0$ or all $M_{\mathfrak{p}} \neq 0$.

Note that for any finitely generated module N, $N_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{ass} R$ means

$$\operatorname{ann} N \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{ass} R} \mathfrak{p} = \operatorname{z.div} R.$$

In our case, $\mathfrak{a}$ contains a non zero divisor, or $\operatorname{ann} \mathfrak{a}$ contains. But the latter means $\mathfrak{a} = 0$.

(B.11) Vasconcelos, [7] *For a polynomial ring R over field, the homogenous ideal I is generated by a regular sequence of homogenous elements if I/I^2 is free over R/I.*

PROOF. By Hilbert's syzygy theorem, any finitely generated graded module admits a finite finitely generated projective resolution. Since we will make induction, let us make clear whet we are going to prove.

■ Claim *If the ring R is graded noetherian, the homogenous ideal I admits admits a finite finitely generated projective resolution of equal rank such that I/I^2 is free over R/I, then I is generated by a regular sequence.*

Firstly, R/I also admits a finite finitely rank free (twisted) resolution. So as the annihilator of R/I, I does not fully consist of zero divisor.

I claim, there is some $x \in I \setminus R_+I$ which is non zero divisor of R. Otherwise, $I \subseteq R_+I \cup \bigcup_{\mathfrak{p}\in \mathrm{ass}\, R} \mathfrak{p}$. Since $I \neq R_+I$, and by prime avoidance, I lies in some $\mathfrak{p}$, full of zero divisor, a contradiction.

Pick a nonzero divisor $x \in I \setminus R_+I$, then consider $\overline{R} = R/xR$, and $\overline{I}$ the image of I. It is clear, now $\overline{I}/\overline{I}^2 = I/(I^2 + xR)$ is free of less rank than $\overline{R}/\overline{I} = R/I$. Also, $\overline{I}$ is of finite projective dimension.

(B.12) Λemma (Prime avoidance) *If $\mathfrak{a} \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$ with at most two of $\mathfrak{p}_i$'s are not prime, then $\mathfrak{a}$ lies in one of $\mathfrak{p}_i$.*

(B.13) Λemma *Under the assumption of above theorem (B.11), any permutation of regular sequence is still a regular sequence.*

PROOF. It suffices to prove for two elements. Assume x, y are regular sequence, then y, x forms. Since if $yz = 0$, pick the z to be homogenous and of minimal degree, we see $y(z \bmod xR) = 0$, so $z \in xR$, say $z = xz_0$, but x is not zero divisor, then we find a smaller z. Then, assume $x(w \bmod yR) = 0$, then $xw = yz$ for some z, so $y(z \bmod xR) = 0$, hence $z \in xR$. Since we have proven x is not a zero divisor, so $w \in yR$, i.e. $(w \bmod yR) = 0$.

(B.14) Theorem *Under the assumption of above theorem (B.11), any element presenting a set of basis of I/I^2 over R/I forms a regular sequence.*

PROOF. By our process, the regular sequence presenting a set of basis. So for any choice of basis, it differs in each degree by an invertible matrix over field. Write the invertible matrix into product of elementary transform, it turns out it suffices to show the permutation of two elements in regular sequence.

B.3 Flag Manifolds

(B.15) !! Notation— Let k be a field, we will write $\mathsf{H}(-) = \mathsf{H}(-; k)$ for short. Let G a compact lie group, T its maximal torus. The homogenous space G/T, known as **flag manifold**. Let Δ be its root system, and $M = L \otimes k$ where L is the weight lattice, W its Weyl group.

