Equivariant Version (II)

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§ Fixed Points and Tori §

Remind

- Let G be a group, X be a G-set, the equivariant cohomology

$$H_G^*(X) = H^*(EG \times_G X).$$

- In particular, when the action of X is trivial, then by definition,

$$H_G^*(X) = H^*(BG \times X).$$

- In particular, by Künneth theorem,

$$H_G^*(X;\mathbb{Q}) = H^*(BG;\mathbb{Q}) \otimes H^*(X;\mathbb{Q}).$$

Fixed Points

- Let X be a G-space. We denote X^G the fixed points of X. There is a map induced by the inclusion X^G, called **Localization**

$$H_G^*(X) \longrightarrow H_G^*(X^G).$$

- Note that X^G is a trivial G-space. So

$$H_G^*(X^G;\mathbb{Q}) = H_G^*(\mathrm{pt};\mathbb{Q}) \otimes H^*(X^G;\mathbb{Q}).$$

- Usually, the restriction loss much information using ordinary cohomology. But in equivariant case, in good case, it restores most of information.

Examples

Let G/B be the flag manifold.

- Note that $(G/B)^G = \varnothing$, so $H_G^*((G/B)^G) = 0$.
- Let T be the maximal torus of G contained in B, then

$$\begin{aligned}(G/B)^T &= \{xB : TxB = xB\} = \{xB : xTx^{-1} \subseteq B\} \\ &= \{xB : xTx^{-1} = T \subseteq B\} \\ &= N_G(T) \cdot B/B = N_G(T)/T = \text{Weyl group } W.\end{aligned}$$

So the localization

$$H_T^*(G/B) \longrightarrow H_T^*\Big(\bigcup\nolimits_{w \in W} wB/B\Big) = \bigoplus_{w \in W} H_T^*(\text{pt}).$$

Both sides have the same rank over $H_T^*(\text{pt})$.

Tori

- Today, the main role is the case G is a torus $(\mathbb{C}^\times)^n$. It is equivalent to consider $(S^1)^n$ as its maximal compact subgroup.
- Recall that

$$\begin{array}{ccc|ccc} ET & = & \mathbb{C}^\infty \setminus 0 & ET & = & S^\infty \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ BT & = & \mathbb{C}P^\infty & BT & = & \mathbb{C}P^\infty \\ & G = \mathbb{C}^\times & & & G = S^1 & \end{array}$$

- For points, $H_T^*(\mathrm{pt}) = H^*(BT)$,

$$H_T^*(\mathrm{pt}) = \mathbb{Z}[t_1, \ldots, t_n].$$

  Note that $\deg t_i = 2$.

Tori

- However, we should work for abstract torus, without a specific choice of isomorphism. We say

$$\text{an } {}^{\text{algebraic group}}_{\text{Lie group}} \ T \text{ is an } {}^{\text{algebraic torus}}_{\text{topological torus}} \text{ if it is } {}^{\text{isomorphism to } (\mathbb{C}^\times)^n}_{\text{homoemorphism to } (S^1)^n}$$

- Let its character group

$$\mathrm{Ch}(T) = \{{}^{\text{algebraic}}_{\text{continuous}} \text{ group homomorphism } T \to \mathbb{C}^\times\}.$$

- Note that $\mathrm{Ch}(T)$ is a free abelian group of finite rank $\dim T$. Say, if $T = \mathbb{C}^\times$, then $\mathrm{Ch}(T) = \{[z \mapsto z^n] : n \in \mathbb{Z}\}$.

- Let $\mathfrak{t}$ be the complexification of the Lie algebra of T, consider the dual space $\mathfrak{t}^*$.
- We can think $\mathrm{Ch}(T) \subseteq \mathfrak{t}^*$. By the following diagram

$$\begin{array}{ccc} \mathfrak{t} & \to & \mathbb{C} \\ \exp\downarrow & & \downarrow\exp \\ T & \to & \mathbb{C}^\times \end{array} \quad \Bigg| \quad \begin{array}{cccc} \mathfrak{t} & \to & i\mathbb{R} & \subseteq \mathbb{C} \\ \exp\downarrow & & \downarrow\exp & \\ T & \to & S^1 & \subseteq \mathbb{C}^\times \end{array}$$

- Say, we usually write $\lambda \in \mathrm{Ch}(T)$, where formally $\lambda \in \mathfrak{t}^*$, the map is given by

$$[x \mapsto e^{\lambda(t)}], \qquad x = \exp(t) \in T.$$

So the product is always written additively (be careful!).

