Borel–Weil Theorem and Applications

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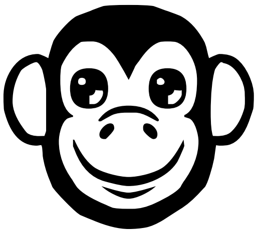

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1 Lecture 1 — Borel–Weil Theorem

1.1 Let G be a reductive group over $\mathbb{C}$, and B be its Borel subgroup. We call G/B the **flag variety** of G.

- G/B only depends on the Dynkin type of G.
- If K is the compact form of G, then $G/B \cong K/S$ with $S = K \cap B$ the maximal torus of K.
- G/B is a projective variety. An explicit embedding can be constructed by Plücker embedding.

For example, $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ has the same flag variety. One can also construct the flag manifold from $U(n)$ or $SU(n)$.

1.2 For type A, we take GL_n, we take $B = \left(\begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{smallmatrix}\right)$ the group of invertible upper triangular matrices then we can identify G/B with

$$\mathcal{F}\ell(n) = \mathcal{F}\ell(\mathbb{C}^n) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim V_i = i. \right\}$$

1.3 For other classic types, we take the symmetric form over $\mathbb{C}^n$ defining $\mathrm{SO}(n)$ to be

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{y}^{\mathrm{t}} \left(\begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix}\right) \mathbf{x} = x_1 y_n + \cdots + x_n y_1,$$

and the symplectic form over $\mathbb{C}^n$ defining $\mathrm{Sp}(n)$ to be

$$\omega(\mathbf{x}, \mathbf{y}) = \mathbf{y}^{\mathrm{t}} \left(\begin{smallmatrix} & & 1 \\ & \iddots & \\ -1 & & \end{smallmatrix}\right) \mathbf{x} = x_1 y_n + \cdots - x_n y_1$$

Then the Borel subgroup is exactly of the form $B = \left(\begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{smallmatrix}\right)$. In this case, G/B can be identifies with

$$\left\{ V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n : \dim V_i = i,\ V_i^{\perp} = V_{n-i}. \right\}.$$

1.4 Denote the maximal torus of B to be T, and the unipotent radical of B to be U. Recall that $B = U \rtimes T$, that is, we have a split short exact sequence of groups

$$0 \longrightarrow U \longrightarrow B \longrightarrow T \longrightarrow 0.$$

As a result, any representation of T can be extended to B (with trial U-action).

1.5 We denote $\mathbb{G}_m = \mathbb{C}^\times$ the algebraic group with natural multiplication. Let T be a torus. An algebraic group homomorphism $\lambda : T \to \mathbb{G}_m$ is called a **character** of T. We denote $X(T)$ the group of all character, we will write them additively

$$(\lambda + \mu)(t) = \lambda(t)\mu(t), \qquad (-\lambda)(t) = \lambda(t)^{-1}.$$

Sometimes, we may write e^λ to avoiding abuse of notations.

1.6 Let λ be a character of T, that is an algebraic group homomorphism $T \to \mathbb{G}_m = \mathbb{C}^\times$. It corresponds to a one-dimenisonal representation $\mathbb{C}(\lambda)$ with $t \in T$ acts by $\lambda(t)^{-1}$. It naturally extended to B.

Consider the space $\xi(\lambda) = G \times_B \mathbb{C}(\lambda)$. It is a G-equivariant line bundle over G/B. Let us denote the corresponding sheaf to be $\mathcal{O}(\lambda)$.

Actually, all the G-equivariant line bundle over G/B comes from this construction. since the fibre of $1 \cdot B/B$ is an one-dimensional representation of B (thus factor through T).

1.7 For $G = \mathrm{GL}_n$, the maximal torus $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is the group of diagonal matrices. We denote $x_1, \ldots, x_n \in X(T)$ the coordinate of indices.

Let us denote the **tautological bundle** ϕ_k over $\mathcal{F}\ell(n)$ to be the k-dimensional vector bundle whose fibre at the flag $(V_0 \subseteq \cdots \subseteq V_n)$ is V_k. Then by explicit computation $\phi_k/\phi_{k-1} \cong \mathcal{O}(-x_k)$.

In particular, for $n = 2$, $\mathcal{F}\ell(2) = \mathbb{P}^1$, $\mathcal{O}(x_1) = \mathcal{O}(1)$.

1.8. Borel–Weil Theorem For any character $\lambda \in X(T)$,

$$H^0(G/B; \mathcal{O}(\lambda))^* = \begin{cases} L(\lambda) & \lambda \text{ is dominant} \\ 0 & \text{otherwise} \end{cases}$$

where $L(\lambda)$ the the finite dimensional representation of G with the highest weight λ.

1.9. Proof We have a G-bimodule decomposition

$$\mathbb{C}[G] = \bigoplus_{\lambda \text{ dominant}} L(\lambda)^* \otimes L(\lambda).$$

Since $\mathrm{Hom}_G(V(\lambda), \mathbb{C}[G]) \cong \mathrm{Hom}_{\mathbb{C}}(V(\lambda), \mathbb{C})$. On the other hand, a section of $\mathcal{O}(\lambda)$ is exactly a map $f : G \to \mathbb{C}$ with $f(g) = \lambda^{-1}(b)f(gb) = \lambda^{-1}(b)(r_b f)(g)$ where r_b is the right multiplication by b.

$$\begin{array}{ccc} G \times \mathbb{C}(\lambda) & \longrightarrow & G \times_B \mathbb{C}(\lambda) \\ \uparrow\downarrow & & \uparrow\downarrow \\ G & \longrightarrow & G/B \end{array}$$

As a result, there only rest $L(\lambda)^*$. Q.E.D.

1.10 The tangent bundle of G/B is given by $G \times_B \mathfrak{g}/\mathfrak{b}$ with the action by adjoint action. Note that U does not acts $\mathfrak{g}/\mathfrak{b}$ trivially, but there is a filtration, such that

$$\mathrm{gr}\, \Omega^1_{G/B} = \bigoplus_{\alpha_i \in \Delta^+} \mathcal{O}(-\alpha_i)$$

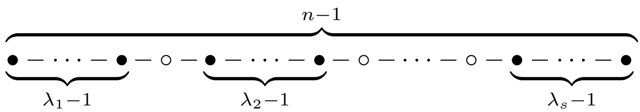
where Δ^+ the set of positive roots. In particular, the canonical bundle $\omega = \mathcal{O}(-2\rho)$ where ρ is the half sum of positive roots. By Serre duality,

$$H^{N-i}(G/B; \mathcal{O}(-2\rho - \lambda)) = H^i(G/B; \mathcal{O}(\lambda))^*,$$

where $N = \dim G/B$. The dual is the dual of G-representation when G is semi-simple.

1.11 Let P be a standard parabolic subgroup. That is, there is a subset $J \subseteq \mathbb{I}$ such that $P = \bigcup_{w \in W_J} BwB$, where W_J is the Weyl group generated by $\{s_j : j \in J\}$. We denote $P_i = B \cup Bs_iB$ the **minimal parabolic subgroup**.

1.12 For the case of type A. A subset of $\mathbb{I} = \{1, \dots, n-1\}$ cuts the Dynkin diagram into pieces. Assume it is



Then $n = \lambda_1 + \cdots + \lambda_s$, and the corresponding

$$P = \begin{pmatrix} \mathrm{GL}_{\lambda_1} & * & \cdots & * \\ & \mathrm{GL}_{\lambda_2} & \cdots & * \\ & & \ddots & \vdots \\ & & & \mathrm{GL}_{\lambda_s} \end{pmatrix}.$$

Furthermore, G/P is identified with the partial flag variety

$$\mathcal{F}\ell_\lambda(n) = \mathcal{F}\ell_\lambda(\mathbb{C}^n) = \Big\{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_s : \dim V_i/V_{i-1} = \lambda_i. \Big\}.$$

In particular, G/P_i is identified with

$$\Big\{ 0 \subseteq V_1 \subseteq \cdots \widehat{V_i} \cdots \subseteq V_n : \dim V_i = i. \Big\}$$

For the case $n = k + (n-k)$, then G/P is identified with the **Grassmannian**

$$\mathcal{G}r(k,n) = \Big\{ V \subseteq \mathbb{C}^n : \dim V = k \Big\}.$$

1.13. Plücker Embedding Let ρ be the half sum of simple roots. Let $L(\rho)$ be the finite dimensional representation of G with the highest vector v_0. The orbit map

$$G \longrightarrow \mathbb{P}(L(\rho)) \qquad g \longmapsto g[v_0]$$

factors through an embedding of G/B. This is called the **Plücker embedding**. In general, for any $\lambda \in X(T)$,

$$G \longrightarrow \mathbb{P}(L(\lambda)) \qquad g \longmapsto g[v_0]$$

factors through an embedding of G/P for P the stablizer of $[v_0]$.

