Abstract Harmonic Analysis

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Contents

Main References

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— Hewitt Ross. *Abstract Harmonic Analysis.*
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— Graham, McGekee. *Commutative Harmonic Analysis.*

Topological Group

A **topological group** G is a topological space and also a group such that

$$G \times G \longrightarrow G \qquad (x, y) \longmapsto x^{-1}y$$

is continuous. A **locally compact group** (LC group) is a top group which is locally compact. A **locally compact abelian group** (LCA groups) is a LC group which is abelian.

We will only concentrate on locally compact groups (LCgroups), and mostly locally compact abelian groups (LCA groups).

1 Haar Measure

(1.1) !! Notation— Denote

$$L_a : G \longrightarrow G \qquad x \longmapsto ax$$

$$R_a : G \longrightarrow G \qquad x \longmapsto xa$$

(1.2) Theorem *For any LC group G, there exists a (Borel) measure $\mu \geq 0$ over G, such that*

$$\mu \neq 0, \quad \mu(y^{-1}E) = \mu(E), \forall y \in G.$$

and it is unique up to a scalar.

(1.3) Definition (Haar Measure) We call this uo-to-scalar-unique measure the **Haar measure** of G. Similarly, we have right invariant measure, to clarify if necessary, we will say **left/right Haar measure**.

(1.4) EXAMPLE For discrete group G, the counting measure serves.

(1.5) EXAMPLE For $\mathbb{R}^n$ and the circle $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the Lebsgue measures serve.

(1.6) EXAMPLE For $\mathbb{R}^\times$, the measure $\mathrm{d}^\times x = \frac{\mathrm{d}x}{x}$ serve.

(1.7) EXAMPLE For any Lie group G, there exists a left-invariant integral form ω by left translation from the unit. Then we can take the measure to be $\mathrm{d}\mu = \omega$.

(1.8) EXAMPLE In particular, for $\mathrm{GL}_n(\mathbb{R})$, the measure is $\frac{\bigwedge \mathrm{d}x_{ij}}{\det(x_{ij})}$.

1.1 The existence

Representation theorem

Assume S is a locally compact space, denote $C_c^+(S)$ the space of nonnegative, continuous and compactly supported functions on S. Then the linear functional

$$\Phi : C_c^+(S) \longrightarrow \mathbb{R}_{\geq 0}$$

is uniquely represented by $\Phi(f) = \int_S f \mathrm{d}\mu$ for some regular nonnegative measure μ on S.

(1.9) Definition We say $I : C_c^+(G) \to \mathbb{R}_{\geq 0}$ is
(1) **left invariant** if $I(f \circ L_{y^{-1}}) = I(f)$;
(2) **homogenous** if $I(\lambda f) = \lambda I(f)$;
(3) **subadditive** if $I(f + f') \leq I(f) + I(f')$;
(4) **monotone** if $f \leq g \Rightarrow I(f) \leq I(g)$.
Let us reform the theorem of existence.

(1.10) Theorem *There exists a left invariant non-negative nonzero additive homogenous functional Λ on $C_c^+(G)$.*

Construction I

Fix $g \in C_c^+(G)$, and $g \neq 0$. For $f \in C_c^+$, define

$$I_g(f) = \inf \left\{ s : \begin{array}{l} \text{there exists a finite subset } A \subseteq G, \text{ and } \{c_a \geq 0 : a \in A\} \text{ such that } f \leq \sum_{a \in A} c_a (g \circ L_{a^{-1}}), \\ \text{and } \sum c_a \leq s \end{array} \right\}$$

Note that $I_g(f) < \infty$ by compactness, and I_g is left invariant, homogenous and monotone by definition.
(1) $I_g(f + f') \leq I_g(f) + I_g(f')$.
(2) $I_g(u) \leq I_g(\varphi) I_\varphi(u)$ for all u.
To normalize, fix some function $\varphi \in C_c^+(G) \setminus 0$, and $g \in C_c^+(G) \setminus 0$, put the "average"

$$\Lambda_g = \frac{1}{I_g(\varphi)} I_g \leq I_\varphi.$$

The functional I_g (Λ_g) is a rough (average) approximation of translation of the "local ruler" g. We want to make $\mathsf{supp}\, g$ shrink to identity, and g nonsingular as possible.

(1.11) !! Notation— Let V be a neighborhood (nbd) of 1,

$$\mathcal{P}(V) = \{f \in C_c^+(G) : \mathsf{supp}\, f \subseteq V, f \neq 0\},$$

and $\mathcal{P}_*(V) = \{f \in \mathcal{P}(V) : f(g) = f(g^{-1})\}$.

(1.12) Lemma *For $f_1, \ldots, f_n \in C_c(G)$, and $r > 1$, there exists a nbd V of 1, such that*

$$I_g(f' + f') \leq I_g(f) + I_g(f') \leq rI_g(f' + f')$$

for all $g \in \mathcal{P}(V)$.

(1.13) Lemma *For all $f \in C_c^+(G)$, and $r > 1$, there exists a nbd U of 1, such that for all $g \in \mathcal{P}_*(U)$, there exists a nbd W of 1 with*

$$I_h(f) \leq I_g(f)I_h(g) \leq rI_h(f)$$

for all $h \in \mathcal{P}(W)$.

(1.14) Corollary *For all $f \in C_c^+(G)$, and $r > 1$, there exists a nbd U of 1, such that for all $g \in \mathcal{P}_*(U)$, there exists a nbd W of 1 with*

$$\frac{1}{r}\Lambda_g(f) \leq \frac{I_h(f)}{I_h(\varphi)} \leq r\Lambda_g(f)$$

for all $h \in \mathcal{P}(W)$.

Proof. One can find a W serves for both f and φ. So for any $h \in \mathcal{P}(W)$,

$$I_h(f) \leq I_g(f)I_h(g) \leq rI_h(f),$$
$$I_h(\varphi) \leq I_g(\varphi)I_h(g) \leq rI_h(\varphi).$$

Then divide them each other, we get the desired inequality.

Construction II

Let

$$\mathcal{H}_r(f) = \left\{ g \in \mathcal{P}(G) : \begin{array}{l} \text{there exists nbd } W \text{ of } 1 \\ \text{such that for all } h \in \\ \mathcal{P}(W),\ \Lambda_g(f) \leq r\Lambda_h(f) \end{array} \right\},$$

$$\overline{\Lambda}_r(f) = \sup\{\Lambda_g(f) : g \in \mathcal{H}_r(f)\}, \quad \Lambda(f) = \lim_{r \searrow 1} \overline{\Lambda}_r(f).$$

Since $\overline{\Lambda}_r(\varphi) = 1$, $\Lambda \neq 0$.

By the lemmate above, for $f, f' \in C_c^+(G)$, we have

$$\frac{1}{r}\overline{\Lambda}_r(f + f') \leq \overline{\Lambda}_r(f) + \overline{\Lambda}_r(f') \leq r^2\overline{\Lambda}_r(f + f').$$

So Λ is left-invariant and additive.

So everything is done except the proof of lemma (1.12) and lemma (1.13).

Proof of lemma (1.12). By Urysohn lemma and uniformly continuity, for $\epsilon > 0$, one can find F satisfying

$$I_g(f + f') \leq I_g(F) \leq (1 + \epsilon)I_g(f + f').$$

for any g, and a nbd V of 1 whenever $a^{-1}y \in V$,

$$\frac{f^{(\prime)}(y)}{F(y)} \leq \frac{f^{(\prime)}(a)}{F(a)} + \epsilon.$$

Assume

$$F \leq \sum_{a \in A} c_a(g \circ L_{a^{-1}}),$$

then

$$\begin{aligned} f^{(\prime)} &= \frac{f^{(\prime)}}{F} \cdot F \\ &\leq \textstyle\sum_{a \in A} c_a \cdot \frac{f^{(\prime)}}{f} \cdot (g \circ L_{a^{-1}}) \\ &\leq \textstyle\sum_{a \in A} c_a \cdot \left(\frac{f^{(\prime)}(a)}{F(a)} + \epsilon\right) \cdot (g \circ L_{a^{-1}}). \end{aligned}$$

As a result,

$$\begin{aligned} I_g(f) + I_g(f') &\leq \textstyle\sum_{a \in A} c_a \cdot \left(\frac{f(a) + f'(a)}{F(a)} + \epsilon\right) \\ &\leq (1 + 2\epsilon) \textstyle\sum_{a \in A} c_a \\ &\leq (1 + 2\epsilon)I_g(F) \\ &\leq (1 + 2\epsilon)(1 + \epsilon)I_g(f + f'). \end{aligned}$$

The proof is complete.

Proof of lemma (1.13). Firstly, by Urysohn Lemma, there exists $\overline{f}$ and nbd U of 1 such that $f(x) \leq \overline{f}(y)$ whenever $x^{-1}y \in U$, and

$$I_h(\overline{f}) \leq (1 + \epsilon)I_h(f)$$

for any h. Now, if $g \in \mathcal{P}_*(U)$, one can find $\overline{g}$ and nbd V of 1 such that $g(x^{-1}y) \leq \overline{g}(x^{-1}a) = \overline{g}(a^{-1}x)$ whenever $a^{-1}y \in V$, and

$$I_h(\overline{g}) \leq (1 + \epsilon)I_h(g).$$

Now, assume $\operatorname{supp} \overline{f} \subseteq \bigcup_{a\in A} aV$. By decomposition of unity, write $\overline{f} = \sum_{a\in A} \overline{f}_a$. Then

$$\begin{aligned} f(x)g(x^{-1}y) &\le \overline{f}(y)g(x^{-1}y) \\ &\le \textstyle\sum_{a\in A} \overline{f}_a(y)g(x^{-1}y) \\ &\le \textstyle\sum_{a\in A} \overline{f}_a(y)\overline{g}(a^{-1}x), \end{aligned}$$

So

$$\begin{aligned} I_g(f)I_h(g) &\le \textstyle\sum_{a\in A} I_h(\overline{f}_a)I_g(\overline{g}) \\ &\le (1+\epsilon)I_g(\overline{f})I_g(g) \\ &\le (1+\epsilon)^2 I_g(f) \end{aligned}$$

The proof is complete.

(1.15) Remark If we take

$$X = \prod_{g\in C_c^+} [(I_\varphi g)^{-1}, I_\varphi g]$$

and $K(V)$ the closure of $\{\Lambda_g : \operatorname{supp} g \in V\}$. Then $\bigcap_{\mathrm{nbd} V} K(V) \neq 0$ by Tychonoff theorem. In this case, lemma (1.13) is not needed. The element in the intersection is a desired linear functional. But this argument uses the axiom of choice which is not needed.

1.2 The uniqueness

Let μ be a left-invariant measure, and ν a right-invariant measure,

$$\begin{aligned} \textstyle\int_G f\mathrm{d}\mu \cdot \int_G g\mathrm{d}\nu &= \textstyle\int_G f(x)\left(\int g(yx)\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x) \\ &= \textstyle\int_{G\times G} f(x)g(yx)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \\ &= \textstyle\int_{G\times G} f(y^{-1}x)g(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \\ &= \textstyle\int_{G\times G} g(x)\left(\int f(y^{-1}x)\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x) \end{aligned}$$

So

$$\frac{\int f(y^{-1}x)\mathrm{d}\nu(y)}{\int_G f\mathrm{d}\mu}$$

is continuous and independent with respect to $f \neq 0$. Then apply $x = 1$, this is desired uniqueness.

1.3 Modular character

(1.16) Definition For a LC group G, μ its Haar measure, we define **Modular character** $\Delta_G : G \to \mathbb{R}_{>0}$ by

$$\mathrm{d}\mu(x) = \Delta_G(g)\mathrm{d}\mu(gxg^{-1}).$$

Equivalently,

$$\Delta_G(g)\int f(xg)\mathrm{d}\mu(x) = \int f(x)\mathrm{d}\mu(x).$$

It is easy to see Δ_G is a character, i.e.

$$\Delta_G(gh) = \Delta_G(g)\Delta(h).$$

It is also easy to see that the left right Haar measure coincide if and only if Δ_G is trivial.