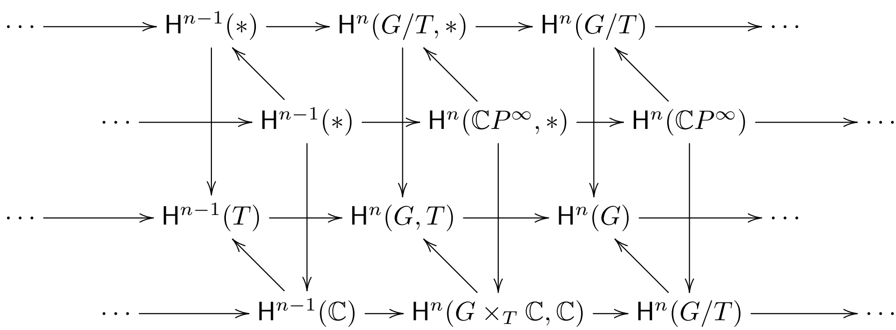
(B.16) Theorem *The odd degree of $\mathsf{H}^\bullet(\mathcal{F}\ell(G); k)$ is zero.*

PROOF. Due to the theory of lie groups (Bruhat decomposition), $\mathcal{F}\ell(G)$ admits a cellular decomposition whose cells have only even dimensions.

(B.17) Definition For any $\chi \in L$, it defines $T \xrightarrow{\chi} \mathbb{C}$, we can get a line bundle $G \times_T \mathbb{C}$ over G/T. Define its Chern class by $\psi(\chi)$. Then it extends to $S(M) \xrightarrow{\psi} \mathsf{H}^\bullet(G/T)$.

(B.18) Theorem *The ψ coincides the transgression.*

PROOF. Consider the following diagram



when $n = 2$, it is exactly what we want Chern class.

(B.19) Theorem *Let $\alpha \in \Delta$, there is a unique $\Delta_\alpha x$ such that*

$$x - s_\alpha x = \psi(\alpha)\Delta_\alpha x$$

for all $x \in \mathsf{H}^\bullet(G/T)$.

PROOF. For each $\alpha \in \Delta$, denote S_α the subgroup of G of rank 1 corresponding to α. Then $S_\alpha T/T = S_\alpha / S_\alpha \cap T \cong \mathbb{C}P^1$. Consider the fibration

$$S_\alpha T/T \to G/T \xrightarrow{p} G/S_\alpha T.$$

By a Leray-Hirsch (or spectral sequence) argument,

$$\mathsf{H}^\bullet(G/T) \cong \mathsf{H}^\bullet(G/S_\alpha T) \otimes \mathsf{H}^\bullet(\mathbb{C}P^1)$$

So any $x \in \mathsf{H}^\bullet(G/T)$ can be written uniquely as

$$x = x_1\phi(R_\alpha) + x_2, \qquad x_i \in p_*(\mathsf{H}^\bullet(G/S_\alpha T))$$

where R_α is any weight $\langle R_\alpha, \alpha^\vee\rangle = 1$. Then, since x_i is stable under the reflection s_α,

$$\begin{aligned} s_\alpha x &= s_\alpha x_1 \cdot s_\alpha \phi(R_\alpha) + s_\alpha x_2 \\ &= x_1 \phi(s_\alpha R_\alpha) + x_2 \\ &= x_1 \phi(R_\alpha - \alpha) + x_2. \end{aligned}$$

Take the difference, then we get what we want.

(B.20) Definition (Difference operator) Let $\alpha \in \Delta$ be a root.

- Over $\mathsf{H}^\bullet(G/T)$, we defined above Δ_α the unique element such that
$$x - s_\alpha x = \psi(\alpha)\Delta_\alpha x.$$
- Over $S(M)$ we also have algebraic difference operator
$$\Delta_\alpha x = \frac{x - s_\alpha x}{\alpha}.$$

Clearly, $\psi\Delta_\alpha(x) = \Delta_\alpha\psi(x)$.