1.14 For example, when $\lambda = \omega_i$ the fundamental weight, then the corresponding P is maximal parabolic. In GL_n, for $\lambda = \omega_k = x_1 + \ldots + x_k$, $L(\omega_k) = \Lambda^k \mathbb{V}$ where $\mathbb{V}$ is the natural representation. It gives the classic Plücker embedding for $\mathcal{G}r(k,n)$.

1.15 For each i, we have a natrual map $\mathrm{SL}_2 \to G$ with image in P_i. This inducing an isomorphism $\mathbb{P}^1 \cong \mathrm{SL}_2 / \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \cong P_i/B$. The restriction of $\mathcal{O}(\lambda)$ to P_i/B corresponds to $\mathcal{O}(d)$ over $\mathbb{P}^1$ with $d = \langle \alpha_i^\vee, \lambda \rangle$.

The natrual projection $G/B \to G/P$ is a fibre bundle with fibre P/B. In particular, when $P = P_i$, it is a $\mathbb{P}^1$ bundle.

1.16 Recall that over $\mathbb{P}^1$, we have

$\mathcal{O}(n)$	$\cdots$	$\mathcal{O}(-4)$	$\mathcal{O}(-3)$	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}(0)$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\cdots$
$\dim H^0$	$\cdots$	0	0	0	0	1	2	3	$\cdots$
$\dim H^1$	$\cdots$	3	2	1	0	0	0	0	$\cdots$

Actually the pairing

$$H^i(\mathbb{P}^1; \mathcal{O}(-1+d)) \times H^{1-i}(\mathbb{P}^1; \mathcal{O}(-1-d)) \to H^1(\mathbb{P}^1; \mathcal{O}(-2))$$

is a perfect pairing.

1.17. Borel–Weil Theorem When $\langle \alpha_i^\vee, \lambda \rangle \geq -1$,

$$H^i(G/B; \mathcal{O}(\lambda)) = H^{i+1}(G/B; \mathcal{O}(s_i \bullet \lambda)).$$

Recall: for $w \in W$ and $\lambda \in X(T)$, we denote $w \bullet \lambda = w(\lambda + \rho) - \rho$.

1.18. Proof of the case $\langle \alpha_i^\vee, \lambda \rangle = -1$ Consider the Serre–Leray spectral sequence for

$$\begin{array}{ccc} G/B & \longrightarrow & \operatorname{Spec} \mathbb{C} \\ & \searrow \quad \nearrow & \\ & G/P_i & \end{array}$$

Since $G/B \to G/P_i$ is a fibre bundle, it suffices to see the cohomology of the fibre. But by the computation of $\mathbb{P}^1$, it is identical zero. Q.E.D.

1.19. Proof of the case $\langle \alpha_i^\vee, \lambda \rangle = 0$ Denote $p : G/B \to G/P$. Consider the natural map

$$p^* p_* \mathcal{O}(\lambda + \rho) \longrightarrow \mathcal{O}(\lambda + \rho).$$

This is surjective by fibrewise computation. The kernel of this map is $\mathcal{O}(s_i(\lambda + \rho))$ by direct computation. So we get

$$0 \longrightarrow \mathcal{O}(s_i \bullet \lambda) \longrightarrow p^* p_* \mathcal{O}(\lambda + \rho) \otimes \mathcal{O}(-\rho) \longrightarrow \mathcal{O}(\lambda) \longrightarrow 0.$$

Use the spectral sequence argument again, we get from the long exact sequence that

$$H^i(G/B; \mathcal{O}(s_i \bullet \lambda) = H^{i+1}(G/B; \mathcal{O}(\lambda)).$$

We get the assertion. Q.E.D.

1.20. Proof of the general case The general case is similar, but technical. We can construct a filtration of $p^*p_*\mathcal{O}(\lambda+\rho)$ with subquotients

$$\mathcal{O}(s_i(\lambda+\rho)), \quad p^*p_*\mathcal{O}(\lambda-\alpha_i+\rho), \quad \mathcal{O}(\lambda+\rho).$$

By the spectral sequence argument, we can ignore $p^*p_*\mathcal{O}(\cdots)$ after tensoring with $-\rho$. Q.E.D.

1.21. Principal Block Assume G is semisimple. We denote $\mathcal{O}(w) = \mathcal{O}(w \bullet 0)$, then

$$\dim H^i(G/B;\mathcal{O}(w)) = \begin{cases} 1 & i=\ell(w) \\ 0 & \text{otherwise} \end{cases}$$

1.22 Assume the smooth projective variety X is acted by algebraic torus T with discrete fixed points X^T. For a T-equivariant vector bundle $\mathcal{F}$ over X, we have the **Atyiah–Bott Localization** for $t \in T$,

$$\sum (-1)^i \operatorname{tr}(t; H^i(X;\mathcal{F})) = \sum_{x\in X^T} \frac{\operatorname{tr}(t;\mathcal{F}|_x)}{\det(1-t|_{T_x^*X})}$$

where T_x^*X is the cotangent space of X at x, and $\mathcal{F}|_x = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ is the fibre at x.

1.23 At any point $xB/B \in G/B$, the tangent space is naturally identified with $\operatorname{ad}_x \mathfrak{g}/\mathfrak{b}$. We know at point $1\cdot B/B$, $T_x^* = \bigoplus_{\alpha\in\Delta^+}\mathbb{C}(-\alpha)$ as T-space. So

$$\det(1-t|_{T_x^*X}) = w \cdot \prod_{\alpha\in\Delta^+}\left(1-e^{\alpha_i}\right).$$

Similarly, $\operatorname{tr}(t;\mathcal{O}(\lambda)|_x) = w\cdot e^{-\lambda}$. Thus

$$\operatorname{tr}(t;H^i(X;\mathcal{O}(\lambda))) = \sum_{w\in W} w\frac{e^{-\lambda}}{\prod_{\alpha\in\Delta^+}\left(1-e^{\alpha_i}\right)} = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(-\lambda-\rho)}}{\prod_{\alpha\in\Delta^+}(e^{-\alpha_i/2}-e^{\alpha_i/2})}.$$

Then taking the dual, we get

$$\operatorname{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha_i/2}-e^{-\alpha_i/2})}.$$

We get the **Weyl character formula**.

1.24 In the case GL_n. We denote $X_i = e^{x_i}$. Then the Weyl character formula gives

$$\mathrm{ch}(L(\lambda)) = \frac{\sum(-1)^{\ell(w)} X^{w(\lambda+\rho)}}{\prod_{i<j}(X_i - X_j)} = \frac{\det(X_i^{\lambda_j+n-j})}{\det(X_i^{n-j})}$$

the Schur polynomial.

References

- Knutson. Lie groups. [notes]
- Sepanski. Compact Lie Groups.

2 Lecture 2 — Demazure Character Formula

2.1 Let w_0 be the longest word in Wely group. Then the opposite Borel subgroup B^- is $w_0 B w_0$. We denote the **Schubert variety** to be

$$\Sigma_w = \overline{BwB/B} \subseteq G/B, \qquad \Sigma^w = \overline{B^- wB/B} \subseteq G/B.$$

Then $\dim \Sigma_w = \operatorname{codim} \Sigma^w = \ell(w)$. In particular, $\Sigma_{s_i} = P_i/B$, $\Sigma_{\mathrm{id}} = \Sigma^{w_0}$ is the point $1 \cdot B/B$, and $\Sigma_{w_0} = \Sigma^{\mathrm{id}} = G/B$.

2.2 For standard parabolic subgroup P defined by $J \subseteq \mathbb{I}$, define the **Schubert variety** for w which is shortest among $wW_J \in W/W_J$

$$\Sigma_w = \overline{BwP/P} \subseteq G/P, \qquad \Sigma^w = \overline{B^- wP/P}.$$

Then $\dim \Sigma_w = \operatorname{codim} \Sigma^w = \ell(w)$.

2.3 Denote $K_G(G/B)$ the G-equivariant K-theory. It is naturally isomorphic to the group algebra of $X(T)$. We denote the class of $\mathcal{O}(\lambda)$ by e^λ.

Assume P is standard parabolic corresponding to $J \subseteq \mathbb{I}$. Then $K_G(G/P)$ is the W_J-invariant subalgebra of $K_G(G/B)$.

2.4 Let $p_i : G/B \to G/P_i$ be the natural projection. We define the **Demazure operator** π_i to be the composition

$$K_G(G/B) \xrightarrow{(p_i)_*} K_G(G/P_i) \xrightarrow{(p_i)^*} K_G(G/B).$$

We denote the class of $\mathcal{O}(\lambda)$ by e^λ. By the compuation in cohomology and Grothendieck–Riemann–Roch, we have

$$\forall f \in K_G(G/B), \qquad \pi_i f = \frac{f - e^{-\alpha_i} s_i f}{1 - e^{-\alpha_i}}.$$

By direct computation, π_i satisfies Braid relations and $\pi_i^2 = \pi_i$. Thus we can define π_w for any element $w \in W$ by

$$\pi_w = \pi_{i_1} \circ \cdots \circ \pi_{i_r}, \qquad w = s_{i_1} \cdots s_{i_r} \quad \text{(any reduced word)}$$

2.5. Demazure Character Formula For dominant $\lambda \in X(T)$,

$$\operatorname{ch}\big(H^0(\Sigma_w; \mathcal{O}(\lambda))^*\big) = \pi_w e^\lambda,$$

and

$$\forall i \geq 1, \quad H^i(\Sigma_w; \mathcal{O}(\lambda)) = 0.$$

The proof is difficult and will not be given here.