(1.17) Theorem *For compact group G, left right Haar measure coincide.*

PROOF. Let μ and ν be the left and right Haar measure respectively. Take $f = 1$, then

$$\Delta_G(g)\mu(G) = \mu(G).$$

As a result, $\Delta_G(g) = 1$.

▶ **(1.18) Exercise.** For Haar measure μ, show that if $f \in C_c^+(G)\setminus 0$, then $\int_G f\mathrm{d}\mu > 0$. `Hint: If` $f \neq 0$, `WLOG we can assume there is a nbd` V `of 1 such that` $0 \notin f(V)$. `Using` V `to cover any compact subset, we find that` $\mathrm{d}\mu = 0$.

▶ **(1.19) Exercise.** If $G/[G,G]$ is compact, show that left right Haar measure coincide. `Hint: The modular character factors through` $G/[G,G]$, `but` $\mathbb{R}_{>0}$ `has no compact subgroup except the trivial one.`

▶ **(1.20) Problem.** About modular character, show that

$$\mathrm{d}x = \Delta(x)\mathrm{d}(x^{-1}).$$

`Hint: Since`

$$\begin{aligned} \textstyle\int f(xy)\Delta(xy)\mathrm{d}(x^{-1}) &= \Delta(y)\textstyle\int f(x^{-1})\Delta(x^{-1})\mathrm{d}x \\ &= \textstyle\int f((xy^{-1})^{-1})\Delta((xy^{-1})^{-1})\mathrm{d}x \\ &= \textstyle\int f(yx^{-1})\Delta(yx^{-1})\mathrm{d}x \\ &= \textstyle\int f(x^{-1})\Delta(x^{-1})\mathrm{d}x \\ &= \textstyle\int f(x)\Delta(x)\mathrm{d}(x^{-1}) \end{aligned}$$

`so` $\Delta(x)\mathrm{d}(x^{-1})$ `is left-invariant. To determine the scalar it suffices to check a symmetric compact neighborhood.`

▶ **(1.21) Problem.** Let $G = \{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right) : x > 0, y \in \mathbb{R}\}$, calculate the left/right Haar measures which are different. `Hint: Since` $\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} X & Y \\ & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} xX & xY+y \\ & 1\end{smallmatrix}\right)$. `Left Haar measure` $\lambda(x,y)\mathrm{d}x\mathrm{d}y$ `satisfy`

$$\begin{aligned} &\textstyle\int f(xx_0, x_0y+y_0x)\lambda(xx_0, x_0y+y_0x)\mathrm{d}x\wedge\mathrm{d}y \\ =& \textstyle\int f(x,y)\lambda(x,y)\mathrm{d}x\wedge\mathrm{d}y \\ =& \textstyle\int f(xx_0, x_0y+y_0x)\lambda(x,y)\mathrm{d}(xx_0)\wedge\mathrm{d}(x_0y+y_0) \\ =& \textstyle\int f(xx_0, x_0y+y_0x)\lambda(x,y)x_0^2\mathrm{d}x\wedge\mathrm{d}y \end{aligned}$$

`So` $\lambda(x_0,y_0) = \lambda(1,1)x_0^2$, `so left Haar measure is` $\frac{\mathrm{d}x\wedge\mathrm{d}y}{x^2}$. `Similarly,`

$$\mathrm{d}(x_0x)\wedge\mathrm{d}(xy_0+y) = x_0\mathrm{d}x\wedge\mathrm{d}y$$

`so right Haar measure is` $\frac{\mathrm{d}x\wedge\mathrm{d}y}{x}$.

2 LCA Groups

Commutative Banach Algebra

If A is a commutative Banach algebra with unity, we denote $\mathcal{M}(A)$ the spectrum (of Gelfand space) by

$$\mathcal{M}(A) = \left\{ \varphi : A \to \mathbb{C} : \begin{array}{l} \varphi \neq 0 \text{ is an algebra homo-} \\ \text{morphism.} \end{array} \right\}$$

with ω^*-topology, i.e. the smallest topology such that the evaluation

$$\mathrm{eva}_a : \mathcal{M}(A) \longrightarrow \mathbb{C} \qquad \varphi \longmapsto \varphi(a)$$

is continuous. We know that

(1) any $\varphi \in \mathcal{M}(A)$ is continuous with norm 1;

(2) $\mathcal{M}(A)$ is ω^*-closed subset of A^*, i.e. $\sigma(A^*, A)$, therefore $\mathcal{M}(A)$ is compact.

(3) under the imbedding

$$\pi : A \longrightarrow C(\mathcal{M}(A)) \qquad a \longmapsto \mathrm{eva}_a,$$

$\pi(A)$ is a separating subalgebra. One can define

$$\begin{aligned} \|a\| &= \| \mathrm{eva}_a \| \\ &= \text{spectral norm of } a \\ &= \lim_{n\to\infty} \sqrt[n]{\|a\|^n} \leq \|a\|. \end{aligned}$$

What happens if there is no unity ?

Let

$$\tilde{A} = \{(a, x) : a \in A, x \in \mathbb{C}\}$$

with norm $\|a + xe\| = \|a\| + |x|$. Then $S = \mathcal{M}(\tilde{A})$ is compact. It has the infinity point

$$\varphi_0 : \tilde{A} \longrightarrow \mathbb{C} \qquad a + xe \longmapsto e,$$

which is the only element φ with $\varphi(A) = 0$. We define $\mathcal{M}(A) = \mathcal{M}(\tilde{A}) \setminus \varphi_0$.

(2.1) !! Notation— In this section, the G is a LCA group. We will write it additively with Haar measure m if we need to clarify. For a function over G, $u \in G$, we denote the **translation** f_u defined by

$$\tau_u f = f_u(x) = f(x - u).$$

(2.2) Definition (Convolution) For two functions f, g over LCA group G, we define the **convolution**

$$f * g(x) = \int_G f(x-y)g(y)\mathrm{d}y = \int_G f(y)g(x-y)\mathrm{d}y.$$

This makes sense whenever $\displaystyle\int |f(x-y)g(y)|\mathrm{d}y < \infty$.

(2.3) Proposition *For fixed $f \in L^p(G)$, the map*

$$G \longrightarrow L^p(G) \qquad u \longmapsto f_u$$

is uniformly continuous.

Proof. It is true if $f \in C_c(G)$. For $f \in L^p(G)$, one can find $g \in C_c(G)$ such that $\|f - g\|_p \leq \epsilon$.

Properties of Convolution

1. If $f \in L^1$ and $g \in L^\infty$, then $f * g$ is bounded and uniformly continuous.

$$\begin{aligned} |f * g(x)| &\leq \int_G |f(x-y)| \cdot |g(y)|\mathrm{d}y \\ &\leq \|g\|_\infty \cdot \|f\|_1. \end{aligned}$$

So $\|f * g\|_\infty \leq \|g\|_\infty \|f\|_1$. Similarly,

$$|f * g(x_1) - f * G(x_2)| \leq \|f - \tau_{\Delta x} f\|_1 \cdot \|g\|_\infty.$$

2. If $1 < p < \infty$, $q = p'$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then $f * g$ is is bounded and uniformly continuous.

$$\begin{aligned} |f * g(x)| &\leq \int_G |f(x-y)| \cdot |g(y)|\mathrm{d}y \\ &\leq \left(\int_G |f(x-y)|^p \mathrm{d}y\right)^{1/p} \cdot \left(\int_G |g(y)|^q \mathrm{d}y\right)^{1/q} \\ &= \|f\|_p \cdot \|g\|_q. \end{aligned}$$

3. If $f, g \in L^1(G)$, then $f * g \in L^1$.

$$\begin{aligned} \int |f * g(x)|\mathrm{d}x &\leq \iint |f(x-y)| \cdot |g(y)|\mathrm{d}x\mathrm{d}y \\ &= \int |g(y)| \cdot \left(\int |f(x-y)|\mathrm{d}x \right) \mathrm{d}y \\ &\leq \|f\|_1 \cdot \|g\|_1. \end{aligned}$$

The fact that $|f(x-y)g(y)|$ is integrable follows from when f, g is characteristic functions of measurable sets, and the continuity of addition.

4. $f * g = g * f$.

So $L^1(G)$ is a Banach algebra under convolution. It has a unity if and only if G is discrete.

(2.4) Approximation of identity *Let $f \in L^1(G)$, $C > 0$. For all $\epsilon > 0$, there exists a* nbd *V of 0 such that for any $v \in L^1(G)$ which vanishes outside of V, and*

$$\int v = 1, \quad \int |v| = C,$$

we have $\|f * v - f\|_1 \leq \epsilon$.

2.1 Characters

(2.5) Definition (Character) A continuous function $\gamma : G \to \mathbb{T} \subseteq \mathbb{C}^\times$ is called a **character** if γ is a homomorphism, that is $|\gamma(x)| = 1$ and $\gamma(x + y) = \gamma(x)\gamma(y)$.

So $\gamma(0) = 1$ and $\gamma(-x) = \overline{\gamma(x)}$.

(2.6) Definition (Dual Group) Denote all characters of G by $\hat{G}$, which equipped with a group structure. We call $\hat{G}$ the **dual group**.

(2.7) Theorem *Any $\varphi \in \mathcal{M}(G)$ is of the form*

$$\varphi(f) = \int_G f(x)\overline{\gamma(x)}\mathrm{d}x$$

for some character γ.

Proof. Firstly, it is definitely a homomorphism.

$$\begin{aligned}\varphi(f * g) &= \iint f(x-y)g(y)\overline{\gamma(x)}\mathrm{d}x\mathrm{d}y \\ &= \iint f(x-y)\overline{\gamma(x-y)}g(y)\overline{\gamma(y)}\mathrm{d}x\mathrm{d}y \\ &= \left(\int f(z)\overline{\gamma(z)}\mathrm{d}z\right)\left(\int g(y)\overline{\gamma(y)}\mathrm{d}y\right) \\ &= \varphi(f)\varphi(g).\end{aligned}$$

Conversely, since the homomorphism is locally L^1, there exists $\gamma \in L^\infty(G)$ with $\|\gamma\|_\infty = \|\varphi\| = 1$, such that

$$\varphi(f) = \int_G f(x)\overline{\gamma(x)}\mathrm{d}x.$$

Now, we also have $\varphi(f * g) = \varphi(f)\varphi(g)$, i.e.

$$\iint f(x-y)g(y)\overline{\gamma(x)}\mathrm{d}x\mathrm{d}y = \iint f(x)g(y)\overline{\gamma(x)}\,\overline{\gamma(y)}\mathrm{d}x\mathrm{d}y.$$

Since g is arbitrary,

$$\int f(z)\overline{\gamma(z+y)} = \int f(x-y)\overline{\gamma(x)}\mathrm{d}x = \int f(x)\overline{\gamma(x)}\,\overline{\gamma(y)}\mathrm{d}x$$

since f is also arbitrary, $\gamma(x + y) = \gamma(x)\gamma(y)$. Then

$$\gamma(y) = \frac{\int f(x-y)\overline{\gamma(x)}\mathrm{d}x}{\int f(x)\overline{\gamma(x)}}$$

is continuous. Since $|\gamma(x)| \leq 1$ and $\left|\frac{1}{\gamma(x)}\right| = |h(-x)| \leq 1$, $\gamma(x) \in \mathbb{T} \subseteq \mathbb{C}^\times$.

(2.8) Remark So we find a bijection between $\hat{G}$ and $\mathcal{M}(G)$. We can topologize $\hat{G}$ by the topology of $\mathcal{M}(G)$.