(B.21) Lemma *As notations above,*

$$\ker\psi = \left\{ f \in S(M) : \begin{array}{l} \forall \alpha_1, \dots, \alpha_n \in \Delta, \\ \Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_n} f \in S_+(M). \end{array} \right\}.$$

PROOF. If $\psi(f) = 0$, then of course $\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_n} f \mapsto 0$, so $\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_n} f \in S_+(M)$. Conversely,it suffices to prove that $\mathsf{H}^\bullet(G/T)^W = \mathsf{H}^0(G/T)$. For such an $x \in \mathsf{H}^\bullet(G/T)$, it must come from $G/S_\alpha T$ for all α. The cellular decomposition do not allow this except the constant.

(B.22) Lemma *As notations above,* $\ker\psi$ *is generated by a regular sequence.*

PROOF. Denote $I = \ker\psi$. By (B.11), it suffices to show I/I^2 is free over $S(M)/I$. Let $y_1, \dots, y_n$ be a set of generator with

$$\deg y_1 \le \deg y_2 \le \cdots, \qquad y_k \notin \sum_{i<k} y_i S(M) + I^2. \tag{$*$}$$

So $\Delta_\alpha y_k \in \sum_{i<k} y_i S(M) + I^2$. If

$$\sum s_i y_i \in I^2$$

Pick the $s_i \notin I$ with $\deg y_i$ maximal. Then applying Δ_α several times reduce s_i to some nonzero constant, but then it contradicts to $(*)$.

(B.23) Lemma *As notations above,* $\mathsf{H}^\bullet(G/T)$ *is a free module over* $\operatorname{im}\phi$.

PROOF. Note that $\mathsf{H}^\bullet(G/T)/\langle\psi(M)\rangle$ is an $S(M)/\langle M\rangle = k$-module. So pick a representative y_i of basis. So $\sum y_i \operatorname{im}\psi = \mathsf{H}^\bullet(G/T)$. If

$$\sum \psi(a_i) y_i = 0$$

If $\psi(a_i) \neq 0$, we can apply some Δ_α to get a smaller relation, a contradiction.

(B.24) Remark For field of characteristic zero,

$$\ker\psi = \text{the ideal generated by } S_+^W(M).$$

Denote the right hand side by I. For fg with $g \in S_+^W(M)$, $\Delta_\alpha(fg) = (\Delta_\alpha f)\cdot g$, so $I \subseteq \ker\psi \subseteq$. Conversely, if $\alpha \in \Delta$, with $\Delta_\alpha f \in I$, then $f \equiv s_\alpha f \bmod I$, so $f \equiv \frac{1}{|W|}\sum_{w\in W} f \bmod I$, where $\frac{1}{|W|}\sum_{w\in W} f$ is of course invariant.

It is well-known that $S(M)$ is a free $S^W(M)$-module, and $S^W(M)$ isomorphic to a polynomial ring, say of **fundamental invariants** $e_1,\dots,e_n$. Assume $S(W) = S(M)^W \otimes k^n$, then the ideal generated by $S(M)_+^W$ is exactly $S(M)_+^W \otimes k^n$. So as $S(M)^W$-module,

$$S(W) = S(M)^W \otimes S(M)_W.$$

Then the Poincaré series

$$\mathcal{P}(S(M)_W) = \frac{\mathcal{P}(S(W))}{\mathcal{P}(S(W)^W)} = \frac{\prod_{i=1}^n \frac{1}{1-t}}{\prod_{i=1}^n \frac{1}{1-t^{\deg e_i}}} = \prod_{i=1}^n \frac{1-t^{\deg e_i}}{1-t}.$$

B.4 Cohomology Computation

Let k be a field.

(B.25) The Collapse Theorem *For a compact Lie group G, with its maximal torus T, the Leray-Serre spectral sequence E for*

$$T \to G \to G/T$$

collapse for $\mathsf{E}_{\geq 3}$, that is, $\mathrm{d}_{\geq 3} = 0$. Moreover,

$$\mathsf{E}_3 = \mathsf{H}^\bullet(G/T;k)/\langle\psi(M)\rangle \otimes_k \Lambda_k(\xi_{2d_1-1},\cdots,\xi_{2d_\ell-1})$$

where the isomorphism preserves the multiplication structure, and $d_1,\dots,d_\ell$ are the degrees of regular sequence claimed in (B.22).