2.6 Roughly speaking, push forward is "taking global section along fibres". Actually, when $\ell(ws_i) = \ell(w) + 1$, $p : \Sigma_w \to \Sigma_w^P$ is birational, and $\Sigma_{ws_i} = p^{-1}(\Sigma_w^P)$. But K-theory is very sensitive with respect to birational morphisms.

- When $w = \mathrm{id}$, $H^0(1 \cdot B/B; \mathcal{O}(\lambda))$ is nothing but $\mathbb{C}(\lambda)$. So the character is e^λ.
- When $w = w_0$, one can compute

  $$\pi_{w_0} f = \frac{\sum (-1)^{\ell(w)} w(f e^\rho)}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

  Thus with Borel–Weil theorem, we have

  $$\operatorname{ch}(L(\lambda)) = \frac{\sum (-1)^{\ell(w)} w(f e^\rho)}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

  So this recovers Weyl character formula.

2.7 For the case SL_2, denote $e = e^{\omega_1} = e^{x_1}$, then $e^{\alpha_1} = e^2$.

$$\begin{array}{cccccccccccc}
{\scriptstyle -n} & {\scriptstyle -n+2} & & {\scriptstyle n-2} & {\scriptstyle n} & & {\scriptstyle -n} & {\scriptstyle -n+2} & & {\scriptstyle n-2} & {\scriptstyle n} \\
0 & 0 & \cdots & 0 & 1 & \xmapsto{\pi_1} & 1 & 1 & \cdots & 1 & 1
\end{array}$$

$$e^n \qquad \xmapsto{\pi_1} \qquad \frac{e^n - e^{-2} e^{-n}}{1 - e^{-2}} = e^{-n} + \cdots + e^n$$

$$\begin{array}{cccccc} \scriptstyle -n & \scriptstyle -n+2 & & \scriptstyle n-2 & \scriptstyle n \\ 1 & 1 & \cdots & 1 & 1 \end{array} \quad \stackrel{\pi_1}{\longmapsto} \quad \begin{array}{cccccc} \scriptstyle -n & \scriptstyle -n+2 & & \scriptstyle n-2 & \scriptstyle n \\ 1 & 1 & \cdots & 1 & 1 \end{array}$$

$$e^{-n} + \cdots + e^{n} \quad \stackrel{\pi_1}{\longmapsto} \quad e^{-n} + \cdots + e^{n}$$

Then for example,

$$\begin{array}{cccccc} \scriptstyle -5 & \scriptstyle -3 & \scriptstyle -1 & \scriptstyle 1 & \scriptstyle 3 & \scriptstyle 5 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array} \quad \stackrel{\pi_1}{\longmapsto} \quad \begin{array}{cccccc} \scriptstyle -5 & \scriptstyle -3 & \scriptstyle -1 & \scriptstyle 1 & \scriptstyle 3 & \scriptstyle 5 \\ 1 & 2 & 4 & 4 & 2 & 1 \end{array}$$

since we can decompose $(001211) = (001100) + (000100) + (000010) + (000001)$.

2.8 Consider the case SL_3, see Figure 1.

2.9 For GL_n a series (i.e. composition) $\lambda = (\lambda_1, \ldots, \lambda_n)$, we can define the **Key polynomial** by

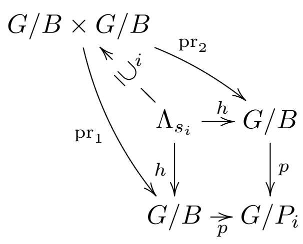
$$\kappa_\lambda = X_1^{\lambda_1} \cdots X_n^{\lambda_n} \qquad \text{if } \lambda_1 \geq \cdots \geq \lambda_n$$
$$\kappa_{s_i\lambda}(X) = \pi_i \kappa_\lambda(X) \qquad \text{if } \lambda_i \geq \lambda_{i+1}$$

This is essentially the Demazure character formula $\pi_w e^\lambda$. Note that in this case, $\pi_i f = \frac{X_i f - s_i(X_i f)}{X_i - X_{i+1}}$.

2.10 Let us also lift everything to G-version. The G-orbit of $G/B \times G/B$ are one-to-one corresponding to B-orbit of G/B. Let us denote

$$\Lambda_w = \overline{\{(xB, yB) : xy^{-1} \in BwB\}} \subseteq G/B \times G/B.$$

Note that when $w = s_i$, we have a pull back square



$$\begin{aligned} \pi_i \alpha &= p^* p_* \alpha = h_* h^* \alpha \\ &= (\mathrm{pr}_1)_* i_* i^* \, \mathrm{pr}_2^* \, \alpha \\ &= (\mathrm{pr}_1)_* \big([\mathcal{O}_{\Lambda_{s_i}}] \cdot \mathrm{pr}_2^* \, \alpha \big) \\ &:= [\mathcal{O}_{\Lambda_{s_i}}] * \alpha \end{aligned}$$

So the Demazure operator $\pi_i : K_G(G/B) \longrightarrow K_G(G/B)$ is actually given by convolution with $[\mathcal{O}_{\Lambda_{s_i}}] \in K_G(G/B)$. In general, the Demazure operator π_w is given by convlution with $[\mathcal{O}_{\Lambda_w}]$.

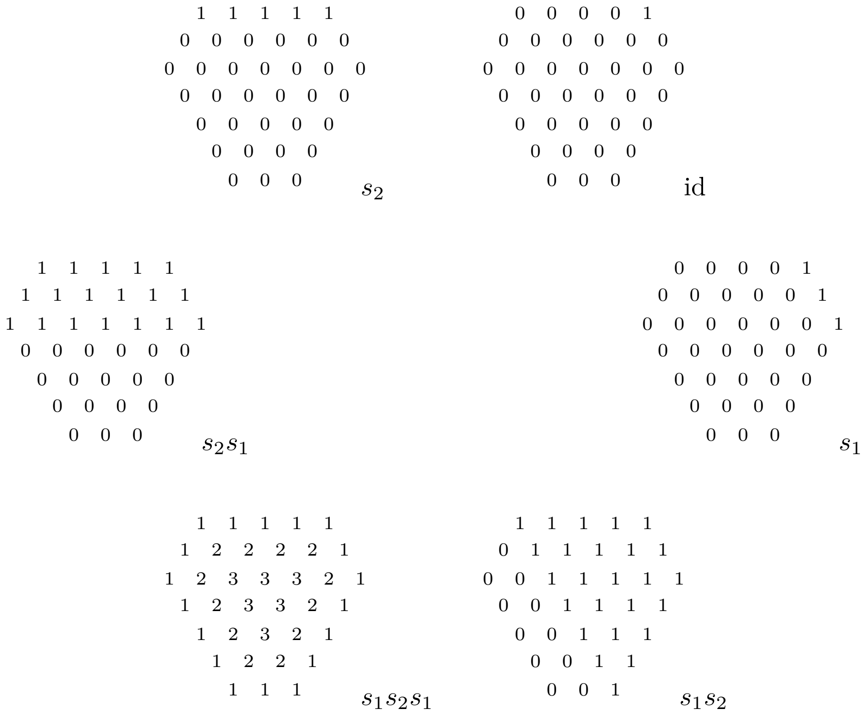


Figure 1: Example of SL_3

2.11 In the case of GL_n,

$$\Lambda_{s_i} = \left\{ 0 \subseteq V_1 \subseteq \cdots \begin{array}{c} \rotatebox{-45}{$\subseteq$} \; V_i^1 \; \rotatebox{45}{$\subseteq$} \\ \rotatebox{45}{$\subseteq$} \; V_i^2 \; \rotatebox{-45}{$\subseteq$} \end{array} \cdots \subseteq V_n : \dim V_i^{\cdots} = i \right\}$$

2.12. Tits system Recall Tits system

$$Bs_iB \cdot BwB = \begin{cases} Bws_iB & \ell(ws_i) = \ell(w) + 1 \\ BwB \cup Bws_iB & \text{otherwise} \end{cases}$$

Actually, we can say more that if $\ell(uv) = \ell(u) + \ell(v)$,

$$BuB \times_B BvB \longrightarrow BuvB$$

is an isomorphism.

2.13 For an element $w \in W$, we pick a reduced word $\underline{w} = (s_{i_1}, \ldots, s_{i_r})$ for w. Define the **Bott–Samelson variety** to be

$$\mathrm{BS}(\underline{w}) = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r}/B.$$

- Note that $\mathrm{BS}(\underline{w})$ is smooth, since it is iterated $\mathbb{P}^1$ bundle over $\mathbb{P}^1 \cong P_{i_r}/B$.
- the map $\mu : \mathrm{BS}(\underline{w}) \longrightarrow \Sigma_w$ induced by multiplication is birational by Tits system.