(2.9) Proposition *If G is discrete, then $\hat{G}$ is compact. If G is compact, then $\hat{G}$ is discrete.*

Proof. If G is compact, then all characters in $L^1(G)$. We normalize Haar measure m such that $m(G) = 1$. Now,

$$\mathrm{eva}_1(\gamma) = \int 1\overline{\gamma}(x)\mathrm{d}x = \begin{cases} 1, & \gamma = 1, \\ 0, & \gamma \neq 1, \end{cases}$$

since $\int \gamma(x)\mathrm{d}x = \int \gamma(x+x_0)\mathrm{d}x = \gamma(x_0)\int \gamma(x)\mathrm{d}x$ for any $x_0 \in G$. But eva_1 is continuous, so $\hat{G}$ is discrete.

Conversely, if G is discrete, then $L^1(G)$ has unity, so $\hat{G}$ is compact.

(2.10) EXAMPLE For $G = \mathbb{R}$, then $\hat{G} = \mathbb{R}$ given by the pairing

$$B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C} \qquad (x, a) \longmapsto e^{ixa}.$$

More exactly, if we denote the character corresponding to $a \in \hat{G} = \mathbb{R}$, then $\gamma_a(x) = B(a, x)$.

More generally, for $G = \mathbb{R}^n$, then $\hat{G} = \mathbb{R}^n$ given by the pairing

$$B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C} \qquad (x, a) \longmapsto e^{i\langle x, a\rangle},$$

where $\langle \cdot, \cdot \rangle$ is standard inner product.

(2.11) EXAMPLE For $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\hat{G} = \mathbb{Z}$ given by the pairing

$$B : \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C} \qquad (x, a) \longmapsto e^{2\pi ixa},$$

where $xa \bmod \mathbb{Z}$ is well-defined. Conversely, for $G = \mathbb{Z}$, $\hat{G} = \mathbb{R}/\mathbb{Z}$ by the same pairing.

For $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then $\hat{G} = \mathbb{Z}^n$ given by the pairing

$$B : \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{C} \qquad (x, y) \longmapsto e^{i\langle \alpha, x\rangle},$$

Here $\langle \cdot, \cdot \rangle$ is standard inner product. Conversely, for $G = \mathbb{R}/\mathbb{Z}$, $\hat{G} = \mathbb{Z}$ by the same pairing.

(2.12) EXAMPLE For $G = \mathbb{Z}/n\mathbb{Z}$, then $\hat{G} = \mathbb{Z}/\mathbb{Z}$ given by the pairing

$$B : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C} \qquad (x, y) \longmapsto e^{2\pi i\frac{xy}{n}},$$

Here $\frac{xy}{n} \bmod \mathbb{Z}$ is well-defined.

(2.13) Definition (Fourier transform) Let $f \in L^1(G)$, define its **Fourier transform**

$$\hat{f} : \hat{G} \longrightarrow \mathbb{C} \qquad \gamma \longmapsto \int_G f(x)\overline{\gamma(x)}\mathrm{d}x.$$

(2.14) !! Notation— Denote

$$\mathcal{A} = \widehat{L^1(G)} = \{\hat{f} : f \in L^1(G)\}.$$

Properties of $\mathcal{A}$

1. $\mathcal{A}$ is an subalgebra of $\subseteq C_0(\hat{G})$. Since $\hat{G} = \mathcal{M}(\widetilde{L^1(G)}) \setminus \varphi_0$, and $\hat{f}(\gamma)$ is actually $\mathrm{eva}_f\,\Gamma$, with $\Gamma \in \mathcal{M}(L^1(G))$. But $\mathrm{eva}_f\,\varphi_0 = 0$. This is known as Riemann-Lebsgue lemma.

2. $\mathcal{A}$ separate points of $\hat{G}$.

3. $\mathcal{A}$ is self adjoint. Actually, $\overline{\widehat{f(\bullet)}} = \widehat{\overline{f(-\bullet)}}$. As a result, $\mathcal{A}$ is dense in $C_0(\hat{G})$.

4. $\mathcal{A}$ is translation invariant, and is invariant under multiplication by characters.

$$\begin{aligned} \widehat{f_y}(\gamma) &= \int f(x-y)\overline{\gamma}(x)\mathrm{d}x \\ &= \int f(s)\overline{\gamma(s+y)}\mathrm{d}s \\ &= \overline{\gamma(y)}\hat{f}(\gamma) \end{aligned}$$

$$\begin{aligned} \widehat{\rho f}(\gamma) &= \int f(x)\rho(x)\overline{\gamma}(x)\mathrm{d}x \\ &= \int f(x)\overline{\gamma\overline{\rho}(x)}\mathrm{d}s \\ &= \hat{f}(\gamma\overline{\rho}). \end{aligned}$$

(2.15) Theorem *The dual group $\hat{G}$ is a topological group under Gelfand topology.*

(2.16) !! Notation— Let $K \subseteq G$ be a compact set, $\epsilon > 0$, denote

$$U_{K,\epsilon} = \{\gamma \in \hat{G} : \forall x \in K, |\gamma(x) - 1| < \epsilon\}.$$

Let $C \subseteq \hat{G}$ be a compact set, $\epsilon > 0$, denote

$$V_{C,\epsilon} = \{x \in G : \forall \gamma \in C, |\gamma(x) - 1| < \epsilon\}.$$

(2.17) Lemma *The pairing*

$$G \times \hat{G} \longrightarrow \mathbb{C} \qquad (x, \gamma) \longmapsto \gamma(x)$$

is continuous.

Proof. Find some good $f \in L^1(G)$. Note that $\hat{f}(\gamma)\overline{\gamma(x)} = \widehat{f_x}(\gamma)$, so $\gamma(x) = \frac{\widehat{f_x}(\gamma)}{\hat{f}(\gamma)}$. Then

$$\begin{aligned} |\widehat{f_x}(\gamma) - \widehat{f_y}(\delta)| &\le |\widehat{f_x}(\gamma) - \widehat{f_x}(\delta)| + |\widehat{f_x}(\delta) - \widehat{f_y}(\delta)| \\ &\le |\widehat{f_x}(\gamma) - \widehat{f_x}(\delta)| + \|\widehat{f_x - f_y}\| \\ &\le |\widehat{f_x}(\gamma) - \widehat{f_x}(\delta)| + \|f_x - f_y\|_1. \end{aligned}$$

The proof is complete.

(2.18) Proposition *The set $\{U_{K,\epsilon}$: K compact, $\epsilon > 0\}$ forms a basis of* nbd *of unity of $\hat{G}$.*

Proof. It is open by tube lemma.

Pick $\gamma_0 \in \hat{G}$, $f_1, \ldots, f_N \in L^1(G)$. The open subset of the form $W = \{\gamma \in \hat{G} : |\hat{f}_j(\gamma_0) - \hat{f}_j(\gamma)| < \epsilon, j = 1, \ldots, N\}$ is a basis of $\hat{G}$.

We need to find $U_{K,\epsilon}$ such that $\gamma_0 + U_{K,\epsilon} \subseteq W$. By shifting, it may be assumed that $\gamma_0 = 0$. Note that $C_c(G)$ is dense in $L^1(G)$, we can also assume $f_i \in C_c(G)$, since $|(\hat{f}_j - \hat{g}_j)(\gamma)| \le \|f_j - g_j\|_1$. Let $K = \bigcup \operatorname{supp} f_i$. Suppose $|\gamma(x) - 1| < \delta$ for all $x \in K$, then

$$\begin{aligned} |\hat{f}_i(0) - \hat{f}_j(\gamma)| &= \left|\int f_j(x)(1 - \overline{\gamma(x)})\mathrm{d}x\right| \\ &\le \delta \int |f_j(x)|\mathrm{d}x = \delta\|f_j\|_1. \end{aligned}$$

So we can take $\epsilon = \delta / \max_j \|f_j\|_1$.

Proof of (2.15). That is, the map

$$\hat{G} \times \hat{G} \longrightarrow \hat{G} \qquad (\gamma_1, \gamma_2) \longmapsto \gamma_1 - \gamma_2$$

is continuous, since $U_{K_1,\epsilon}U_{K_2,\epsilon} \subseteq U_{K_1 \cap K_2, 2\epsilon}$. More precisely, $|\delta_1\overline{\delta_2}(x) - 1| = |\delta_1(x) - \delta_2(x)| \le |\delta_1(x) - 1| + |\delta_2(x) - 2|$.

▶ **(2.19) Exercise.** Prove example (2.10). Hint: Consider

$$\int \gamma(y)u(x-y)\mathrm{d}y = \int \gamma(x-y)u(y)\mathrm{d}y = \gamma(x)\int u(y)\overline{\gamma}(y)\mathrm{d}y$$

so $\gamma(x)$ is differentiable. Now $\gamma'(x) = \gamma'(1)\gamma(x)$.

▶ **(2.20) Exercise.** Prove example (2.11). Hint: Use the example (2.10). Conversely, the homomorphism from $\mathbb{Z}$ is determined by the image of 1.

▶ **(2.21) Problem.** Find a direct proof of Riemann-Lebsgue lemma over $\mathbb{R}$. Hint: $2\hat{f}(t) \le \int (f(x) - f(x + t/2t^2))e^{ixt}\mathrm{d}x$.

3 Fourier Analysis over LCA Groups

(3.1) !! Notation— Let $\mathcal{M}(G)$ the space of finite regular Borel measures over G.

(3.2) Definition (Convolution) Let $C_c(G)$ be space of functions of compact support. Given two finite regular Borel measure μ, ν on G, the functional

$$C_c(G) \longrightarrow \mathbb{C} \qquad f \longmapsto \iint f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$$

is generated by some measure λ, i.e.

$$\int f\mathrm{d}\lambda = \iint f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

We will denote $\lambda = \mu * \nu$ the **convolution** of μ and ν. It makes $\mathcal{M}(G)$ a Banach algebra (with unity δ_0).

(3.3) Remark Let $\lambda = \mu * \nu$.

- Note that

  $$\lambda(E) = \iint \mathbb{1}_E(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int \mu(E-y)\mathrm{d}\nu(y).$$

  So if μ is absolutely continuous wrt Haar measure, then so is $\mu * \nu$.

- If μ is absolutely continuous wrt Haar measure, say $\mu(E) = \int_E f\mathrm{d}x$, then

  $$\begin{aligned}\lambda(E) &= \int_G \int_{E-y} f(x)\mathrm{d}x\mathrm{d}\nu(y) \\ &= \int_E \underbrace{\left(\int_G f(x-y)\mathrm{d}\nu(y)\right)}_{\text{density}} \mathrm{d}x\end{aligned}$$

- If ν is also absolutely continuous wrt Haar measure, say $\mu(E) = \int_E g\mathrm{d}x$, then the density of λ is $\int f(x-y)g(y)\mathrm{d}y = f * g$.

By above, the embedding $L^1(G) \hookrightarrow M(G)$ makes $L^1(G)$ an ideal.

(3.4) Definition (Fourier transform) For $\mu \in \mathcal{M}(G)$, we define its **Fourier transform**

$$\hat{\mu} : \hat{G} \longrightarrow \mathbb{C} \qquad \gamma \longmapsto \int \overline{\gamma}(x)\mathrm{d}\mu(x).$$

Then clearly, $(\mu * \nu)^\wedge = \hat{\mu} \cdot \hat{\nu}$.

3.1 Bochner's theorem

(3.5) Definition (Positive definite) A complex continuous function φ on G is said to be **positive definite** if

$$\sum_{n,m=1}^{N} c_n \cdot \overline{c_m} \cdot \varphi(x_n - x_m) \geq 0$$

for any $c_i \in \mathbb{C}$ and $x_i \in G$.