Tori

- The conclusion is, $H^2(BT) = \mathrm{Ch}(T)$, by

$$\mathrm{Ch}(T) \longrightarrow H^2(BT) \qquad \lambda \longmapsto -c_2\left[\begin{array}{c} ET\times_T\mathbb{C}\lambda \\ \downarrow \\ BT \end{array}\right]$$

  where $\mathbb{C}\lambda$ is a copy of $\mathbb{C}$ acted by T through character λ.

- This can be checked easily from a choice of isomorphism. Say, from the isomorphism we know

$$H^*(B(\mathbb{C}^\times)^n) = \mathbb{Z}[t_1, \ldots, t_n],$$

  t_i is the character of i-th projection $(\mathbb{C}^\times)^n \to \mathbb{C}^\times$.

Tori

- And $H^*(BT)$ is generated freely by $H^2(BT)$. Formally,

$$H^*(BT) = \text{Symmetric power of } \operatorname{Ch}(T).$$

- In particular,

$$H^*(BT; \mathbb{C}) = \mathbb{C}[\mathfrak{t}] = \text{polynomials functions over } \mathfrak{t}.$$

$\gg$ *Questions?* $\ll$

§ LOCALIZATION THEOREM (I) §

Let T be a $\substack{\text{algebraic torus}\\\text{topological torus}}$, X be a $\substack{\text{variety}\\\text{manifold}}$ acted by T $\substack{\text{algebraically}\\\text{smoothly}}$.

Theorem (Borel)

The equivariant cohomology $H_T^*(X \setminus X^T; \mathbb{Q})$ is a torsion $H_T^*(\mathrm{pt})$-module. As a result, if X^T is a submanifold, the kernel and cokernel of

$$H_T^*(X; \mathbb{Q}) \longrightarrow H_T^*(X^T; \mathbb{Q})$$

are both torsion modules.

In particular, denote F the fraction field of $H_T^*(\mathrm{pt})$, then

$$H_T^*(X; \mathbb{Q}) \otimes F \longrightarrow H_T^*(X^T; \mathbb{Q}) \otimes F$$

is an isomorphism.

Proof

- Firstly, we only need to work in compact group, since any Lie group is homotopy equivalent to its maximal compact group.
- In compact case, we have "equivariant tubular neighborhood theorem". We can pick a neighborhood U of X^T, then

$$H_T^*(X, X^T) = H_T^*(X, U) = H_T^*(X \setminus X^T, U \setminus X^T).$$

Since $H_T^*(X \setminus X^T)$ and $H_T^*(U \setminus X^T)$ are all torsion module.

Proof

- Then to prove $H_T^*(X \setminus X^T; \mathbb{Q})$ is torsion module. It suffices to show the case $T = S^1$ or $\mathbb{C}^\times$, since

$$X \setminus X^{T_1 \times \ldots \times T_n} = (X \setminus X^{T_1}) \cup \cdots \cup (X \setminus X^{T_n}),$$

the Mayer–Vietoris sequences shows.

Proof

- In the case $T = S^1$ or $\mathbb{C}^\times$, the stablizer of $x \in X \setminus X^T$ is finite. The action is nearly to be free. Actually this is hidden in the $\mathbb{Q}$-coefficients.
- In this case, the map

$$ET \times_T (X \setminus X^T) \longrightarrow \text{orbit space of } X \setminus X^T,$$

has fibre BG_x at orbit of x. Note that $H^*(BG_x; \mathbb{Q})$ is $\mathbb{Q}$-acyclic, since G_x is finite. Some refined topology shows that

$$H_T^*(X \setminus X^T; \mathbb{Q}) = H^*(\text{orbit space of } X \setminus X^T; \mathbb{Q})$$

of finite dimension.