PROOF. Now $\mathsf{H}^\bullet(T) = \Lambda(M)$, so

$$\begin{aligned}\mathsf{E}_2 &= \mathsf{H}^\bullet(G/T) \otimes \mathsf{H}^\bullet(T) \\ &= \mathsf{H}^\bullet(G/T)/\left\langle \operatorname{im}\psi \right\rangle \otimes_k \operatorname{im}\psi \otimes_k \Lambda(M) \\ &= \mathsf{H}^\bullet(G/T)/\left\langle \operatorname{im}\psi \right\rangle \otimes_k S(M)/\ker\psi \otimes_{S(M)} \Lambda(S(M) \otimes_k M)\end{aligned}$$

Over E_2, the differential is generated by the transgression

$$\mathsf{E}_2^{01} = \mathsf{H}^1(T) = M \xrightarrow{\psi} \mathsf{H}^2(G/T) = \mathsf{E}_2^{20}.$$

So there is no differential over $\mathsf{H}^\bullet(G/T)/\left\langle \operatorname{im}\psi \right\rangle \otimes_k$. If we pick any basis $m_1, \dots, m_r$ of M, then

$$\Lambda(S(M) \otimes_k M) = K(m_1, \dots, m_r)$$

Let $x_{d_1}, \dots, x_{d_\ell}$ be homogenous regular sequence generated $\ker\psi$. Then consider the double complex

$$K(m_1, \dots, m_r) \otimes K(x_{d_1}, \dots, x_{d_\ell}).$$

By a standard trick used hundred times, and the fact $S(M)/\left\langle m_1, \dots, m_r \right\rangle = k$, we can find that

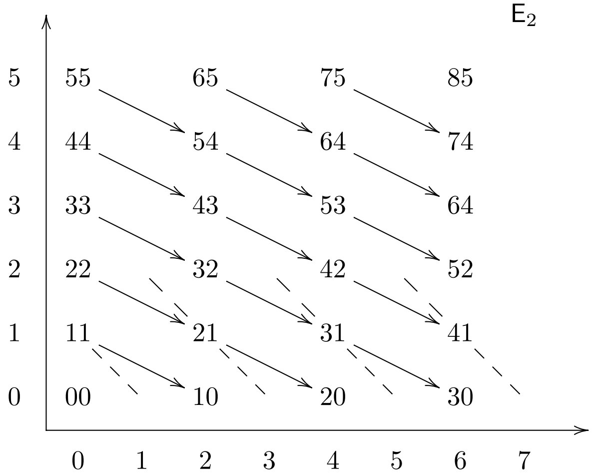
$$\mathsf{H}\Big(S(M)/\ker\psi \otimes_{S(M)} \Lambda(S(M) \otimes_k M)\Big) = \Lambda_k(x_{d_1}, \dots, x_{d_\ell}),$$

with x_i from the i-th homology group of

$$\big(S(M)/\ker\psi\big)_i \otimes_{S(M)} \big(\Lambda(S(M) \otimes_k M)\big)_1.$$

The next task is to find the degree of preimage of x_{d_i}. Actually, since ψ twice the degree, we point the degree of $S(M)/\ker\psi \otimes_{S(M)}$

$\Lambda(S(M) \otimes_k M)$ in this digram



So

$$\mathsf{E}_3 = \mathsf{H}^\bullet(G/T;k)/\langle \operatorname{im}\psi \rangle \otimes_k \Lambda_k(\xi_{2d_1-1},\cdots,\xi_{2d_\ell-1}).$$

Now, E_3 is generated by E_3^{p0} and E_3^{p1}, so d_3 is zero. By induction, E_r is generated by E_3^{p0} and E_3^{p1}, so $\mathrm{d}_{\geq 3}$ is zero.