(3.6) Proposition *If φ is positive definite, then*

$$\iint f(x)\overline{f(y)}\varphi(x+y)\mathrm{d}x\mathrm{d}y \geq 0$$

(3.7) Bochner's theorem *A complex continuous function φ is positive definite if and only if there is a measure $\mu \in \mathcal{M}(\hat{G})$,*

$$\mu \geq 0, \quad \varphi(x) = \int_{\hat{G}} \gamma(x)\mathrm{d}\mu(\gamma).$$

THE "if" PART. We have

$$\begin{aligned}&\sum_{n,m=1}^{N} c_n \cdot \overline{c_m} \cdot \varphi(x_n - x_m) \\ &= \int \sum_{n,m=1}^{N} c_n \cdot \overline{c_m} \cdot \gamma(x_n - x_m)\mathrm{d}\mu(\gamma) \\ &= \int \left|\sum_{j=1}^{N} c_i\gamma(x_i)\right|^2 \mathrm{d}\mu(\gamma) \geq 0.\end{aligned}$$

(3.8) EXAMPLE For f, we denote $\tilde{f}(x) = \overline{f(-x)}$. If $f \in L^2(G)$, then $f * \tilde{f}$ is positive.

$$\begin{aligned}&\sum c_n\overline{c_m}\varphi(x_m - x_n) \\ &= \sum c_n\overline{c_m} \int f(x_n - x_m - y)\overline{f(-y)}\mathrm{d}y \\ &= \sum c_n\overline{c_m} \int f(x_n - y)\overline{f(x_m - y)}\mathrm{d}y \\ &= \int \left|\sum c_j f(x_i - y)\right|^2 \mathrm{d}y \geq 0.\end{aligned}$$

Script

If $\varphi : G \to \mathbb{C}$ is positive. When $N = 1$. That is, $c\bar{c}\varphi(0) \geq 0$, so $\varphi(0) \geq 0$. When $N = 2$. We have $\begin{pmatrix} \varphi(0) & \varphi(x) \\ \varphi(-x) & \varphi(0)\end{pmatrix}$ is hermitian and of determinant nonnegative, so

$$\varphi(x) = \overline{\varphi(-x)}, \quad |\varphi(x)| \leq |\varphi(0)|.$$

In particular, φ is bounded.

(3.9) !! Notation— By above, we may assume $\varphi(0)=1$. Define

$$T_\varphi : L^1(G) \longrightarrow \mathbb{C} \qquad f \longmapsto \int f\varphi \mathrm{d}x.$$

Then $\|T_e\| = \operatorname{ess\,sup}|\varphi| = 1$. For $f,g \in L^1(G)$, set

$$[f,g] = T_\varphi(f * \tilde{g}),$$

where $\tilde{f}(x) = \overline{f(-x)}$.

Properties of $[f,g]$

Firstly, note that

$$\begin{aligned}[f,g] &= \int \varphi(x) \int f(u)\tilde{g}(x-u)\mathrm{d}u\mathrm{d}x \\ &= \iint f(u)\overline{g}(y)\varphi(u-y)\mathrm{d}u\mathrm{d}y.\end{aligned}$$

So, $[-,-]$ is hermitian, i.e. $[f,g] = \overline{[g,f]}$, and so it has Cauchy inequality, $\left|[f,g]\right|^2 \le [f,f]\cdot[g,g]$.

Proof of (3.7). We are going to take some approximation of identity for g. Let $g = \chi_V = \frac{\mathbb{1}_V}{|V|}$, with V a symmetric compact nbd of 1. Then

$$\begin{aligned}[f,\chi_V] &= \int \frac{1}{|V|}\int_V f(x)\varphi(x-y)\mathrm{d}y\mathrm{d}x \\ &\overset{V \text{ shrinks}}{\longrightarrow} \int f(x)\varphi(x)\mathrm{d}x = T_\varphi f. \\ [\chi_V,\chi_V] &\overset{V \text{ shrinks}}{\longrightarrow} \varphi(0) = 1.\end{aligned}$$

So

$$|T_\varphi f|^2 \le [f,f] = T_\varphi(f*\tilde{f}).$$

Put $h = f * \tilde{f}$, then $\tilde{h} = h$, and

$$\begin{aligned}|T_\varphi(h)| &\le |T_\varphi(h*h)|^{1/2} \\ &\le |T_\varphi(h*h*h*h)|^{1/4} \\ &\le \cdots \le \Big|T\big(\underbrace{h*\cdots*h}_{2^n}\big)\Big|^{1/2^n} \\ &\le \big\|\underbrace{h*\cdots*h}_{2^n}\big\|^{1/2^n}.\end{aligned}$$

So $|T_\varphi(h)| \le \|h\| = \|\hat{h}\|_\infty$. But we know $\hat{h} = \hat{f}\hat{\tilde{f}} = |\hat{f}|^2$, so

$$|T_\varphi(f)| \le \|\hat{f}\|_\infty.$$

Since $\widehat{L(G)}$ is dense in $C_0(\hat{G})$, by Banach extension theorem, and the fact that $f \mapsto \hat{f}$ is an embedding by Gelfand theory, there is a Borel measure μ on $\hat{G}$ such that

$$\begin{aligned}T_\varphi f &= \int_{\hat{G}} \hat{f}(-\gamma)\mathrm{d}\mu(\gamma) \\ &= \int_G f(x)\int_{\hat{G}}\gamma(x)\mathrm{d}\mu(x)\mathrm{d}x\end{aligned}$$

So $\varphi(x) = \int_{\hat{G}}\gamma(x)\mathrm{d}\mu(x)$. Now, $1 = \varphi(0) = \int \mathrm{d}\mu = \mu(\Gamma) \le \|\mu\| \le 1$, so μ is nonnegative.

3.2 Inversion formula

We want this theorem on uniqueness.

(3.10) Uniqueness *If $\mu \in \mathcal{M}(G)$ and $\hat{\mu} = 0$, then $\mu = 0$.*

(3.11) Dual Uniqueness *Suppose $\nu \in \mathcal{M}(\hat{G})$, and $\int_{\hat{G}}\gamma(x)\mathrm{d}\nu(\gamma) = 0$ for all $x \in G$, then $\nu = 0$.*

Proof. Let $f \in L^1(G)$.

$$\begin{aligned}0 &= \iint_{G\times\hat{G}} f(x)\overline{\gamma(x)}\mathrm{d}\nu(\gamma)\mathrm{d}x \\ &= \int_{\hat{G}} \hat{f}(\gamma)\mathrm{d}\nu(\gamma).\end{aligned}$$

But we know that $\widehat{L^1(G)}$ is dense in $C_0(\hat{G})$.

The set of $\{\hat{\mu} : \mu \in \mathcal{M}(G)\}$ is translation invariant, and is stable under multiplication by $\gamma(x)$.

(3.12) !! Notation— Denote

$$\mathcal{B}(G) = \left\{ f : \begin{array}{l} f(x) = \int_{\hat{G}}\gamma(x)\mathrm{d}\mu(\gamma) \text{ for some} \\ \text{finite regular Borel measure.}\end{array}\right\}$$

By uniqueness (3.11) above, we can write $f = \int_{\hat{G}}\gamma(x)\mathrm{d}\mu_f(\gamma)$.

(3.13) Inversion Formula *If $f \in L^1(G)\cap\mathcal{B}(G)$, then*

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)\gamma(x)\mathrm{d}\gamma,$$

for some Haar measure over $\hat{G}$.

Proof. Let $f \in L^1(G)\cap\mathcal{B}(G)$. For $h \in L^1$,

$$\begin{aligned}h*f(0) &= \int_G h(-x)f(x)\mathrm{d}x \\ &= \int_G\int_{\hat{G}} h(-x)\gamma(x)\mathrm{d}\mu_f(\gamma)\mathrm{d}x \\ &= \int_{\hat{G}} \hat{h}(\gamma)\mathrm{d}\mu_f(\gamma).\end{aligned}$$

If $g \in L^1(G)\cap\mathcal{B}(G)$, then

$$\begin{aligned}\int_{\hat{G}} \hat{h}\cdot\hat{g}\mathrm{d}\mu_f &= \textstyle\int_{\hat{G}} \widehat{h*g}\mathrm{d}\mu_f \\ &= h*g*f(0) = \int_{\hat{G}} \hat{h}\cdot\hat{f}\mathrm{d}\mu_g.\end{aligned}$$

As a result, $\hat{g}\mathrm{d}\mu_f = \hat{f}\mathrm{d}\mu_g$. So $\left\{\frac{\mathrm{d}\mu_g}{\hat{g}} : g \in L^1(G)\cap\mathcal{B}(G)\right\}$ glues up a global nonzero measure on $\hat{G}$. But

$$\int_{\hat{G}} f(\gamma+\tau)\frac{\mathrm{d}\mu_g(\gamma)}{\hat{g}(\gamma)} = \int_{\hat{G}} f(\gamma+\tau)\frac{\mathrm{d}\mu_g(\gamma+\tau)}{\hat{g}(\gamma+\tau)}$$

so $d\mu_g/g$ is $\hat{G}$-invariant, so there is some constant c such that

$$c \cdot \hat{g} d\gamma = d\mu_g$$

where $d\gamma$ is the Haar measure of $\hat{G}$. Since $f(x) = \int_{\hat{G}} \gamma(x) d\mu(\gamma)$, by a normalization, we get what we need.

(3.14) !! Notation— Recall what we defined in (2.16). Let $C \subseteq \hat{G}$ be a compact set, $\epsilon > 0$, denote

$$V_{C,\epsilon} = \{x \in G : \forall \gamma \in C, |\gamma(x) - 1| < \epsilon\}.$$

(3.15) Proposition *The set $\{V_{C,\epsilon} : C$ compact, $\epsilon > 0\}$ forms a basis of nbd of unity of G.*

PROOF. These are all open subsets by tube lemma.

Let V be a nbd of 0 in G. Let W be a compact nbd of 0 with $W - W \subseteq V$. Consider $f = \frac{\mathbb{1}_W}{|W|^{1/2}}$ and $g = f * \tilde{f}$. Then g is positive definite and $\hat{g} = |\hat{f}|^2 \geq 0$. ($g$ is so-called Fejér kernel). By Bochner theorem (3.7) and inversion formula (3.13),

$$g(x) = \int_{\hat{G}} \hat{g}(\gamma)\gamma(x) d\gamma.$$

We have the following.

- $\int \hat{g}(\gamma) d\gamma = g(0) = 1$.
- There exists a compact subset $C \subseteq \hat{G}$, such that

$$\int_C \hat{g}(\gamma) d\gamma > \frac{2}{3}$$

Assume x is such that $|1 - \gamma(x)| < 1/3$. Now

$$\begin{aligned} |g(x)| &= \left| \left(\int_C + \int_{\hat{G} \setminus C} \right) \hat{g}(\gamma)\gamma(x) d\gamma \right| \\ &= \left| \int_C \hat{g}(\gamma)\gamma(x) d\gamma + \int_{\hat{G} \setminus C} \hat{g}(\gamma)\gamma(x) d\gamma \right| \\ &\geq \left| \int_C \hat{g}(\gamma)\gamma(x) d\gamma \right| - \left| \int_{\hat{G} \setminus C} \hat{g}(\gamma)\gamma(x) d\gamma \right| \\ &\geq \tfrac{2}{3} \cdot \tfrac{2}{3} - \tfrac{1}{3} > \tfrac{1}{9}. \end{aligned}$$

So $x \in V$. We proved that $V_{C,1/3} \subseteq V$.

3.3 Plancherel theorem

(3.16) Plancherel theorem *The Fourier transform maps $L^2(G) \cap L^1(G)$ isometrically into a dense subset of $L^2(\hat{G})$.*

(3.17) Corollary *The Fourier transform uniquely extended to a unitary operator from $L^2(G)$ to $L^2(G)$.*

PROOF OF (3.16). Let $f \in L^2(G) \cap L^1(G)$. Consider $g = f * \tilde{f}$. Then g is positive definite and $\hat{g} = |\hat{f}|^2 \geq 0$. So by Bochner theorem (3.7) and inversion formula (3.13),

$$\int_G f(x)\tilde{f}(x-u) du = g(x) = \int_{\hat{G}} \gamma(x)|\hat{f}(\gamma)|^2 d\gamma.$$

Apply $x = 0$, we get $\hat{f} \in L^2(\hat{G})$, and $\|\hat{f}\|_2 = \|f\|_2$.