2.14 When $\ell(ws_i) = \ell(w) + 1$, then we have the following pull back square

$$\begin{array}{ccccccc} \cdots \times P_\bullet \times_B P_i/B & = & \mathrm{BS}(\underline{w} \oplus s_i) & \longrightarrow & \mathrm{BS}(\underline{w}) & = & \cdots \times P_\bullet/B \\ & & \downarrow & & \downarrow & & \\ & & G/B & \longrightarrow & G/P_i & & \end{array}$$

2.15 We may also consider

$$\begin{aligned} \widehat{\mathrm{BS}}(\underline{w}) &= G/B \underset{G/P_{i_1}}{\times} G/B \underset{G/P_{i_r}}{\times} \cdots \underset{G/P_{i_r}}{\times} G/B \\ &= P_{i_1} \times_B G/B \underset{G/P_{i_r}}{\times} \cdots \underset{G/P_{i_r}}{\times} G/B = \cdots \\ &= P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r} \times_B G/B \end{aligned}$$

So $\mathrm{BS}(\underline{w})$ is the fibre at $1 \cdot B/B$ of

$$\widehat{\mathrm{BS}(\underline{w})} \longrightarrow G/B.$$

2.16 We can also define the line bundle $\mathcal{O}(\lambda)$ on $\mathrm{BS}(\underline{w})$ by pull back from G/B. Actually, its total space is

$$P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r} \times_B \mathbb{C}(\lambda).$$

2.17. Demazure Character Formula For any reduced word $\underline{w}$ for w, for dominant $\lambda \in X(T)$,

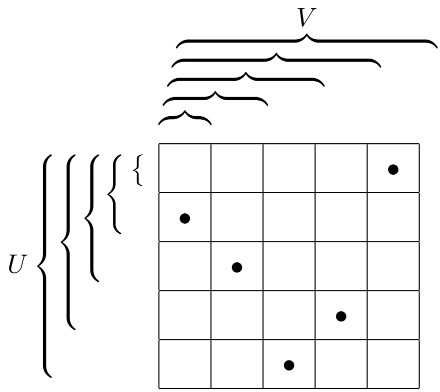
$$\operatorname{ch}\left(H^0(\mathrm{BS}(\underline{w}); \mathcal{O}(\lambda))^*\right) = \pi_w e^\lambda,$$

and

$$\forall i \geq 1, \quad H^i(\mathrm{BS}(\underline{w}); \mathcal{O}(\lambda)) = 0.$$

2.18. Sketch of the Proof Actually, the second assertion can be proved by spectral sequence argument as before. The first argument follows from the definition of Demazure operator — Bott–Samelson variety is the variety-theoretical composition of push forward and pull back.

2.19 For two flags $(0 \subseteq V_1 \subseteq \cdots \subseteq V_n)$ and $(0 \subseteq U_1 \subseteq \cdots \subseteq U_n)$, we can assume a permutation $w(U, V)$ as follows. There exists a set of basis $v_1, \ldots, v_n$ such that $V_i = \operatorname{span}(v_1, \ldots, v_i)$, and $U_i = \operatorname{span}(v_{w^{-1}(1)}, \ldots, v_{w^{-1}(i)})$. See Figure 2. Equivalently, $w(U, V)$ is the unique permutation w with



$$\begin{pmatrix} 12345 \\ 23541 \end{pmatrix}$$

Figure 2: Relative Position

$$\dim \frac{V_i + U_{j+1} \cap V_{i+1}}{V_i + U_j \cap V_{i+1}} = \begin{cases} 1, & w(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Besides, it is also equivalent to the condition

$$\dim(U_j \cap V_i) = \#\{b \le j, a \le i : w(a) = b\}.$$

We pick a standard flag $(0 \subseteq V_1^0 \subseteq \cdots \subseteq V_n^0)$. Then

$$BwB/B = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : w(U, V^0) = w \right\}.$$

Its closure

$$\Sigma_w = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : \dim(U_j \cap V_i^0) \ge \#\{b \le j, a \le i : w(a) = b\} \right\}.$$

If we pick the opposite standard flag $(0 \subseteq V_1' \subseteq \cdots \subseteq V_n')$, then

$$\Sigma^w = \left\{ 0 \subseteq U_1 \subseteq \cdots \subseteq U_n : (\dim U_j \cap V_i') \ge \#\{b \le j, a \le i : w_0 w(a) = b\} \right\}.$$

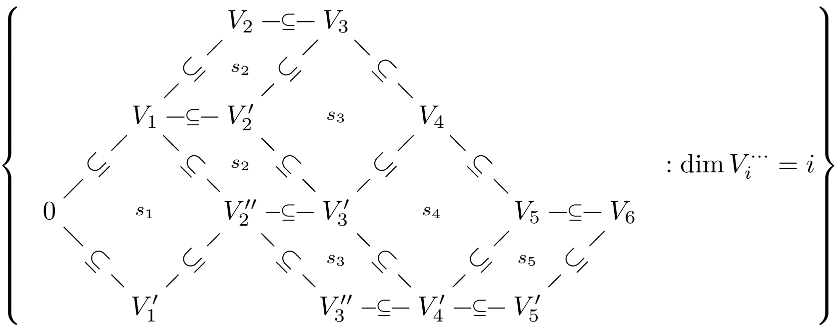
2.20 For the case $\mathcal{G}r(k,n)$, the shortest representive are in one-to-one correspondence with Young diagrams inside $k \times (n-k)$. To be exact, for a partition $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$, the map $i \mapsto \lambda_{k+1-i} + i$ naturally extends to a permutation which is monotonous on $\{k+1, \ldots, n\}$. In this case,

$$\Sigma_\lambda = \left\{ V \in \mathcal{G}r(k,n) : \dim(V \cap V^0_{\lambda_{k+1-i}+i}) \ge i \right\},$$

$$\Sigma^\lambda = \left\{ V \in \mathcal{G}r(k,n) : \dim(V \cap V'_{n-k+i-\lambda_i}) \ge i \right\}.$$

See Figure 3

2.21 In the case GL_n, we may regard $\widehat{\mathrm{BS}}(\underline{w})$ as flags of a given shape. For example, for GL_6, for $\underline{w} = s_5 s_3 s_4 s_1 s_2 s_3 s_2$, $\widehat{\mathrm{BS}}(\underline{w})$ is



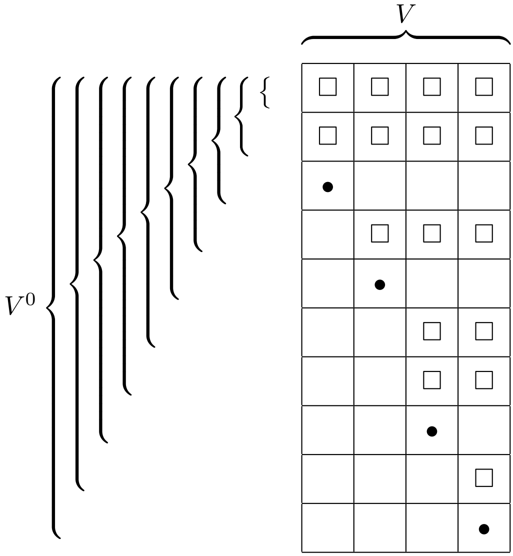
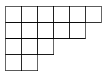


Figure 3: Schubert Cells 

The map $\widehat{\mathrm{BS}(\underline{w})} \to G/B$ corresponds to the topmost flag.

References

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3 Lecture 3 — Schur–Weyl Modules

3.1 Let $\mathcal{G}r(k,n)$ be the Grassmaniann. There is a line bundle $\mathcal{O}(1)$ defining plücker embedding. Let $\mathcal{V}$ be the tautological bundle of $\mathcal{G}r(k,n)$, that is, the fibre at $V \in \mathcal{G}r(k,n)$ is V itself. Then $\mathcal{O}(1) = \Lambda^k \mathcal{V}^*$.

Denote the natural map $p : G/B \longrightarrow G/P$, that is, $p : \mathcal{F}\ell(n) \longrightarrow \mathcal{G}r(k,n)$ by picking k-th space. Then $p^*\mathcal{V}$ is exactly the k-th tautological bundle ϕ_k. Thus it is not hard to $p^*\mathcal{O}(1) = \mathcal{O}(\omega_k)$, with $\omega_k = x_1 + \cdots + x_k$.

3.2 There is a

$$H^0(\mathcal{G}r(k,n);\mathcal{O}(d)) \longrightarrow H^0(\mathcal{F}\ell(n);\mathcal{O}(d\omega_k)).$$

It is injective, G-equivariant, and nonzero. So

$$H^0(\mathcal{G}r(k,n);\mathcal{O}(d))^* = L(d\omega_k)$$

This can also be seen from the proof of Borel–Weil theorem.