Remarks

- We have $\chi(X) = \chi(X^T)$. Actually, for any $H_T^*(\mathrm{pt})$, we can define its equivariant Euler character

$$\chi_T(X) = \sum (-1)^i \dim_F H_T^i(X;\mathbb{Q}) \otimes F.$$

- Then by the Serre–Leray spectral sequences,

$$\begin{aligned} \chi(X) &= \sum (-1)^i \dim_{\mathbb{Q}} H^i(X;\mathbb{Q}) \\ &= \sum (-1)^i \dim_F E_2^{pq} \otimes F \\ &= \sum (-1)^i \dim_F E_3^{pq} \otimes F \\ &= \cdots = \sum (-1)^i \dim_F E_\infty^{pq} \otimes F \\ &= \sum (-1)^i \dim_F H_T^i(X;\mathbb{Q}) \otimes F \\ &= \chi_T(X). \end{aligned}$$

Examples

- Recall $(G/B)^T$ is the Weyl group W, so

$$H_T^*(G/B) \longrightarrow H_T^*(w\cdot B/B) = \bigoplus_{w\in W} H_T^*(\mathrm{pt}).$$

  Since we computed $H_T^*(G/B)$ is free $H_T^*(\mathrm{pt})$-module, so this map is injective.

- One can see that for each Schubert cells BwB/B, it has one fixed point w. It is the evidence that $\chi(G/B) = \chi((G/B)^T)$.
- This is not a coincidence, it is called the Białynicki-Birula decomposition. See Milne.

$\gg$ <u>*Questions?*</u> $\ll$

§ Localization Theorem (II) §

Support

- Recall the concept support of commutative algebra. For a module M over ring R,

$$\mathrm{supp}(M) = \{\mathfrak{p} \in \operatorname{spec} R : M_{\mathfrak{p}} \neq 0\}.$$

- In our case,

$$R = H_T^*(\mathrm{pt}; \mathbb{C}) = \mathbb{C}[\mathfrak{t}]$$

  its spectrum is exactly $\mathfrak{t}$.

- For a T-space X, we denote

$$\mathrm{supp}(X) = \mathrm{supp}(H_T^*(X; \mathbb{C})) \subseteq \mathfrak{t}.$$

Let T be a $\substack{\text{algebraic torus}\\ \text{topological torus}}$, X be a $\substack{\text{projective variety}\\ \text{compact manifold}}$ acted by T $\substack{\text{algebraically}\\ \text{smoothly}}$.

Theorem (Atiyah–Segal)

The stablizer of $x \in X$ has only finite possibility, and

$$\mathrm{supp}(X) \subseteq \bigcup_{x \in X} \mathfrak{t}_x \subseteq \mathfrak{t},$$

where $\mathfrak{t}_x$ is the Lie algebra of stablizer of x.

Proof

- Similarly, it suffices to prove for compact torus. Actually, for each orbit, we can find a tubular neighborhood, since we assume X to be compact, we can find a finite subcovering. So it suffices to show for each conormal bundle.
- For each conormal bundle of orbit, the stablizer has only finite many choice. But the projection of conormal to itself is a homotopy equivalence, thus has the same equivariant cohomology. Moreover, the orbit has bigger stablizer. So finally, it reduces to show for one orbit.

Proof

- Let T_0 be the stablizer of this orbit.

$$ET \times_T X = ET \times_T T/T_0 \times_{T_0} X = BT_0 \times_{T_0} X$$

it is a fibre bundle of X with fibre BT_0.

- We see that the algebra map factors through

$$H_T^*(\mathrm{pt}) \xrightarrow{\text{augment}} H_{T_0}^*(\mathrm{pt}) \longrightarrow H_T^*(X).$$

This finishes the proof.

$\gg$ *Questions?* $\ll$

§ Localization Theorem (III) §

The inverse

- Remind in the case of ordinary cohomology, if $Y \subseteq X$ is a closed submanifold of codimension n, the composition of push forward and pull back

$$H^*(Y) \xrightarrow{i_*} H^{*+n}(X) \xrightarrow{i^*} H^{*+n}(Y),$$

factors through

$$\begin{array}{ccccc} H^*(Y) & \xrightarrow{\text{Thom}} & H^{*+n}(U, U \setminus Y) & \xrightarrow{i^*} & H^{*+n}(Y) \\ & & \downarrow & & \| \\ & & H^{*+n}(X, X \setminus Y) & \xrightarrow[i^*]{} & H^{*+n}(Y) \end{array}$$

- Therefore it is given by the cup product with Euler class of normal bundle of Y in X.