So it remains to prove the image is dense. Note that the image is stable under translation and by a multiplication of a character. If $\psi \in L^2(\hat{G})$ such that

$$\int_{\hat{G}} \varphi(\gamma)\overline{\psi}(\gamma) d\gamma = 0$$

for any φ lying in the image. Replace φ by $\varphi(\gamma)\gamma(x)$. By (3.11), $\varphi \cdot \overline{\psi} = 0$, so by a translation of any $\varphi \neq 0$, $\psi = 0$.

(3.18) Corollary *The image of Fourier transform*

$$\widehat{L^1(G)} = \{f_1 * f_2 : f_1, f_2 \in L^2(\hat{G})\}.$$

PROOF. Note that, for $f, g \in L^2(G)$

$$\int f(x)g(x) dx = \int \hat{f}(\gamma)\hat{g}(-\gamma) d\gamma.$$

Replace g by $\overline{\gamma_0(x)}$,

$$\begin{aligned} \widehat{fg}(\gamma_0) &= \int f(x)g(x)\overline{\gamma_0}(x) dx \\ &= \int \hat{f}(\gamma)\hat{g}(\gamma_0 - \gamma) d\gamma = \hat{f} * \hat{g}(\gamma_0). \end{aligned}$$

Note that $h \in L^1(G)$ if and only if $h = fg$ with $f, g \in L^2(G)$.

▶ **(3.19) EXERCISE.** If μ is continuous, show that $\mu * \nu$ is continuous. Hint: Assume $d\mu = f dx$, then $d\lambda = \int_G f(x-y) d\nu(y) dx$. Now, $\int_G f(x-y) d\nu(y)$ is certainly continuous.

▶ **(3.20) PROBLEM.** Find an elementary proof of Bochner theorem (3.7) when $G = \mathbb{Z}/n\mathbb{Z}$.

▶ **(3.21) EXERCISE.** If $\varphi \in L^1(\hat{G})$ with $f = \check{\varphi}(x) = \int_\Gamma \varphi(x)\gamma(x) d\gamma \in L^1(G)$, show that $\hat{f} = \varphi$.

▶ **(3.22) PROBLEM.** If E is a nonempty open subset of $\hat{G}$, show that there exists $\hat{f} \in L^1(\hat{G})$ for some $f \in L^1(G)$, such that $\hat{f} \neq 0$ but $\hat{f} = 0$ outside E. Hint: Take some compact subset K of E, and some compact nbd V such that $K + V \subseteq E$, then $\chi_K * \chi_V$ serves.

4 Structure of LCA Groups

4.1 Pontryagin Duality

(4.1) !! Notation— Let G be a LCA group, Γ its dual, and $\hat{\Gamma}$ the dual of Γ. Denote

$$\alpha : G \longrightarrow \hat{\Gamma} \qquad x \longmapsto [\gamma \mapsto \gamma(x)].$$

(4.2) Pontryagin *The map α is a topological group isomorphism.*

PROOF. In view of (2.16) and (3.15), α is an embedding (note that Γ separate points of G and $\hat{\Gamma}$).

By (3.22), if $\alpha(G)$ is not dense, then one can find an $f \in L^1(G)$ such that $\hat{f}(\alpha(G)) = 0$ but $\hat{f} \neq 0$. Then for $\mathtt{x} \in \hat{\Gamma}$,

$$\widehat{\hat{f}}(\mathtt{x}) = \int_\Gamma \hat{f}(\gamma)\mathtt{x}(\gamma)\mathrm{d}\gamma.$$

In particular, when $\mathtt{x} = \alpha(x)$, it shows $\int_\Gamma \hat{f}\gamma(x)\mathrm{d}\gamma = 0$, then $\hat{f} = 0$ by dual uniqueness (3.11).

So it rests to prove that $\hat{\Gamma} = \alpha(G)$. For any $\mathtt{x} \in \hat{G}$, pick a basis of compact nbd of $\hat{\Gamma}$, say $\mathcal{V}$. For any $V \in \mathcal{V}$, pick some $x_v \in (\mathtt{x} + V) \cap \alpha(G)$. Now $\{x_V : V \in \mathcal{V}\}$ forms a Cauchy net in $\alpha(G)$,

> for any nbd V of 0 in $\alpha(G)$, there exists $U \in \mathcal{V}$, such that for any $W, WW \in \mathcal{V}$ with $W, WW \subseteq U$, we have $x_W - x_{WW} \in V$.

More precisely, $U \in \mathcal{V}$ such that $(U-U)\cap\alpha(G) \subseteq V$. But $\alpha(G)$ is locally compact, so $x_V \to x_0 \in \alpha(G)$. So $x_0 \in \bigcap_{V\in\mathcal{V}}(\mathtt{x} + V) \cap \alpha(G)$, i.e. $x_0 = \mathtt{x}$.

(4.3) Corollary *If G is not discrete, then $L^1(G)$ has no unity.*

(4.4) Corollary *If $\hat{G}$ is not compact, then $1 \notin \widehat{L^1(G)} \subseteq C_0(\Gamma)$.*

(4.5) Corollary *If μ is a finite regular Borel measure over G, with $\hat{\mu} \in L^1(\hat{G})$, then μ is absolutely continuous with density $f(x) = \int_{\hat{G}} \gamma(x)\hat{\mu}(\gamma)\mathrm{d}\gamma$.*

Summary

Let G be LCA group, and Γ be its dual group. We denote $\mathcal{M}(X)$ the space of finite Borel measures over space X. For $f \in L^1(G)$ and $m \in \mathcal{M}(G)$, denote the Fourier transform

$$\mathcal{F}f = \hat{f}(\gamma) = \textstyle\int_G f(x)\overline{\gamma(x)}\mathrm{d}x,$$
$$\mathcal{F}m = \hat{m}(\gamma) = \textstyle\int_G \overline{\gamma(x)}\mathrm{d}m(x).$$

For $\varphi \in L^1(\Gamma)$ and $\mu \in \mathcal{M}(\Gamma)$, denote the inverse Fourier transform

$$\overline{\mathcal{F}}\varphi = \check{\varphi}(x) = \textstyle\int_\Gamma \varphi(\gamma)\gamma(x)\mathrm{d}\gamma,$$
$$\overline{\mathcal{F}}\mu = \check{\mu}(x) = \textstyle\int_\Gamma \gamma(x)\mathrm{d}\mu(\gamma).$$

If $m \in \mathcal{M}(G)$, with $\mathcal{F}m = \hat{m} =\in L^1(\Gamma)$, then

$$\mathrm{d}m = \overline{\mathcal{F}}(\mathcal{F}m)\mathrm{d}g.$$

If $\mu \in \mathcal{M}(\Gamma)$, with $\mathcal{F}\mu \in L^1(G)$, then

$$\mathrm{d}\mu = \mathcal{F}(\overline{\mathcal{F}}\mu)\mathrm{d}\gamma.$$

In particular, for $f \in L^1(G)$ and $\varphi \in L^1(\Gamma)$, then

$$\varphi = \hat{f} \iff f = \check{\varphi}.$$

The classic Fourier invention formula.

(4.6) Theorem *If H is a closed subgroup of G, denote*

$$\Lambda = H^\perp = \{\gamma \in \Gamma : \forall x \in H, \gamma(x) = 1\}.$$

then Γ is the dual group of G/H, and Γ/Λ is the dual group of H.

PROOF. Since we have proved the dual theorem (4.2), so it suffices to show the first assertion. Algebraically, there is no problem. To show they are homeomorphism, look at (2.16) and (3.15).

4.2 Structure theorem

(4.7) Structure Theorem *Let G be a LCA group, then there exists $n \in \mathbb{Z}_{\geq 0}$, and G contains an open subgroup of the form $H \oplus \mathbb{R}^n$, where H is a compact group.*

(4.8) Remark This means the compactness is due to the discrete G/G_0, and locally $\mathbb{R}^n$, where G_0 is the connected component of $0 \in G$.

(4.9) Λemma *Suppose that there is a homomorphism $\mathbb{Z} \xrightarrow{\rho} G$ so that $\rho(\mathbb{Z})$ is dense. Then G is compact or $G = \mathbb{Z}$.*

Proof. Suppose G is discrete, then G is quotient group of $\mathbb{Z}$ there is nothing to prove. So assume G is not discrete. Denote $x_k = \rho(k)$. Pick V be a symmetric nbd of 0 in G, with $\overline{V}$ compact. Now any nbd of 0 meets infinite many x_k, so in particular,

$$G = \bigcup_{i=1}^{\infty}(x_i + V).$$

More precisely, if $y \in G$, then $y \in x_k + W$, with W symmetric and $W + W \subseteq V$, say $y - x_k \in W$, but some $|m| \gg 0$, $x_{\pm m} \in W$, so $y - x_k + x_m \in W + W \subseteq V$.

Assume $\bigcup_{i=1}^{N}(x_i + V)$ covers $\overline{V}$. For any y, find the smallest $n > 0$, such that $y \in x_n + \overline{V}$. Then $y - x_n \in \bigcup_{i=0}^{N}(x_i + V)$, if $n > N$, then we will get a smaller n, a contradiction. So $G = \bigcup_{i=1}^{n}(x_i + \overline{V})$.

(4.10) Λemma *Suppose G is generated by a compact nbd V of 0. Then there is a closed subgroup H of G isomorphic to $\mathbb{Z}^n$ such that G/H is compact and $V \cap H = 0$.*

Proof. We can firstly assume $V = -V$, define $V_n = V + \overset{n}{\cdots} + V$. Then by assumption, $G = \bigcup V_n$. Assume $V + V \subseteq \bigcup_{i=1}^{p}(x_i + V)$. Let H be the subgroup generated by $x_1, \ldots, x_p$. Then $V + H = G$, since

$$\begin{aligned} V_{n+1} &= V + V_n \subseteq V + (V + H) \\ &= (V + V) + H \subseteq (V + H) + H \\ &= V + H. \end{aligned}$$

by induction. Let $H_i = \overline{\mathbb{Z}x_i}$. If all H_i are compact, then G itself is compact, so $n = 0$ serves. Suppose some i such that $\overline{H}_i$ is not compact, then $\overline{H}_i = H_i \cong \mathbb{Z}$ by lemma above (4.9). Then take the subgroup H' of H isomorphic to $\mathbb{Z}^r$ of maximal rank r. Then $H' \cap V$ is finite, so we can replace H' by a finite index subgroup so that $H' \cap V = 0$. Now, the image of H under $G \to G/H'$ is compact by our choice. So this is desired subgroup.

(4.11) !! Locally isomorphic— We say two topological group is locally isomorphic, if there exists nbd V and W respectively, and homeomorphism $V \xrightarrow{f} W$ with $f(x + y) = f(x) + f(y)$ whenever $x, y, x + y \in V$.

(4.12) Λemma *Let G be connected. Assume G is locally isomorphic to $\mathbb{R}^n$, if G does not contain the infinite compact subgroup, then $G = \mathbb{R}^n$.*

Proof. One can extend to a group homomorphism $\mathbb{R}^n \xrightarrow{\varphi} G$, by set $\varphi(x) = n\varphi(x/n)$. Then φ is injective (by the assumption that G does not contain the infinite compact subgroups), and open, so $\varphi(\mathbb{R}^n) = G$.

We may use some knowledge of totally disconnected spaces/groups, which is outlined in exercises, see (4.18).

Proof of (4.7). Firstly, let us deal with the components. Let G_0 be component of 0 in G. Then G_0 is closed, and G/G_0 is totally disconnected.

So by (4.18), there exists an open compact subgroup K of G/G_0. Denote $G_1 = \pi^{-1}(K)$ where $\pi : G \to G/G_0$. Then G_1 does not have open subgroup of infinite index, i.e. G_1 does not have any infinite discrete quotient groups.

Since π is open, there exists a compact nbd V of 0 such that $\varphi(V) = K$. Then V generates G_1, since it is open, and intersects all cosets of G_0. The by (4.10), G_1 contains a subgroup $H \cong \mathbb{Z}^n$, with G_1/H compact, and $H \cap V = 0$.