3.3 We have another point view of Grassmannian, that

$$\mathcal{G}r(k,n) = \mathrm{St}(k,n)/\,\mathrm{GL}_k$$

where $\mathrm{St}(k,n)$ is Stiefel variety, the space of $n \times k$ full rank matrices (geometrically, the space of all k-frames in $\mathbb{C}^n$). Then the total space of $\mathcal{O}(1)$ is

$$\mathrm{St}(k,n) \times_{\mathrm{GL}_k} \mathbb{C}(\det),$$

where $\mathbb{C}(\det)$ is the one-dimensional representation with $g \in \mathrm{GL}_k$ acts by $(\det g)^{-1}$.

3.4 Let us fix the coordinate

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \in \mathrm{St}(k,n).$$

A section of $\mathcal{O}(1)$ is then a map $f : \mathrm{St}(k,n) \to \mathbb{C}$ with $f(Xg) = \det(g)f(X)$. Such f has to be linear combination of $\Delta^I := \det\left(x_{ij} : {}^{i\in I}_{j\in[k]}\right)$ for a subset $I \in \binom{[n]}{k}$. That is, f can be view as alternating k-linear map on $\mathbb{C}^n$, sending $\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k$ to $f(X)$ where the j-th column of X is $\mathbf{x}_j$. Thus, taking the dual, we get

$$L(\omega_k) = \Lambda^k \mathbb{C}^n,$$

as we expected.

3.5 In general, a section of $\mathcal{O}(d)$ is linear combination of $\Delta^{I_1}\Delta^{I_2}\cdots\Delta^{I_d}$ for $I_1,\ldots,I_d\in\binom{[n]}{k}$. Thus

$$L(d\omega_k)^* = \operatorname{span}(\Delta^{I_1}\Delta^{I_2}\cdots\Delta^{I_d}) \subseteq \mathbb{C}\big[x_{ij} : {}^{1\le i\le n}_{1\le j\le k}\big] \subseteq \mathbb{C}[\mathrm{St}(k,n)]$$

Note that Δ^I's are not linear independent, the relation defining them is Plücker relations. So formally,

$$L(d\omega_k) = \mathsf{S}^d\big(\Lambda^k\mathbb{C}^n\big)/\text{Plücker relations}.$$

This is also as we expected.

3.6 Then consider the diagonal embedding

$$\mathcal{F}\ell(n) \longrightarrow \mathcal{G}r(1,n)\times\cdots\times\mathcal{G}r(n,n) =: \mathcal{G}r(1,\ldots,n).$$

The the pull back $\mathcal{O}(c_1,\ldots,c_n) := \mathcal{O}(c_1)\boxtimes\cdots\boxtimes\mathcal{O}(c_n)$ is exactly $\mathcal{O}(c_1\omega_1+\cdots+c_n\omega_n)$. Denote $\lambda = c_1\omega_1+\cdots+c_n\omega_n$. We get a restriction map

$$H^0(\mathcal{G}r(1,\cdots,n);\mathcal{O}(c_1,\ldots,c_n)) \longrightarrow H^0(\mathcal{F}\ell(n);\mathcal{O}(\lambda)).$$

This is nonzero, G-equivariant, thus is surjective. Then we can compute over a dense subset. One choice is $\mathfrak{n}^- \cong w_0Bw_0B/B$. But it turns out, the next choice is the most convenient.

3.7 We use the map for the dense orbit

$$\kappa : B \longrightarrow G/B \qquad b \longmapsto bw_0\cdot B/B.$$

Then clear

$$H^0(\mathcal{F}\ell(n);\mathcal{O}(\lambda)) \xrightarrow{\kappa^*} H^0(B;\kappa^*\mathcal{O}(\lambda))$$

is injective. We use the coordinate

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn} \end{pmatrix} \in B.$$

Then the map $B\to\mathcal{F}\ell(n)\to\mathcal{G}r(k,n)$ factor through $\mathrm{St}(k,n)$ by

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} x_{1n} & \cdots & x_{1,n-k} \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & x_{n-k,n-k} \\ \vdots & \iddots & \\ x_{nn} & & \end{pmatrix}$$

Thus as T-module,

$$L(\lambda)^* \cong \operatorname{span}\left(\Delta^{I_1}(x_{ij}w_0)\cdots\Delta^{I_h}(x_{ij}w_0)\right)\Big|_{i>j\Rightarrow x_{ij}=0} \subseteq \mathbb{C}\big[x_{ij}\big]_{1\leq i\leq j\leq n} \subseteq \mathbb{C}[B],$$

where $h = c_1 + \ldots + c_{n-1}$, $I_1, \ldots, I_{c_1} \in \binom{[n]}{1}$, $I_{c_1+1}, \ldots, I_{c_1+c_2} \in \binom{[n]}{2}$, etc.

3.8 For example, consider the case $\lambda = \yng(2,1) = 2x_1 + x_2$ whose $c_1 = c_2 = 1$. Then the two maps factor through $\mathrm{St}(1,3)$ and $\mathrm{St}(2,3)$ is $\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$, $\begin{pmatrix} x_{13} & x_{12} \\ x_{23} & x_{22} \\ x_{33} & \end{pmatrix}$, hence

$$L(\lambda)^* = \operatorname{span}\begin{pmatrix} x_{13}\cdot(x_{13}x_{22} - x_{12}x_{23}) & x_{13}\cdot x_{22}x_{33} & x_{13}\cdot x_{12}x_{33} \\ x_{23}\cdot(x_{13}x_{22} - x_{12}x_{23}) & x_{23}\cdot x_{22}x_{33} & x_{23}\cdot x_{12}x_{33} \\ x_{33}\cdot(x_{13}x_{22} - x_{12}x_{23}) & x_{23}\cdot x_{22}x_{33} & x_{23}\cdot x_{12}x_{33} \end{pmatrix}$$

Note that we have one relation

$$(x_{33})(x_{13}x_{22} - x_{12}x_{23}) = (x_{13})(x_{22}x_{33}) - (x_{23})(x_{12}x_{33}).$$

The dimension is $9 - 1 = 8$. The action of T is on left, so $t \cdot X = t^{-1}X$. The character of is just the row number,

$$\mathrm{ch}(L(\lambda)) = (e^{x_1} + e^{x_2} + e^{x_3})(e^{x_1}e^{x_2} + e^{x_2}e^{x_3} + e^{x_1}e^{x_3}) - e^{x_3}(e^{x_1}e^{x_2})$$

$$\ytableaushort{11,2},\ \ytableaushort{11,3},\ \ytableaushort{12,2},\ \ytableaushort{12,3},\ \ytableaushort{13,2},\ \ytableaushort{13,3},\ \ytableaushort{22,3},\ \ytableaushort{23,3}$$

3.9 If we use the natural map $G \to G/B$, with coordinate

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in G$$

Then as G-module,

$$L(\lambda)^* \cong \operatorname{span}\left(\Delta^{I_1}(x_{ij}w_0)\cdots\Delta^{I_h}(x_{ij}w_0)\right) \subseteq \mathbb{C}\big[x_{ij}\big]_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} \subseteq \mathbb{C}[G],$$

where $h = c_1 + \ldots + c_{n-1}$, $I_1, \ldots, I_{c_1} \in \binom{[n]}{1}$, $I_{c_1+1}, \ldots, I_{c_1+c_2} \in \binom{[n]}{2}$, etc. As a result, taking the dual, we get

$$L(\lambda) \cong \mathsf{S}^{c_1}\left(\Lambda^1\mathbb{C}^n\right) \otimes \cdots \otimes \mathsf{S}^{c_n}\left(\Lambda^n\mathbb{C}^n\right) \Big/ \text{Plücker relations.}$$

This is exactly how Weyl construct representations.

3.10 Lastly, let us consider the analogy for Demazure characrer. In this case, we only need to exchange w_0 by any w. Note $\Sigma_w \longrightarrow \mathcal{G}r(1,\ldots,n-1)$ factor through $\mathcal{F}\ell(n)$, and by Demazure character formula,

$$H^0(G/B;\mathcal{O}(\lambda)) \longrightarrow H^0(\Sigma_w;\mathcal{O}(\lambda))$$

is also surjective (by induction). So theoretically, there is no problem.

3.11 But to be general, assume $D=(D_1,\ldots,D_h)$ is a series of subsets of $[n]$. Let us denote **flagged Weyl module**

$$M_D = \operatorname{span}\left(\prod_{i=1}^{h} \Delta_{D_i}^{C_i} : C_i \subseteq \binom{[n]}{\#D_i}\right) \subseteq \mathbb{C}[x_{ij}]_{1\le i\le j\le n},$$

where Δ_D^I is the determinant of sub-matrix $I \times D$ in

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ & & x_{nn} \end{pmatrix}.$$

We define the **character** to be

$$\operatorname{ch}(M_D) = \sum_{\lambda\in X(T)} \dim(M_D)_\lambda e^\lambda, \qquad \operatorname{ch}^*(M_D) = \overline{\operatorname{ch}(M_D)}.$$

3.12 A hint:

$$\operatorname{ch}^*(\mathbb{C}\cdot x_{i_1j_1}\cdots x_{i_rj_r}) = e^{x_{i_1}}\cdots e^{x_{i_r}} = X_{i_1}\cdots X_{i_r}.$$

3.13 For a composition λ (i.e. a series of numbers), define the **skyline diagram**

$$D(\lambda) : D(\lambda)_j = \{\bullet : \alpha_\bullet \ge j\}.$$

For example, $\lambda=(3,2,1,0,1)$,

3	1	1	1
2	2	2	
1	3		
0			
1	5		

$D_1 D_2 D_3 D_4 D_5$

Then $\kappa_\lambda(X) = \mathrm{ch}^*(M_{D(\lambda)})$. It involves some careful combinatorial translation which is left to readers.