(B.26) Theorem *If k is of characteristic zero, then*

- $\mathsf{H}^\bullet(G/T;k) = S(M)_W$.
- $\mathsf{H}^\bullet(G;k) = \Lambda_k(\xi_{2d_1-1},\cdots,\xi_{2d_\ell-1})$, *with $d_1,\ldots,d_\ell$ the degree of fundamental invariants.*

PROOF. In view of Borel–Hopf theorem (A.15), the cohomology group $\mathsf{H}^\bullet(G;k)$ has no place for even dimensional generating stuff. So $\mathsf{H}^\bullet(G/T;k)/\langle \psi(M) \rangle = k$.

(B.27) EXAMPLE For $\mathsf{U}(n)$, the Weyl group acts by permutating indices on variables of $k[x_1,\ldots,x_n]$. So its fundamental invariants are just symmetric polynomials. So for characteristic zero field k,

$$\mathsf{H}^\bullet(G;k) = \Lambda_k(\xi_1, \cdots, \xi_{2n-1}).$$

(B.28) Remark The degree of fundamental invariants can be computed.

A_n	$2, 3, 4, \ldots, n+1$	F_4	$2, 6, 8, 12$
B_n, C_n	$2, 4, 6, \ldots, 2n$	E_6	$2, 5, 6, 8, 9, 12$
D_n	$2, 4, 6, \ldots, 2n-1, n$	E_7	$2, 6, 8, 10, 12, 14, 18$
G_2	$2, 6$	E_8	$2, 8, 12, 14, 18, 20, 24, 30$

(B.29) Remark The computation over characteristic zero can be done by Schubert calculus.

Exercises

▶ **(B.30)** Problem. Over characteristic zero field, determine $\mathsf{H}^\bullet(G) \to \mathsf{H}^\bullet(T)$ induced by the inclusion. Assuming G is simply connected. Hint: Remind our theorem (A.10), and (B.25). If $\mathsf{H}^1(G) = \Lambda(x_\bullet)$ only the $x_\bullet$ of degree 1 maps to a nozero element of $\mathsf{H}^\bullet(T)$. But it never exists due to simple-connectedness.

Appendix C

Cohomology for Discrete Groups

The material of this chapter is mainly from [3].

C.1 Equivariant cohomology

(C.1) !! Notation— Let G be a discrete group.

(C.2) Definition Given a chain complex $C_\bullet$ in G-Mod, we can define the **equivariant homology group**

$$\mathsf{H}_\bullet(G; C) = \mathsf{H}_\bullet(\mathrm{Tot}(F_\bullet \otimes_G C_\bullet)),$$

where $F_\bullet \to \mathbb{Z}$ a resolution.

For a cochian complex $C^\bullet$ in G-Mod, we can similarly define the **equivariant cohomology group**

$$\mathsf{H}^\bullet(G; C) = \mathsf{H}_\bullet(\mathrm{Tot}(\mathrm{Hom}_G(F_\bullet, C^\bullet))).$$

This can be understood as the homology theory in G-Mod.

(C.3) Theorem *Given a chain complex $C_\bullet$ in G-Mod, then there is two spectral sequences* E*'s with*

$$\mathsf{E}^2_{pq} = \mathsf{H}_q(\mathsf{H}_p(G; C_\bullet)) \quad \textit{and} \quad \mathsf{E}^2_{pq} = \mathsf{H}_p(G; \mathsf{H}_q(C_\bullet))$$

respectively which both converge to $\mathsf{H}_\bullet(G; C)$.

PROOF. Actually,

$$F_p \otimes_G C_q \overset{p}{\Big|} \mathsf{H}_p(G; C_q) \overset{q}{\Big|} \mathsf{H}_q(\mathsf{H}_p(G; C_\bullet))$$

$$F_p \otimes_G C_q \overset{q}{\Big|} F_p \otimes_G \mathsf{H}_q(C_\bullet) \overset{p}{\Big|} \mathsf{H}_q(G; \mathsf{H}_p(C_\bullet))$$

The above says everything.