The consider the dual group, $H^\perp = \widehat{G_1/H}$ is discrete, $\widehat{H}$ is a torus $\mathbb{T}^n$. So by (4.6), $\Gamma_1 = \hat{G_1}$ is locally isomorphic to $\mathbb{R}^n$. Let Γ_0 the component of 0 in Γ_1, and Γ_0 has no infinite compact subgroups. So $\Gamma_0 = \mathbb{R}^n$. In conclusion, Γ_1 contains $\mathbb{R}^n$ as an open subgroup.

Now $\Lambda = \Gamma_1/\Gamma_0$ is discrete, since $H^\perp + \Gamma_0 = \Gamma_1$. Since $\mathbb{R}^n$ is injective module, there is section

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \Gamma_1 \rightleftarrows \Lambda \longrightarrow 0$$

since Λ is discrete, so it is continuous, thus, $\Gamma_1 = \mathbb{R}^n \oplus \Lambda$.

Then $G_1 = \mathbb{R}^n \oplus \hat{\Lambda}$ with $\hat{\Lambda}$ compact. The proof is complete.

(4.13) EXAMPLE Consider

$$G = \left\{ \{\zeta_n\} \in (\mathbb{Z}/4)^{\mathbb{Z}} : \begin{array}{l} \text{only finitely many } \zeta_n\text{'s are} \\ \text{equal to 1 or 3.} \end{array} \right\}$$

Then $K = \{x \in G : 2x = 0\} = \{0, 2\}^{\mathbb{Z}}$ is compact and open. We do not in general have $G \cong G_1 \oplus G/G_1$. In this example, if so, G/G_1 intersects K by a infinite sets, contradicts to the fact K is compact. (Since G/G_1 cannot be finite, since G is not.)

(4.14) EXAMPLE Let $\{G_\alpha : \alpha \in A\}$ be a family of topological groups. Define its **direct sum**

$$\sum_{\alpha \in A} G_\alpha = \left\{ (x_\alpha) \in \prod_{\alpha \in A} G_\alpha : \begin{array}{l} \text{only finitely many} \\ x_\alpha \neq 0. \end{array} \right\}$$

If all G_α's are compact, then $\prod_{\alpha \in A} G_\alpha$ is locally compact and $\widehat{\prod_\alpha G_\alpha} = \sum_\alpha \widehat{G_\alpha}$.

▶ **(4.15) Exercise.** For locally compact group G, and a family $\mathfrak{F}$ of open compact subsets, show that

$$\mathfrak{F} \text{ forms a basis of } \mathsf{nbd} \iff \bigcap_{F \in \mathfrak{F}} F = \{1\}.$$

Hint: For any open nbd V of 1, there is some $F \in \mathfrak{F}$ such that $(G \setminus V) \cap F = \varnothing$, since the intersection serves. Then $F \subseteq V$.

▶ **(4.16) Exercise.** For a compact space X, show that the component of $x \in X$ is the intersection of all open compact subsets containing x. Hint: It suffices to show the intersection C is connected. Write it as disjoint union of two compact subsets $C = X \sqcup Y$, then use two open subsets to separate them, say $X \subseteq U, Y \subseteq V$. By compactness, some C_0 open compact subsets contains x such that $C_0 = (U \cap C_0) \sqcup (V \cap C_0)$ (since C does). Now $U \cap C_0$ and $V \cap C_0$ are both compact open, one of them contains x. so $X = \varnothing$ or $Y = \varnothing$.

▶ **(4.17) Problem.** Show that, for a local compact totally disconnected space X, the open compact subsets containing x forms a nbd basis of x. Hint: It is right for compact case. Generally, consider $x \in V \subseteq \overline{V}$, find some clopen $U \subseteq V$, since they form nbd basis.

▶ **(4.18) Exercise.** For a totally disconnected locally LC group G. Any open compact subspace E, show that there is an open compact group H such that $H \cdot E = E$, so that E is a union of cosets of H. In particular, the family of open compact subgroup forms a basis of nbd of 1. When G is compact, then the family of open normal subgroup forms a basis of nbd of 1. Hint: Let $x \in E$, and take some V_x such that $V_x V_x x \subseteq E$, then finite many $V_x x$ covers E. Let $W = \bigcap V_x$. So $WE = E$. Denote H be the group generated by W, it is open with $HE \subseteq E$. Now E is a union of cosets of H, and therefore H is also closed. In the compact case, the index is finite, thus $\bigcap_{xH} xHx^{-1} \subseteq H$ serves.

▶ **(4.19) Problem** *(Pro-finite groups)*. If a topological group G is compact and totally disconnected, show that $G = \varprojlim_i G_i$ with each G_i finite. Hint: With $G_i = G/N$ and N goes through all open normal subgroups. Then $G \to \varprojlim_i G_i$ is injective since such N forms a basis, and is surjective by net-completeness.

▶ **(4.20) Problem.** For topological group G and compact subgroup H, show that G/H is (locally) compact if and only if G is (locally) compact. Hint: Firstly, show the analogy of tube lemma. For any open cover $\mathcal{U}$ of H, we can find a open nbd V with $V = \pi^{-1}(\pi(V))$, and a finite subcover cover V. Assume $\mathcal{U} = \{U_i x_i : x_i \in H\}$. Let $\mathcal{V} = \{V_i x_i : V_i \cdot V_i \subseteq U_i\}$ a refine of $\mathcal{U}$, and $\mathcal{V}_0$ be the finite subcover of H, take

$$V = \bigcap_{V' \in \mathcal{V}_0} \pi^{-1}(\pi(V')) = \bigcap_{V' \in \mathcal{V}_0} V' \cdot H,$$

Then any $x \in V$, so $x = y_i \cdot h_i$ for $x_i \in V_i$ and $h_i \in H$. So $y_i^{-1} x \in H$ for any i, assume $y_i^{-1} x \in V_j x_j$ for some j, then take $i = j$ makes $x \in V_j V_j x_j \subseteq U_j x_j$.

▶ **(4.21) Problem** *("Fubini")*. Let H be a closed subgroup of a LC group G. If we take a left Haar measure ν on H, and assume there exists a left G-invariant measure υ on G/H. Show that

$$\mu(E) = \int_{G/H} \nu(x^{-1} E \cap H) \mathrm{d}\upsilon(xH)$$

is nonzero and left invariant, so a scaler of left Haar measure over G.

▶ **(4.22) Problem** *(Poisson summation formula)*. Let H be a closed subgroup of G. Suppose that $f \in L^1(G)$, such that

(1) for each x, $[h \mapsto f(x+h)] \in L^1(H)$;
(2) $[x + H \mapsto \int_H f(x+h) \mathrm{d}h] \in L^1(G/H)$;
(3) $[\gamma \mapsto \hat{f}(\gamma)] \in L^1(H^\perp)$.

show that

$$\int_H f(h) \mathrm{d}h = \int_{H^\perp} \hat{f}(\lambda) \mathrm{d}\lambda$$

with $\mathrm{d}h$ and $\mathrm{d}\lambda$ some Haar measure on H and $H^\perp$.

5 Compact Groups

5.1 Representations

(5.1) Definition Let H be a Hilbert space, a continuous group homomorphism from G to $U(H)$ the unitary operators over H, is called a **unitary representation** (and will be called directly a **representation** (rep)) of G .

(5.2) Definition (Irreducible representation) Let H be a rep of G, if there is no G-invariant closed subspace, then H is called an **irreducible rep**.

(5.3) Theorem *For a compact group G, any rep is direct sum of irreducible rep s, and each irreducible rep is finite dimensional.*

Infinite sum

Let $\{H_i : i \in I\}$ be a family of Hilbert spaces. Denote

$$\widehat{\bigoplus_{i\in I}} H_i = \left\{ (x_i) \in \prod_{i\in I} H_i : \sum_{i\in I} \|x_i\|^2 < \infty \right\},$$

with inner product

$$\langle (x_i), (y_i) \rangle = \sum_{iI} \langle x_i, y_i \rangle .$$

It is a Hilbert space.

If $\{H_i\}$ is a family of pairwise orthogonal closed subspaces in some big Hilbert space H. Then the map

$$\widehat{\bigoplus_{i\in I}} H_i \longrightarrow \overline{\sum_{i\in I} H_i} \qquad (x_i) \longmapsto \sum x_i$$

is continuous bijection, so it is an isomorphism (by open map theorem).

(5.4) Lemma *For a compact group G, any repH contains a finite dimensional rep.*

PROOF. Let V be any nonzero finite dimensional subspace of H. Let p be the projection on V. Then

$$P : H \longrightarrow H \qquad x \longmapsto \int_G g \cdot p(g^{-1} \cdot x) \mathrm{d}g$$

is a nonzero bounded operator over H. It is also unitary and compact, since it is a limit of finite dimensional unitary operator.

Then, by spectral theorem, there is some eigenvalue λ such that $H_\lambda = \{v \in H : Pv = \lambda v\}$ is nonzero and finite dimensional. Now

$$Pv = \lambda v \Rightarrow Pgv = \lambda gv$$

since $gPg^{-1} = P$. So H_λ is a finite dimensional G-invariant subspace.

(5.5) Lemma *For a compact group G, any rep H and G-invariant closed subspace has its orthogonal complement invariant.*

PROOF. Since g acts as unitary operator, $\langle gv, w \rangle = \langle v, gw \rangle$.

PROOF OF (5.3). Let V be a rep. Pick the maximal element among by Zorn's Lemma (it is nonempty by above (5.4)).

$$\left\{ (U_i) : \begin{matrix} U_i\text{'s are finite dimensional pairwise} \\ \text{orthogonal, } G\text{-invariant subspaces.} \end{matrix} \right\}$$

Assume the maximal element is (U_i), then $\overline{\sum} U_i$ has complement zero, by maximality, (5.4) and (5.5).

(5.6) Remark For a finite group G, the assertion of direct sum in (5.3) can be algebraic. Where one need to check any short exact sequence of G-rep splits. Say

$$0 \to V \to U \to W \to 0$$

Consider any projection from U on V, say p, then consider

$$P : H \longrightarrow H \qquad x \longmapsto \frac{1}{|G|} \int_G g \cdot p(g^{-1} \cdot x) \mathrm{d}g$$

it still maps U on V, and invariant on V, so $P^2 = 1$. Then $U = V \oplus \ker P$, so the short exact sequence splits.

5.2 Characters

(5.7) Definition (Character) Let V be a finite dimensional G-rep. We define its **character**

$$\chi_V : G \longrightarrow \mathbb{C} \qquad g \longmapsto \operatorname{tr} g$$

(5.8) Dixmier's generalization of Schur Lemma *If V is an irreducible G-rep of at most countable dimension, then any G-endormorphism of V is the scalar product of $\mathbb{C}$.*

PROOF. Let A be such an endormorphism. Then $Ag = gA$ for any $g \in G$. The image of A is G-invariant, so must be 0 or V. The same reason, the kernel of A must be 0 or V. So if A is not a scalar, then it is invertible.

Then so is all $A - \lambda 1$ for $\lambda \in \mathbb{C}$. Let $\frac{1}{A-\lambda 1}$ be its inverse, then $\left\{\frac{1}{A-\lambda 1} : \lambda \in \mathbb{C}\right\}$ is linearly independent, otherwise A will be algebraic over $\mathbb{C}$ thus a scalar.

But V is generated by any nonzero element, so $\{A : gA = Ag\}$ has dimension at most V, which is assumed to be at most countable. A contradiction.

(5.9) Corollary *For two G-irreducible rep s V, W, then any G-homomorphism between V and W is*

$$\begin{cases} \text{an isomorphism or zero} & V \cong W \\ \text{zero} & \text{otherwise}. \end{cases}$$

(5.10) Theorem *For two finite irreducible rep s U, V,*

$$\frac{1}{|G|}\int_G \chi_U(g)\overline{\chi_V(g)}\mathrm{d}g = \begin{cases} 1, & U \cong V, \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. Pick two basis for U and V, and let A be a linear map between them, Consider

$$\hat{A} : U \longrightarrow V \qquad x \longmapsto \frac{1}{|G|}\int_G g \cdot A(g^{-1} \cdot x)\mathrm{d}g$$

which is a G-homomorphism.