3.14 For example $D = \{1, 2, 3, 5\}$,

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \\ \hline 5 \\ \hline \end{array} \qquad \left(\begin{array}{cccc||c} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{22} & x_{23} & x_{24} & x_{25} \\ & & x_{33} & x_{34} & x_{35} \\ & & & x_{44} & x_{45} \\ & & & & x_{55} \end{array}\right)$$

Its maximal minor (i.e. 4×4 minors), i.e. Δ_D^C has only two nonzero values,

$$x_{11}x_{22}x_{23}x_{45}, \qquad x_{11}x_{22}x_{23}x_{55}.$$

3.15 For example, when $D = [i] = \{1, \ldots, i\}$, then

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline i \\ \hline \end{array} \qquad \left(\begin{array}{ccc||c||c} x_{11} & \cdots & x_{1i} & x_{1,i+1} & \cdots \\ & \ddots & \vdots & \vdots & \vdots \\ & & x_{ii} & x_{i,i+1} & \cdots \\ & & & x_{i+1,i+1} & \cdots \\ & & & & \ddots \end{array}\right)$$

$$\mathrm{span}\left(\Delta_D^C : C \in \binom{[n]}{i}\right) = \mathbb{C} \cdot x_{11} \cdots x_{ii}$$

3.16 When λ is weakly decreasing, each member of D is of the form $[i]$. So $\kappa_\lambda = X^\lambda$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{l} M_D = \mathbb{C} \cdot x_{11}^6 x_{22}^4 x_{33}. \\ \kappa_{\tiny\begin{array}{|c|c|c|c|c|c|}\hline & & & & & \\ \hline & & & \\ \cline{1-4} \\ \cline{1-1}\end{array}} = X_1^6 X_2^4 X_3 \end{array}$$

3.17 When $\lambda = (0, 1, 2) = \tiny\begin{array}{|c|c|}\cline{1-1} \\ \hline & \\ \hline \end{array}$, the skyline diagram is

$$\begin{array}{c|c|c|} 0 \\ \cline{2-2} 1 & 2 \\ \cline{2-3} 2 & 3 & 3 \\ \cline{2-3} \end{array}$$

Thus, M_D is spanned by

$$\text{maxmial minors of } \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \\ & x_{33} \end{pmatrix} \cdot \text{maxmial minors of } \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$$

It is essentially the same as we did before (up to a permutation of row indices).

3.18 The modern way to deal with vanishing of higher cohomology in representation and combinatorics is **Frobenius splitting**.

References

- Peter Magyar. Schubert Polynomials and Bott-Samelson Varieties.

4 Lecture 4 — Horn's Problem

4.1 **Horn's problem** concerns the eigenvalues of sum of two Hermitian matrices with given eigenvalues. To be exact, given three Hermitian matrices A, B, C with $A + B = C$, with eigenvalues

$$\begin{gathered}\lambda_1(A) \geq \cdots \geq \lambda_n(A)\\ \lambda_1(B) \geq \cdots \geq \lambda_n(B)\\ \lambda_1(C) \geq \cdots \geq \lambda_n(C)\end{gathered}$$

Horn's problem is to characterize these $3n$ values that appears in this way.

4.2 Let M be a symplectic manifold. For each $f \in \mathcal{C}^\infty(M)$, we can define the **Hamitonian vector field** $\mathfrak{X}_f \in \mathfrak{X}(M)$ with

$$\omega(-, \mathfrak{X}_f) = df.$$

Then $\mathcal{C}^\infty$ has a **Poisson structure** defined by

$$\{f, g\} := \omega(\mathfrak{X}_f, \mathfrak{X}_g).$$

4.3 Let M be a symplectic manifold with Hamitonian G-action. That is, the inducing map $\mathfrak{g} \to \mathfrak{X}(M)$ factor through

$$\mathfrak{g} \xrightarrow{H} \mathcal{C}^\infty(X) \xrightarrow{\mathfrak{X}} \mathfrak{X}(M),$$

with H is a Lie algebra homomorphism. In this case, we can define the **moment map** $\mu : M \to \mathfrak{g}^*$ by dualizing H. That is,

$${}^{M\ni}x \longmapsto \left[{}^{\mathfrak{g}\ni}X \mapsto H_X(x)_{\in\mathbb{R}}\right]_{\in\mathfrak{g}^*}.$$

4.4. Atiyah, Guillemin and Sternberg Theorem In the case $G = T$ is a compact torus and X is compact, the image of moment map is a polytope with vertices $\mu(X^T)$ the image of fixed points of X. Actually, for any point p on a k-face of this polytope, and $x \in \mu^{-1}(p)$, the orbit of x is of dimension k.

4.5 For a Lie subgroup $H \subseteq G$, the restriction of H is also Hamitonian, with moment map

$$\mu_H : M \to \mathfrak{h}^*$$

obtained by composition $\mu_G : M \to \mathfrak{g}^*$ with the restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$.

4.6 For two spaces X, Y with Hamiltonian G-action. Then so is $X \times Y$. The the moment map $\mu_{X\times Y}$ satisfies

$$\mu_{X\times Y}(x, y) = \mu_X(x) + \mu_Y(y).$$

4.7. Kirillov-Kostant-Souriau symplectic structure For any Lie group G, Each coadjoint orbit $\mathbb{O}$ of $\mathfrak{g}^*$ can be equipped with a symplectic structure with moment map the natural inclusion of $\mathbb{O} \to \mathfrak{g}^*$.

4.8 Let $\mathfrak{h}_n$ be the space of Hermitian matrices. Consider $\mathfrak{u}_n$ the space of skew-Hermitian matrices. The pairing

$$\mathfrak{u}_n \times \mathfrak{h}_n \to \mathbb{R} \qquad (A, B) \longmapsto \mathrm{tr}(\mathbf{i} \cdot AB)$$

is perfect. So we can identify $\mathfrak{u}_n^* = \mathfrak{h}_n$.

4.9 Denote $\mathfrak{t} \cong \mathbf{i} \cdot \mathbb{R}^n$ the diagonal subalgebra of $\mathfrak{u}_n$. Then we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$. The restriction map $\mathfrak{u}_n^* \to \mathfrak{t}^*$ is then given by taking diagonal entries

$$\mathfrak{h}_n \longrightarrow \mathbb{R}^n \qquad (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \mapsto (a_{ii})_{i=1}^n.$$

4.10 Each coadjoint orbit is isomorphic to a partial flag veriety G/P_λ where P_λ is a parabolic subgroup.

Note that the T-fixed point of $\mathfrak{h}_n$ is exactly the diagonal matrices. As a result, the orbit of $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is exactly all the permutations of it.

4.11. Schur–Horn's Theorem There exists a Hermitian matrix A, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and diagonal entries $d_1 \geq \ldots \geq d_n$ if and only if

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \lambda_i, \quad (1 \leq k \leq n-1), \qquad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i.$$

Actually, this is equivalent to (d_i) lies in the polytope spanned by permutations of (λ_i).

4.12 For example, $\lambda_1 = 2$ and $\lambda_2 = 1$. Then (d_1, d_2) should lies on the segment between $(1,2)$ and $(2,1)$. That is,

$$d_1 \leq 2, \qquad d_1 + d_2 = 3.$$

Actually, in this case $d_1 d_2 \geq 2$, thus some $z \in \mathbb{C}$ such that $|z|^2 = d_1 d_2 - 2$, then $\left(\begin{smallmatrix} d_1 & z \\ \bar{z} & d_2 \end{smallmatrix}\right)$ is the Hermitian matrix desired.

4.13. Horn's conjecture There exists three Hermitian matrices A, B, C with $A + B = C$, with eigenvalues

$$\begin{gathered} \lambda_1(A) \geq \cdots \geq \lambda_n(A) \\ \lambda_1(B) \geq \cdots \geq \lambda_n(B) \\ \lambda_1(C) \geq \cdots \geq \lambda_n(C) \end{gathered}$$

if and only if

$$\sum_{i=1}^{n} \lambda_i(A) + \sum_{i=1}^{n} \lambda_i(B) = \sum_{i=1}^{n} \lambda_i(C)$$

for any k and $I, J, K \subseteq \binom{[n]}{k}$ with $c^{\lambda(I)\lambda(J)}_{\lambda(J)} \neq 0$

$$\sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B) \geq \sum_{k \in K} \lambda_k(C).$$

Here, $\lambda(I) = \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ is the partition with $I = \{\lambda_1 + k, \ldots, \lambda_k + 1\} \subseteq [n]$, and $c^{\nu}_{\lambda\mu}$ the Littlewood–Richardson coefficient.