(C.4) Theorem *Given a cochain complex $C^\bullet$ in G-Mod, then there is two spectral sequence* E*'s with*

$$\mathsf{E}_2^{pq} = \mathsf{H}^q(\mathsf{H}^p(G; C^\bullet)) \quad \textit{and} \quad \mathsf{E}_2^{pq} = \mathsf{H}^p(G; \mathsf{H}^q(C_\bullet))$$

respectively which both converge to $\mathsf{H}^\bullet(G; C)$.

PROOF. Since

$$\mathrm{Hom}_G(F_p, C^q) \overset{p}{\Big|} \mathsf{H}^p(G; C^q) \overset{q}{\Big|} \mathsf{H}^q(\mathsf{H}^p(G; C^\bullet))$$

$$\mathrm{Hom}_G(F_p, C^q) \overset{q}{\Big|} \mathrm{Hom}_G(F_p, \mathsf{H}^q(C^\bullet)) \overset{p}{\Big|} \mathsf{H}^q(G; \mathsf{H}^p(C^\bullet))$$

A daily computation.

(C.5) EXAMPLE If C is acyclic, say with $\mathsf{H}_0(C) = M$ or $\mathsf{H}^0(C) = M$, then

$$\mathsf{H}_\bullet(G; C) = \mathsf{H}_\bullet(G; M), \quad \mathsf{H}^\bullet(G; C) = \mathsf{H}^\bullet(G; M).$$

(C.6) EXAMPLE If C is trivial G-module, then

$$F_\bullet \otimes_G C_\bullet = (F_\bullet \otimes_G \mathbb{Z}) \otimes_\mathbb{Z} C_\bullet,$$

$$\mathrm{Hom}_G(F_\bullet, C^\bullet) = \mathrm{Hom}_\mathbb{Z}(F_\bullet \otimes_G \mathbb{Z}, C^\bullet).$$

If $C_\bullet$ is flat abelian group, then it can be compute by universal coefficient theorem.

(C.7) EXAMPLE If $C_\bullet$ is projective (or generally co-induced) G-module, then

$$\mathsf{H}^q(\mathsf{H}^p(G; C^\bullet)) = \begin{cases} \mathsf{H}^q(G; (C^\bullet)^G), & p = 1 \\ 0 & p \geq 1 \end{cases}$$

Thus $\mathsf{H}(G; C) = \mathsf{H}(C^G)$. So we get a spectral sequence E with $\mathsf{E}_2^{pq} = \mathsf{H}^q(G; \mathsf{H}^p(C^\bullet))$ converges to $\mathsf{H}(C^G)$.

(C.8) EXAMPLE If $C_\bullet$ is projective (or generally induced) G-module, then

$$\mathsf{H}_q(\mathsf{H}_p(G; C^\bullet)) = \begin{cases} \mathsf{H}_q(G; (C_\bullet)_G), & p = 1 \\ 0 & p \geq 1 \end{cases}$$

Hence $\mathsf{H}(G; C) = \mathsf{H}(C_G)$. So we get a spectral sequence E with $\mathsf{E}^2_{pq} = \mathsf{H}_q(G; \mathsf{H}_p(C_\bullet))$ converges to $\mathsf{H}(C_G)$.