Pick basis for U and V, say $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$. Then

$$\begin{aligned} &\frac{1}{|G|}\int_G \chi_U(g)\overline{\chi_V(g)}\mathrm{d}g \\ &= \frac{1}{|G|}\int_G \sum_{j=1}^m \sum_{i=1}^n \langle gu_j, u_j\rangle\, \overline{\langle gv_i, v_i\rangle}\mathrm{d}g \\ &= \sum_{j=1}^m \sum_{i=1}^n \left\langle v_i, \frac{1}{|G|}\int_G \langle u_j, g^{-1}u_j\rangle\, g \cdot v_i \right\rangle \mathrm{d}g \\ &= \sum_{j=1}^m \sum_{i=1}^n \left\langle v_i, \hat{A}^i_j(u_j)\right\rangle \end{aligned}$$

where $A^i_j = \langle u_j, -\rangle\, v_i$.

If $U \cong V$, or WLOG assume $V = U$ and take $u_i = v_i$. By Schur's lemma (5.8) above,

$$\hat{A} \cdot \dim U = \operatorname{tr} \hat{A} \cdot 1.$$

But

$$\begin{aligned} \operatorname{tr} \hat{A} &= \frac{1}{|G|}\int_G \sum_{i=1}^n \langle A(g^{-1} \cdot e_i), g^{-1}e_i\rangle\, \mathrm{d}g \\ &= \frac{1}{|G|}\int_G \operatorname{tr} A\mathrm{d}g = \operatorname{tr} A. \end{aligned}$$

So

$$\frac{1}{|G|}\int_G \chi_U(g)\overline{\chi_V(g)}\mathrm{d}g = \sum_{i,j=1}^n \frac{1}{|G|}\int_G \frac{\delta_{ij}}{\dim U} = 1.$$

If U is not isomorphic to V, then by Schur's lemma (5.9) above, always $\hat{A} = 0$. So

$$\frac{1}{|G|}\int_G \chi_U(g)\overline{\chi_V(g)}\mathrm{d}g = 0.$$

The proof is complete.

(5.11) EXAMPLE Consider any LCA group G, and any finite dimensional G-rep. Note that, a linear algebra exercise shows the elements of G share a common eigenvector. As a result, the irreducible finite-dimensional rep s of LCA group are all one dimensional with the character of them coincides the character defined for LCA.

(5.12) EXAMPLE For a compact group G, and a finite dimensional space V where G linearly acts on it. Then we can introduce an inner product by

$$\langle x, y\rangle = \int_G (gx, gy)\mathrm{d}g$$

with $(\cdot,\cdot)$ any inner product. Now g is unitary with respect to $\langle\cdot,\cdot\rangle$.

(5.13) EXAMPLE For any LC group G, G acts naturally on its function space, for example $L^2(G)$ and $C(G)$. More exactly, it has two actions

$$sf : t \mapsto f^s(t) = f(ts), \qquad sf : t \mapsto f(s^{-1}t).$$

(5.14) Definition (Class function) If a map $\varphi : G \to \mathbb{C}$ satisfies $\varphi(st) = \varphi(ts)$ for any $s, t \in G$, we will call φ is a **class function**.

(5.15) EXAMPLE Consider the special unitary group

$$\mathsf{SU}(2)=\{\left(\begin{smallmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{smallmatrix}\right):\alpha,\beta\in\mathbb{C},|\alpha|^2+|\beta|^2=1\}.$$

It acts on polynomial by

$$\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)f(z,w)=f\left((z,w)\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\right)=f(\alpha z+\gamma w,\beta z+\delta w).$$

Consider the space of two-variable polynomial of total degree n, say V_n.

Firstly, V_n is irreducible for all $n\geq 0$. This follows by Schur lemma — due to (5.3), it suffices to show $\operatorname{End}(V_n)$ is only scalars. Pick $\varphi\in\operatorname{End}(V_n)$. Note that $\left(\begin{smallmatrix}a & \\ & 1/a\end{smallmatrix}\right)\in\mathsf{SU}(2)$, and $\left(\begin{smallmatrix}a & \\ & 1/a\end{smallmatrix}\right)z^kw^{n-k}=a^{n-2k}\cdot z^kw^{n-k}$. By an argument of homogenous, $z^kw^{n-k}\mapsto c_kz^kw^{n-k}$ for some $c_k\in\mathbb{C}$. Then note that $\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)\in\mathsf{SU}(2)$. Then by a direct computation, $c_k=c_n$. So φ is a scalar, thus V_n is irreducible. They are non-isomorphic by dimension reason.

Then, we are going to compute the characters. Consider $T=\{\left(\begin{smallmatrix}e^{it} & \\ & e^{-it}\end{smallmatrix}\right):t\in\mathbb{R}\}\subseteq\mathsf{SU}(2)$. Consider the map

$$\begin{array}{ccc} G\times T & \longrightarrow & G/\mathrm{conj}\\ \downarrow & \searrow^{(g,t)\mapsto gtg^{-1}} & \uparrow\\ G/T\times T & \longrightarrow & G\end{array}$$

This is surjective by linear algebra. Over G/conj, there is two measures one from the Haar measure, one from $G\times T$. If the union of some conjugation classes E of zero measure, then $\{(g,t):xtx^{-1}\in E\}=\{(x,t):t\in E\}=|E\cap T|$ must has zero measure. Since $|E\cap xTx^{-1}|=|E\cap T|$, so

$$|E\cap T|_T=\int_{G/T}|E\cap xTx^{-1}|\mathrm{d}x=\int_{G/T}\int_T E\mathrm{d}t\mathrm{d}x=|E|_G.$$

So we have some density $g(t)$ with $t\in T$ such that

$$\int_G\phi(x)\mathrm{d}x=\int_T\phi(t)g(t)\mathrm{d}t$$

for class function ϕ.

Note that for $t=\left(\begin{smallmatrix}e^{it} & \\ & e^{-it}\end{smallmatrix}\right)$, its character χ_n for V_n is

$$\chi_n(t)=\sum_{k=0}^{n}e^{i(2k-n)t}\overset{t\neq\pi}{=}\frac{\sin(n+1)t}{\sin t}.$$

Since $\int_G\chi_n\overline{\chi_m}=\int_T\chi_n\overline{\chi_m}g=\delta_{mn}$ and $\chi_n(t)$ is dense in continuous even period functions. So $g=\frac{2}{\pi}\sin^2 t$ is the only choice.

As a corollary, $\{V_n\}$ gives the full list of irreducible rep s of $\mathsf{SU}(2)$. Since

$$\frac{2}{\pi}\int_T\chi(t)\frac{\sin(n+1)t}{\sin t}\sin^2 t\mathrm{d}t=0$$

and class functions are even period function, so this implies $\chi=0$.

5.3 Representative functions

Assume we have an n-dimensional rep V, then $G\to\operatorname{End}(V)=\mathbb{M}_n(\mathbb{C})$ has $n\times n$ entries.

(5.16) Proposition *For a function $G\xrightarrow{f}\mathbb{C}$, the following conditions are equivalent.*

(1) $\operatorname{span}\{f(\bullet s):s\in G\}$ is of finite dimension.

(2) $\operatorname{span}\{f(s^{-1}\bullet):s\in G\}$ is of finite dimension.

(3) f is linear combination of matrix coefficients of some finite dimensional rep.

(5.17) Definition A function $G\xrightarrow{f}\mathbb{C}$, is called **finite dimensional** or **representative** if $\operatorname{span}\{f^s:s\in G\}$ is of finite dimension.

Claim *Any entry is a finite dimensional function.*

PROOF. Since $a_{ij}^s(\bullet)=a_{ij}(\bullet s)=\sum a_{ik}(\bullet)a_{kj}(s)$ lies in the space spanned by the entries.

Let f be finite dimensional function over G. Denote $V=\operatorname{span}\{f^s:s\in G\}$, then V is finite dimensional G-rep.

Claim *The space $V=\operatorname{span}\{f^s:s\in G\}$ is contained in the space spanned by matrix coefficients $G\to\operatorname{End}(V)$.*

PROOF. For any $f\in V$, we take the functional $\mathrm{eva}_1:f\mapsto f(1)$. So $f(s)=\langle\mathrm{eva}_1,f^s\rangle$ is in the space spanned by matrix coefficients $G\to\operatorname{End}(V)$.

▶ **(5.18) EXERCISE.** Prove the classic Schur lemma that if V is an irreducible finite dimensional G-rep, then any G-endormorphism of V is the scalar product of $\mathbb{C}$. Hint: Pick some eigenvalue of A, say λ, then

$$(A-\lambda 1)g=g(A-\lambda 1),$$

then the kernel of $A-\lambda 1$ must be V by the irreducibility.

6 Fourier Analysis over Compact Groups

6.1 Peter–Weyl theorem

(6.1) Peter–Weyl theorem *The set of continuous finite dimensional function is dense in $C(G)$ and $L^2(G)$.*

Proof for $L^2(G)$. Let N the orthogonal of the set of all finite dimensional function. Denote $\tilde{f}(s) = \overline{f(s^{-1})}$. Note that if $f = \frac{f+\tilde{f}}{2} + i\frac{f-\tilde{f}}{2i}$, so it suffices to show for that $\{g \in N : g = \tilde{g}\} = 0$. Let $f \in L^2(G)$. Consider the compact operator

$$T_g f(s) = \int_G g(su^{-1}) f(u) \mathrm{d}u.$$

Then T_g is a self-adjoint operator by purely algebraic computation

$$\begin{aligned} \langle f_1, T_g f_2 \rangle &= \int f_1(s) \overline{\int g(st^{-1}) f_2(t) \mathrm{d}t} \mathrm{d}s \\ &= \int \overline{f_2(t)} \int \overline{g(st^{-1})} f_1(s) \mathrm{d}s \mathrm{d}t \\ &= \int \overline{f_2(t)} \int g(ts^{-1}) f_1(s) \mathrm{d}s \mathrm{d}t \\ &= \langle T_g f_1, f_2 \rangle . \end{aligned}$$

So by spectral theorem, T_g has at most countably many nonzero eigenvalue $\lambda_j \in \mathbb{R} \setminus 0$ with $\{f : T_g f = \lambda_i f\}$ finite dimensional. Assume $T_g \varphi = \lambda \varphi$, then

$$\begin{aligned} T_g \varphi^t(s) &= \int g(su^{-1}) \varphi(ut) \mathrm{d}u \\ &= \int g(stu^{-1}) \varphi(u) \mathrm{d}u \\ &= \int T_g \varphi(st) = (T_g \varphi)^t(s). \end{aligned}$$

As a result, any $\varphi \in \{f : T_g f = \lambda_i f\}$ is finite dimensional. Hence $g = \tilde{g} \in N \subseteq \ker T_g$, but $T_g \tilde{g}(0) = \int \overline{g(u)} g(u) \mathrm{d}u = \|g\|_2$. We get what we desired $g = 0$.

Proof for $C(G)$. Let $f \in C(G)$. For any $\epsilon > 0$, find an open nbd U of $1 \in G$ such that $st^{-1} \in U \Rightarrow |f(s) - f(t)| > \epsilon$. Take $g \in C_c(U)$ such that $g \geq 0$ and $\|g\| = 1$, furthermore without loss of generality assume that $g(s) = g(s^{-1})$. Then $\|f - T_g f\| \leq \epsilon$, where

$$T_g f(s) = \int_G g(su^{-1}) f(u) \mathrm{d}u.$$

Similar to the process above, T_h is a self-adjoint operator. For compactness, take $\varepsilon > 0$, then there is an open nbd U of $1 \in G$ such that $st^{-1} \in U \Rightarrow |g(s) - g(t)| > \varepsilon$. If $f \in L^2(G)$ with $\|f\| \leq 1$, then for any $st^{-1} \in U$,

$$|T_g f(s) - T_g f(t)| \leq \int_G |g(su^{-1}) - g(tu^{-1})| |f(u)| \mathrm{d}u \leq \varepsilon.$$

So $\overline{\{T_g f : \|f\| = 1\}}$ is compact by Ascoli lemma. Similar to the process above, the eigenspace belonging to any nonzero eigenvalue λ consists of finite dimensional functions. So $T_h f$ can be approximated by them. The proof is complete.