4.14. Proof of "$\Rightarrow$" For a Hermintian matrix A, we define the Rayleigh trace over $\mathcal{G}r(k, n)$ by

$$R_A : \mathcal{G}r(d, n) \longrightarrow \mathbb{R} \qquad V \longmapsto \sum_{i=1}^{k} \mathbf{x}_i^{\mathrm{t}} A \mathbf{x}_i$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_k$ a choice of orthogonal normal basis of V. Then

$$\sum_{i \in I} \lambda_i(A) = \min_{x \in \Sigma_{\lambda(I)}(A)} R_A(x),$$

where $\Sigma_{\lambda(I)}(A)$ is the Schubert variety corresponding to the flag $(0 \subseteq V_1 \subseteq \cdots V_n)$ with V_i spanned by the first k eigenvectors (with eigenvalues weakly decreasing).

Note that $\Sigma_{\lambda(I)}(A) \cap \Sigma_{\lambda(I)}(B) \cap \Sigma_{\lambda(I)^c}(C) = \varnothing$ implies $c^{\lambda(K)}_{\lambda(I)\lambda(J)} = 0$. Thus we get the condition stated in the theorem.

4.15. Sketch of "$\Leftarrow$" We should use some convex properties for non-torus. Let $C \subseteq \mathfrak{t}^*$ be any Weyl chamber. We have a map

$$\phi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^* / \operatorname{ad} G \cong \mathfrak{t}^*/W \cong C$$

where C is a closed Weyl chamber. Kirwan's theorem claims that the image $\phi \circ \mu_G$ is convex.

4.16 Let X be a projective variety with an very ample line bundle $\mathcal{L}$. Denote

$$\Gamma^\bullet(\mathcal{L}) = \bigoplus_{n \geq 0} H^0(X; \mathcal{L}^{\otimes n}).$$

Then $\operatorname{Proj} \Gamma^\bullet(\mathcal{L}) = X$. Actually, $\Gamma^\bullet(\mathcal{L})$ is the projective coordinate ring for X in $\mathbb{P}^N$.

4.17 For example, when $X = \mathcal{F}\ell(n)$, and $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$ with $\lambda_1 > \cdots > \lambda_n$, then

$$\Gamma^\bullet(\mathcal{O}(\lambda)) = \bigoplus_{d \geq 0} L(d\lambda)^*.$$

In general, if $\lambda_1 \geq \cdots \geq \lambda_n$, we can find a partial flag variety G/P_λ with the same property.

4.18 Over $\mathbb{P}^{n-1}_{\mathbb{C}}$, there is a natural Fubini–Study symplectic structure

$$\omega = \frac{|dx_1|^2 + \cdots + |dx_n|^2}{|x_1|^2 + \cdots + |x_n|^2}$$

The action of $U(n)$ on $\mathbb{P}^1$ is Hamiltonian with moment map

$$\mu(\ell) = \text{rank one Hermitian matrix projecting to } \ell \in \mathfrak{h}_n.$$

That is, view $\mathbb{P}^{n-1}$ as the orbit of $\mathrm{diag}(1, \ldots, 0) \in \mathfrak{h}_n$. Thus the T action moment map is

$$\mu = \left(\frac{|x_1|^2}{|x_1|^2 + \cdots + |x_n|^2}, \ \ldots, \ \frac{|x_n|^2}{|x_1|^2 + \cdots + |x_n|^2} \right).$$

4.19. Kirwan–Ness Theorem If a compact group $K \subseteq U(N)$ acts on a smooth closed subvariety X of $\mathbb{P}^N_{\mathbb{C}}$. Denote the moment map

$$\mu : X \xrightarrow{\subseteq} \mathbb{P}^N \longrightarrow \mathfrak{u}^* \longrightarrow \mathfrak{k}^*$$

Denote the complexification of K by the reductive group G. Then

$$\mu^{-1}(0)/K \longrightarrow X//G \quad \text{(GIT quotient)}.$$

4.20 Denote $\mathbb{O}(\lambda)$ the space of Hermitian matrices with eigenvalue $\lambda_1, \ldots, \lambda_n$ for a weakly decreasing integer sequence. We then apply above theorem to

$$\mathbb{O}(\lambda) \times \cdots \times \mathbb{O}(\mu)$$

with $\lambda, \ldots, \mu$ weakly decreasing integer sequences. Actually, the moment map cooresponds to the Plücker embedding (by computation)

$$\begin{array}{ccc}
\mathbb{O}(\lambda) \times \cdots \times \mathbb{O}(\mu) & \longrightarrow & \mathfrak{h}_n \\
\downarrow & & \uparrow \\
\mathbb{P}(L(\lambda)) \times \cdots \times \mathbb{P}(L(\mu)) & \longrightarrow & \mathfrak{h}_{??} \times \cdots \times \mathfrak{h}_{??} \\
\downarrow & & \uparrow \\
\mathbb{P}(L(\lambda) \otimes \cdots \otimes L(\mu)) & \longrightarrow & \mathfrak{h}_{??}
\end{array}$$

Thus $\mu^{-1}(0)$ is nonempty if and only if

$$\operatorname{Proj} \bigoplus_{d \geq 0} \Big(L(d\lambda) \otimes \cdots \otimes L(d\mu) \Big)^{\mathrm{GL}_n}$$

is nonempty. Equivalently, $\Big(L(d\lambda) \otimes \cdots \otimes L(d\mu) \Big)^{\mathrm{GL}_n} \neq 0$ for some $d \geq 1$.

4.21 There exist Hermitian matrices $A, \ldots, B$ with $A + \cdots + B = 0$ with eigenvalues $\lambda(A), \ldots, \lambda(B)$ if and only if

$$\Big(L(d\lambda(A)) \otimes \cdots \otimes L(d\lambda(B)) \Big)^{\mathrm{GL}_n} \neq 0$$

for some $d \geq 1$. By a limit argument, this method also solves Horn's problem.

4.22 Since the Littlewood–Richardson coefficients has a combinaotrial model by honeycomb due to Knutson and Tao, see Figure 4.

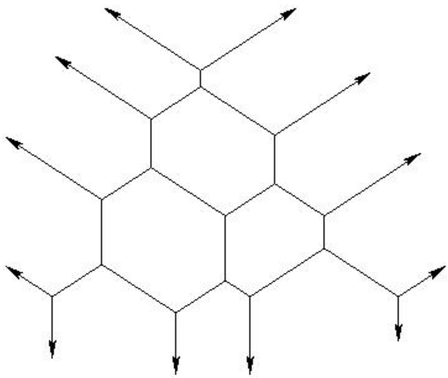


Figure 4: Honeycomb

References

- Knutson. The symplectic and algebraic geometry of Horn's problem. [arXiv]

- Fulton. Eigenvalues, invariant factors, highest weights, and Schubert calculus. [AMS]
- Knutson, Tao. Honeycombs and sums of Hermitian matrices. [arXiv]

5 Appendix: Schubert Calculus

5.1 Actually, by an affine paving argument

$$K_B(G/P) = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\mathcal{O}^w]_B = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\mathcal{O}_w]_B$$

where $\mathcal{O}^w = i_* \mathcal{O}_{\Sigma^w}$ the push forward of regular ring of Σ^w and similarly notation for $\mathcal{O}_w$. It turns out

$$\pi_i[\mathcal{O}^w] = \begin{cases} [\mathcal{O}^{ws_i}] & \ell(ws_i) = \ell(w) - 1 \\ [\mathcal{O}^w] & \text{otherwise} \end{cases}$$

or

$$\pi_i[\mathcal{O}_w] = \begin{cases} [\mathcal{O}_{ws_i}] & \ell(ws_i) = \ell(w) + 1 \\ [\mathcal{O}_w] & \text{otherwise} \end{cases}$$

Note that the second case follows from the first, since $\pi_i^2 = \pi_i$. The first case follows from the fact that the push forward induced by $\mathrm{BS}(\underline{w}) \longrightarrow \Sigma_w$ sending $[\mathcal{O}_{\mathrm{BS}(\underline{w})}]$ to $[\mathcal{O}_{\Sigma_w}]$. To be exact, it has no higher cohomology by a spectral sequence argument, and preserves structure sheaf by applying Zariski connected theorem on Stein decomposition).