C.2 The Cartan–Leray Spectral Sequence

(C.9) Definition For a G-set X, we can define the equivariant homology group and cohomology of X

$$\begin{aligned} \mathsf{H}_\bullet^G(X) &= \mathsf{H}_\bullet(G; C_\bullet(X)) \\ &= \mathsf{H}_\bullet(\mathrm{Tot}(F_\bullet \otimes_G C_\bullet(X))); \\ \mathsf{H}^\bullet_G(X) &= \mathsf{H}^\bullet(G; C^\bullet(X)) = \mathsf{H}^\bullet(G; \mathrm{Hom}_G(F_\bullet, \mathrm{Tot}(\mathrm{Hom}(C_\bullet(X), \mathbb{Z})))) \\ &= \mathsf{H}^\bullet(G; \mathrm{Tot}(\mathrm{Hom}_\mathbb{Z}(F_\bullet \otimes_G C_\bullet(X), \mathbb{Z}))) \\ &= \mathsf{H}^\bullet(G; \mathrm{Hom}_\mathbb{Z}(\mathrm{Tot}(F_\bullet \otimes_G C_\bullet(X)), \mathbb{Z})). \end{aligned}$$

where C is some (co)homology theory, for example, singular (co)homology, cellular (co)homology if X admits a G-cellular structure, etc.

(C.10) Remark Generally, if G is a topological group, X a G-set, then the equivariant cohomology is defined by

$$\mathsf{H}_G^\bullet(X) = \mathsf{H}^\bullet(EG \times_G X)$$

which can be proven by (2.11).

(C.11) EXAMPLE Here list some special cases.

- If X is a trivial G-set, then $C_\bullet(EG) \otimes_G \mathbb{Z} = C_\bullet(BG)$, thus

$$\mathsf{H}_G^\bullet = \mathsf{H}^i(BG \times X).$$

- If X is a G-set with G acting freely, then $C_\bullet(X)$ and $C^\bullet(X)$ are all free $\mathbb{Z}[G]$-modules. If further, X admits a G-celluar structure, then we can compute by cellular homology, then $(C_\bullet(X))_G$ and $C^\bullet(X)$ are nothing but $C_\bullet(X/G)$, and $C^\bullet(X/G)$.

(C.12) Cartan–Leray Spectral Sequence *If X is a connected space on which G acts freely and properly, then there is a spectral sequence E with*

$$\mathsf{E}^2_{pq} = \mathsf{H}_p(G, \mathsf{H}_q(X))$$

and converging to $\mathsf{H}_\bullet(X/G)$.

PROOF. In which case, $X \to X/G$ is a covering map, and we take some CW-approximation of X/G, then we get a CW-structure over X which is compatible with G-action.

(C.13) Cartan–Leray Spectral Sequence *If X is a connected space on which G acts freely and properly, then there is a spectral sequence E with*

$$\mathsf{E}_2^{pq} = \mathsf{H}^p(G, \mathsf{H}^q(X))$$

and converging to $\mathsf{H}^\bullet(X/G)$.

(C.14) EXAMPLE When G is finite, and assume the coefficient is taken to be some field k of characteristic zero, then $\mathsf{H}^{\geq 2} = 0$, and $\mathsf{H}_{\geq 2} = 0$. So

$$\mathsf{H}^{\bullet}(X/G; k) = \mathsf{H}^{\bullet}(X)^{G}, \quad \mathsf{H}_{\bullet}(X/G; k) = \mathsf{H}_{\bullet}(X)_{G}$$

In particular, when $X = EG$, $\mathsf{H}^{\bullet}(BG; k) = 0$ and $\mathsf{H}_{\bullet}(BG; k) = 0$. Actually, $\mathsf{H}^{\bullet}(G; \mathbb{Z})$ and $\mathsf{H}_{\bullet}(G; \mathbb{Z})$ are both finite abelian groups.

Exercises

► **(C.15)** EXERCISE. If $X \xrightarrow{f} Y$ is map between G-set, if $\mathsf{H}^{\bullet}(X) \xrightarrow{H(f)} \mathsf{H}^{\bullet}(Y)$ are all isomorphism, show that so are $\mathsf{H}^{\bullet}_{G}(X) \to \mathsf{H}^{\bullet}_{G}(Y)$.

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