(6.2) Remark The L^2 part is essentially done by (5.3). This is also known as "generalized Peter–Weyl theorem".

(6.3) Theorem *The space generated by characters of irreducible* rep *s forms is dense in the space of continuous class functions.*

Proof. For any continuous class function f, we can find a finite dimensional φ such that $\|f - \varphi\| < \epsilon$. Now $\overline{\varphi} = \int_G f(gxg^{-1}) \mathrm{d}g$ forms a finite dimensional class function with $\|\overline{f} - \varphi\| < \epsilon$.

So it suffices to show any finite dimensional class function f is a linear combination of characters. By (5.16), we can assume f is from some matrix coefficient of some rep. Then we reduce to the case of rep.

Let V be a irreducible rep $E = \bigoplus_\alpha E_\alpha$, and $f(g) = \sum_i \langle w_i, g v_i \rangle$. We can assume that any i, the v_i, w_i comes from one irreducible representation E_i with character φ_i. Then consider

$$\left\langle w_i, \int_G x g x^{-1} v_i \mathrm{d}x \right\rangle .$$

By the proof of (5.10) applying to the linear transform $v \mapsto gv$, the above is exactly $\frac{1}{\dim E_i} \chi_i(g) \langle w_i, v_i \rangle$. Since f is class function, so $f(g) = \int_G \sum_i \langle w_i, x g x^{-1} v_i \rangle \mathrm{d}x$, the proof is complete.

6.2 Fourier transform

(6.4) Definition Let G be a compact group, denote $\hat{G}$ the set of equivalent class of irreducible representation of G. We define

$$\widehat{\mathcal{E}}(\hat{G}) = \widehat{\bigoplus_{V \in \hat{G}}} \operatorname{End}(V).$$

where $\mathrm{End}(V)$ is equipped with the norm

$$\|A\|^2 = \operatorname{trace} AA^*$$

where $\langle Av, w\rangle = \langle v, A^*w\rangle$ for any unitary inner product. Equivalently, if there is a orthogonal basis B, then $\|A\|^2 = \sum_{v\in B}\|Av\|^2$.

(6.5) Definition We define the **Fourier transform**

$$L^2(G) \longrightarrow \widehat{\mathcal{E}}(\hat{G}) \qquad \varphi \longmapsto \hat{\varphi}$$

where

$$\hat{\varphi} = \sum_{V\in\hat{G}}\left[v \mapsto (\dim V)^{1/2}\int_G \varphi(g)\cdot gv\mathrm{d}g\right].$$

We define the **inverse Fourier transform**

$$\begin{array}{ccc} \widehat{\mathcal{E}}(\hat{G}) & \longrightarrow & L^2(G) \\ {[V \xrightarrow{A} V]} & \longmapsto & \left[g \mapsto (\dim V)^{1/2}\operatorname{trace}(g^{-1}A)\right]. \end{array}$$

(6.6) Remark If we use the isomorphism $\operatorname{End} V = V\otimes V^\vee$, then the inverse Fourier transform can be written in the following way

$$\begin{array}{ccc} \widehat{\mathcal{E}}(\hat{G}) & \longrightarrow & L^2(G) \\ V\otimes V^\vee \ni e\otimes f & \longmapsto & \left[g \mapsto (\dim V)^{1/2} f(ge)\right]. \end{array}$$

(6.7) Theorem *The Fourier transform and inverse Fourier transform are well-defined, norm preserving and inverse to each other.*

Proof. If $\varphi(g) = (\dim V)^{1/2} f(ge)$ for some $e\in V$ and $f\in V^\vee$, then

$$\begin{aligned} &(\dim V)^{1/2}\int_G \varphi(g)\cdot gv\mathrm{d}g \\ &= \dim V\int_G f(g^{-1}e)\cdot gv\mathrm{d}g \\ &= \dim V\int_G g\left(f(g^{-1}e)\cdot v\right)\mathrm{d}g \end{aligned}$$

By the proof of (5.10) applying to the linear transform $f(\bullet)\cdot v$, it is $(\operatorname{trace} f(\bullet)v)e = f(v)e$. So $F\circ F'$ is identity, where F the Fourier transform, and F' the inverse Fourier transform.

The image of inverse Fourier transform is exactly the closure of the space generated by matrix coefficients. Hence the map is surjective by Peter–Weyl theorem.

Take a orthogonal basis $B(V)$ for each $V\in\hat{G}$, Let $B = \{\langle v,\cdot\rangle : v\in B(V), V\in\hat{G}\}$ which is also a set of unit orthogonal basis for $\widehat{\mathcal{E}}(\hat{G})$, since

$$\langle\langle v,\cdot\rangle, \langle w,\cdot\rangle\rangle = \sum_{u\in B(V)}\langle v,u\rangle\langle w,u\rangle.$$

Now

$$B' = \{[g\mapsto \dim V^{1/2}\langle gv, v\rangle] : V\in\hat{G}, v\in B(V)\},$$

the inverse Fourier transform of B, is also a set of unit orthogonal basis for $L^2(G)$. Actually,

$$\begin{aligned} \langle\varphi,\psi\rangle &= \dim V\int\langle gv, v\rangle\overline{\langle gw, w\rangle}\mathrm{d}g \\ &= \dim V\int\langle g\langle g^{-1}w, w\rangle v\mathrm{d}g, v\rangle. \end{aligned}$$

By the proof of (5.10) applying to the linear transform $\langle\bullet, w\rangle\cdot v$, it is $\langle\langle v,w\rangle w, v\rangle$.

So $F'f$ is convergent for any $f\in L^2(G)$, and $F'\circ F$ is identity.

(6.8) Theorem *We have the isomorphism of $G\times G$-rep*

$$\widehat{\mathcal{E}}(\hat{G}) = \widehat{\bigoplus_{V\in\hat{G}}} V\otimes V^\vee = \widehat{\bigoplus_{V\in\hat{G}}}\mathrm{End}(V) \cong L^2(G).$$

(6.9) Corollary *The multiplicity of V in $L^2(G)$ is $\dim V$.*

(6.10) Remark We should regard an element of $\widehat{\mathcal{E}}(\hat{G})$ a matrix-value function, where at each point V the fibre of value range is $\mathrm{End}(V)$.

Now, for any $g\in G$, it defines a matrix-value function, say $g(V) = [V\xrightarrow{v\mapsto gv}V]$. For two $A, B\in\mathrm{End}(V)$, we can define the trace form

$$A\circ B = \operatorname{trace}(AB).$$

If we denote the measure μ over $\hat{G}$ with $\{V\}$ of measure $\dim V$.

For $f\in\widehat{\mathcal{E}}(\hat{G})$, we can define

$$f^\wedge(g) = \int_{\hat{G}}\overline{g(V)}\circ f(V)\mathrm{d}\mu(V).$$

Here $\overline{*}$ stands for transposition. For $\varphi\in L^2(G)$, we can define

$$\varphi^\vee(V) = \int_G\varphi(g)\cdot g(V)\mathrm{d}g.$$

These two definitions may be more similar to the abelian case.

Then $f\mapsto f^\wedge$ and $\varphi\mapsto\varphi^\vee$ are mutually inverse. Of course, not isotropic, but differ by our definition by an automorphism.

▶ **(6.11)** Problem. If a compact group G admits a faithful representation V, then the algebra generated by the matrix coefficient of V and $V^\vee$ is dense in $C(G)$. `Hint: Since they satisfy the hypothesis of Stone--Weiestrass theorem.`

▶ **(6.12)** Problem. If a compact group G cannot have infinite descending closed subgroups chain (for example compact Lie group), then

(1) G admits a faithful representation. Hint: Since we can chose for any $g \in G \setminus 1$, and function f separating 1 and g. Then approximate it with finite dimensional function, then the kernel of this representation has smaller kernel.

(2) Every closed subgroup H of G is $\{g \in G : gv = v\}$ for some representation V and $v \in V$. Hint: Find some representative function f which is constant over each coset of H (approximating and then averaging over H) but $f(g) \neq f(1)$. Then the space in $L^2(G)$ generated by f is a representation with H-action on f trivial.

A "Non-abstract" Harmonic Analysis

(A.1) Definition Let $f \in L^1(\mathbb{R}^n)$, denote

$$\hat{f}(t) = \int f(x)\mathrm{e}^{-2\pi i\langle t,x\rangle}\mathrm{d}x$$

the **Fourier transform** of f and

$$\check{f}(x) = \int f(t)\mathrm{e}^{2\pi i\langle t,x\rangle}\mathrm{d}t$$

the **inverse Fourier transform** of f.

Differentiation

(A.2) Proposition *For any polynomial P, we have*

- $(P(\partial)f)^\wedge(t) = P(-it)\cdot\hat{f}(t)$, *if $P(\partial)f$ exists.*
- $(P\cdot f)^\wedge = P(i\partial)(\hat{f})$.

(A.3) Definition (Schwarz space) The **Schwarz space** or **rapid decreasing functions space** is

$$\mathscr{S} = \left\{ f \in \mathscr{C}^\infty : \begin{array}{l} P\cdot\partial^\alpha f \text{ are bounded for all} \\ \alpha \text{ and polynomial } P \end{array} \right\}.$$

Or formally, for all $N > 0$,

$$\sup_{|\alpha|<N}\left(\sup_x |(1+|x|)^N\partial^\alpha f(x)|\right) < \infty.$$

We topologize $\mathscr{S}$ by the countable norms above.

(A.4) Proposition *The space $\mathscr{S}$ is a Fréchet space.*

(A.5) Theorem *Fourier transform is a continuous map from $\mathscr{S}$ to itself.*

(A.6) Inverse formula for $\mathscr{S}$ *The Fourier transform $\mathscr{S} \to \mathscr{S}$ is isomorphism preserving norm with inverse the inverse Fourier transform.*

Holomorphism

(A.7) Paley *Assume f is holomorphic over upper plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, with*

$$\sup_{0<y<\infty}\|f(\bullet + iy)\|_2 < \infty,$$

then there exists $F \in L^2(0,\infty)$ such that

$$f(z) = \int_0^\infty F(t)\mathrm{e}^{itz}\frac{\mathrm{d}t}{2\pi}.$$

(A.8) Wiener *Assume $A, C > 0$, f a entire function with $|f(z)| \leq C\mathrm{e}^{A|z|}$ (known as entire function of order A). Assume that $\int_{\mathbb{R}}|f(x)|^2\mathrm{d}x < \infty$, then there exists $F \in L^2(-A,A)$ such that*

$$f(z) = \int_{-A}^A F(t)\mathrm{e}^{itz}\frac{\mathrm{d}t}{2\pi}.$$

Miscellaneous

(A.9) Poisson Summation *Given $f \in L^1$ and $\hat{f} \in L^1$, assume*

$$f \text{ and } \hat{f} \ll \frac{1}{(1+|x|)^{n+\epsilon}},$$

then

$$\sum_{w\in\mathbb{Z}^n}\hat{f}(w) = \sum_{w\in\mathbb{Z}^n} f(w).$$

(A.10) Hausdorff-Young *If $1 \leq p \leq 2$, assume $f \in L^p$, then $\hat{f} \in L^q$ and*

$$\|\hat{f}\|_q \leq \|f\|_p,$$

where $1/p + 1/q = 1$.

(A.11) Heisenberg uncertain principle *If $\varphi \in \mathscr{S}(\mathbb{R})$ with $\|\varphi\|_2 = 1$. Then*

$$\|x\varphi(x)\|_2 \cdot \|t\hat{\varphi}(t)\|_2 \geq \frac{1}{2}.$$