5.2 Assume $P = \bigcup_{w \in W_J} BwB$ for $J \subseteq \mathbb{I}$. If we denote

$$R_T = \mathbb{Q}[e^\lambda]_{\lambda \in X(T)}, \qquad R_G = R_T^W, \qquad R_P = R_T^{W_J},$$

then

$$K_B(G/B; \mathbb{Q}) = R_T \otimes_{R_G} R_T, \qquad K_B(G/P; \mathbb{Q}) \cong R_T \otimes_{R_G} R_P.$$

For G/B, the class of $\mathcal{O}(\lambda)$ is presented by $1 \otimes e^\lambda \in R_T \otimes_{R_G} R_T$. The class of pull back of $e^\lambda \in K_B(\mathrm{pt}; \mathbb{Q}) = R_T$ is $e^\lambda \otimes 1$.

5.3 The natural map $G/B \longrightarrow G/P$ induces

$$K_B(G/P) \longrightarrow K_B(G/B) \qquad [\mathcal{O}^w] \longmapsto [\mathcal{O}^w]$$

thus an injection. The corresponding $\mathbb{Q}$-efficient map is just the inclusion

$$R_T \otimes_{R_G} R_P \xrightarrow{\subseteq} R_T \otimes_{R_G} R_T.$$

5.4. Atiyah–Bott–Berline–Vergne Let X be a smooth projective variety algebraically acted by an algebraic torus T. Then the localization, i.e. the restriction to the fixed points

$$K_T(X) \longrightarrow K_T(X^T)$$

is an isomorphism after tensoring with $\operatorname{Frac} R_T$.

In particular, if $K_T(X)$ is a free $K_T(\mathrm{pt})$-module, then the localization map is injective.

5.5 The class of $[\mathcal{O}_{\Sigma^w}]_B$ in $K_B(G/B) = R_T \otimes_{R_G} R_T$ is called the **double Grothendieck polynomial** $\mathfrak{G}_w(x,t)$. Here we take the convention that

$$e^{\lambda(t)} = e^\lambda \otimes 1, \qquad e^{\lambda(x)} = 1 \otimes e^\lambda.$$

Then by localization

$$\forall u \not\leq w, \quad \mathfrak{G}_w(ut,t) = 0.$$

Actually, $\mathfrak{G}_w(x,t)$ is uniquely determined by

- $\mathfrak{G}_{\mathrm{id}}(x,t) = 1$;
- $\pi_i \mathfrak{G}_w(x,t) = \mathfrak{G}_{ws_i}(x,t)$ when $\ell(ws_i) = \ell(w) - 1$;
- $\mathfrak{G}_w(t,t) = \delta_{w=\mathrm{id}}$.

5.6 In type A, recall that we denote $X_i = e^{x_i}$, and

$$\pi_i f = \frac{X_i f - s_i(X_{i+1} f)}{X_i - X_{i+1}}.$$

We have the stable choice

$$\mathfrak{G}_{w_0}(X,Y) = \prod_{i+j \leq n} \left(1 - \frac{Y_i}{X_i}\right).$$

5.7 Denote $T_1, \ldots, T_{n-1}$ the symbols with

$$T_i^2 = -T_i, \qquad \begin{cases} T_iT_j = T_jT_i & |i-j| \geq 2 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{cases}$$

Thus T_w can be defined. We consider the generating function

$$\mathfrak{G}(X,Y) = \sum \mathfrak{G}_w(X,Y)T_w,$$

It is amazing that it factors into

$$\begin{array}{ccccc} h_{n-1}(X_1,Y_{n-1}) & h_{n-2}(X_1,Y_{n-2}) & \cdots & h_1(X_1,Y_2) & h_1(X_1,Y_1) \\ & h_{n-1}(X_2,Y_{n-2}) & \cdots & h_3(X_2,Y_2) & h_2(X_1,Y_1) \\ & & \ddots & \vdots & \vdots \\ & & & h_{n-1}(X_{n-2},Y_2) & h_{n-2}(X_{n-2},Y_1) \\ & & & & h_{n-1}(X_{n-1},Y_1) \end{array}$$

where $h_k(X,Y) = 1 + \big(1 - \frac{X}{Y}\big)T_k$.

5.8 The cohomological version is similar. In this case, the cohomological Demazure operator

$$\partial_i : H_G^\bullet(G/B) \xrightarrow{(p_i)_*} H_G^\bullet(G/P_i) \xrightarrow{(p_i)^*} H_G^\bullet(G/B)$$

is given by

$$\partial_i f = \frac{f - s_i f}{\alpha_i},$$

where $\alpha_i = c_1(\mathcal{O}(\alpha_i))$. It satisfies $\partial_i^2 = 0$ and braid relations.

5.9 In the cohomological case, we need to replace

$$R_T^\bullet = \mathsf{S}^\bullet(X(T)_\mathbb{Q}), \qquad R_G = R_T^W, \qquad R_P = R_T^{W_J}.$$

then

$$H_B^\bullet(G/B;\mathbb{Q}) = R_T \otimes_{R_G} R_T, \qquad H_B^\bullet(G/P;\mathbb{Q}) \cong R_T \otimes_{R_G} R_P.$$

For G/B, $c_1(\mathcal{O}(\lambda)$ is presented by $1 \otimes \lambda \in R_T \otimes_{R_G} R_T$. The class of pull back of $\lambda \in H_B^\bullet(\mathrm{pt};\mathbb{Q}) = R_T$ is $\lambda \otimes 1$.

5.10 By a similar affine paving argument

$$H_B^\bullet(G/P) = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\Sigma^w]_B = \bigoplus_{\text{shortest } w \in W/W_J} \mathbb{Z} \cdot [\Sigma_w]_B$$

It turns out

$$\partial_i[\Sigma^w] = \begin{cases} [\Sigma^{ws_i}] & \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$\partial_i[\Sigma_w] = \begin{cases} [\Sigma_{ws_i}] & \ell(ws_i) = \ell(w) + 1 \\ 0 & \text{otherwise} \end{cases}$$

5.11 The natural map $G/B \longrightarrow G/P$ induces

$$H_B^\bullet(G/P) \longrightarrow H_B^\bullet(G/B) \qquad [\Sigma^w] \longmapsto [\Sigma^w]$$

thus an injection. The corresponding $\mathbb{Q}$-efficient map is just the inclusion

$$R_T \otimes_{R_G} R_P \xrightarrow{\subseteq} R_T \otimes_{R_G} R_T.$$

5.12 The class of $[\Sigma^w]_B$ in $K_B(G/B) = R_T \otimes_{R_G} R_T$ is called the **double Schubert polynomial** $\mathfrak{S}_w(x,t)$. Here we take the convention that

$$\lambda(t) = \lambda \otimes 1, \qquad \lambda(x) = 1 \otimes \lambda.$$

Actually, $\mathfrak{S}_w(x,t)$ is uniquely determined by

- $\mathfrak{S}_{\text{id}}(x,t) = 1$;
- $\partial_i \mathfrak{S}_w(x,t) = \mathfrak{S}_{ws_i}(x,t)$ when $\ell(ws_i) = \ell(w) - 1$;
- $\mathfrak{S}_w(t,t) = \delta_{w=\text{id}}$.

5.13 For $\mathcal{G}r(k,n)$, the case w is shortest, $\mathfrak{S}_w(x,t)$ is the corresponding double Schur polynomial.

5.14 In type A,

$$\pi_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

We have the stable choice

$$\mathfrak{S}_{w_0}(x,y) = \prod_{i+j \le n} (x_i - y_j).$$

Denote $T_1, \ldots, T_{n-1}$ the symbols with

$$T_i^2 = 0, \qquad \begin{cases} T_iT_j = T_jT_i & |i-j| \geq 2 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{cases}$$

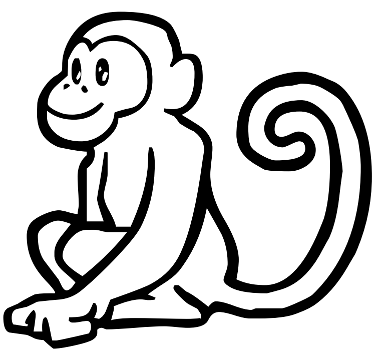
Thus T_w can be defined. We consider the generating function

$$\mathfrak{S}(x,y) = \sum \mathfrak{S}_w(x,y)T_w,$$

It is amazing that it factors into

$$\begin{array}{ccccc} h_{n-1}(x_1,y_{n-1}) & h_{n-2}(x_1,y_{n-2}) & \cdots & h_1(x_1,y_2) & h_1(x_1,y_1) \\ & h_{n-1}(x_2,y_{n-2}) & \cdots & h_3(x_2,y_2) & h_2(x_1,y_1) \\ & & \ddots & \vdots & \vdots \\ & & & h_{n-1}(x_{n-2},y_2) & h_{n-2}(x_{n-2},y_1) \\ & & & & h_{n-1}(x_{n-1},y_1) \end{array}$$

where $h_k(X,Y) = 1 + (x-y)T_k$.



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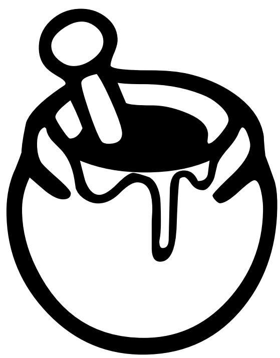

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