Equivariant case

- Let $E \to X$ be a equivariant vector bundle, its **equivariant Euler class** is define by the Euler class of its Borel construction. Say

$$\begin{array}{ccc} E_G & = & EG \times_G E \\ \downarrow & & \downarrow \\ X_G & = & EG \times_G X \end{array}$$

- If it is a complex bundle, then its **equivariant Chern class** is defined by the Chern class of its Borel construction.
- It is clear, the highest Chern class is the Euler class.

Let T be a $\substack{\text{algebraic torus}\\\text{topological torus}}$, X be a $\substack{\text{projective variety}\\\text{compact manifold}}$ acted by T $\substack{\text{algebraically}\\\text{smoothly}}$.

Theorem (Atiayh)

Assume X^T is smooth, then

$$H_T^*(X^T) \xrightarrow{i_*} H_T^*(X) \xrightarrow{i^*} H_T^*(X^T)$$

is given by cup product with equivariant Euler class of normal bundle of X^T.

Denote for a component $\alpha \in \pi_0(X^T)$, denote i^α the inclusion of $\alpha \subseteq X$, and N_α the normal bundle of α in X. As a result, the localization map $H_T^*(X) \otimes F \to H_T^*(X^T) \otimes F$ has an inverse

$$\sum_{\alpha \in \pi_0(X^T)} \frac{i_*^\alpha}{e(N_\alpha)} : H_T^*(X^T) \otimes F \longrightarrow H_T^*(X) \otimes F.$$

Computation for a point

- Let V be a representation of T. Then as an equivariant vector bundle of pt, its equivariant Euler class and Chern class

$$c_T(V) = \det(1 - [V \xrightarrow{t} V]) \in H_T^*(\mathrm{pt}) = \text{Symmetric power of } \mathrm{Ch}(T).$$

- Say, if $V = \bigoplus \lambda_i$ with $\lambda_i \in \mathrm{Ch}(T)$ the 1-dimensional representation. Then

$$c_T(V) = \prod (1 - \lambda_i) \in H_T^*(\mathrm{pt}).$$

This is tautologically from Whitney formula and definition.

Computation for a point

- For Euler class it is the same, since the only real representation of T is trivial.
- So in particular, under the condition of Atiyah localization theorem. If X^T is simply points, then in particular, this map is given by equivariant Euler class is the determinant of T action on cotangent bundle,

$$\bigoplus_{x \in X^T} \det(-[T_x X \xrightarrow{t} T_x X]) = \bigoplus_{x \in X^T} \det([T_x^* X \xrightarrow{t} T_x^* X]).$$

$\gg$ *Questions?* $\ll$

§ Localization Theorem (IV) §

Equivariant K-theory

- In opposite of equivariant cohomology, the equivariant K-theory

$$K_T(\mathrm{pt}) = R(G) = \text{Group algebra of } \mathrm{Ch}(T).$$

  We will write e^λ for $\lambda \in \mathrm{Ch}(T)$ the 1-dimensional representation with character λ.
- It is convenient to view it as a subspace of class functions. Then for a representation V, $[V]$ corresponds its character $\mathrm{tr}([V \xrightarrow{t} V])$.
- Note: the line bundle with the second Chern class λ is $e^{-\lambda}$.

- It also has similar localization theorem (I), (II) and (III). The main difference is the following.
- Note that

$$\begin{aligned} \operatorname{spec} K_T(\mathrm{pt}) \otimes \mathbb{C} &= \text{class functions} \\ &= \text{conj class of closed subgroups.} \end{aligned}$$

- The push forward and pull back (only defined for algebraic K-theory)

$$K(Y) \to K(X) \to K(Y)$$

is given by the product with $\sum(-1)^i \Lambda^i N^*$ where N^* is the dual of normal bundle of X in Y.

Let T be an algebraic torus, X be a projective variety acted by T algebraically.

Theorem (Atiayh–Bott)

Assume X^T is smooth, then

$$K_T^*(X^T) \xrightarrow{i_*} K_T^*(X) \xrightarrow{i^*} K_T^*(X^T)$$

is given by product with $\sum(-1)^i \Lambda^i N^*$ where N^* is the dual of normal bundle of X in Y.

Assume X^T are points, for an equivariant vector bundle ξ over X,

$$\sum(-1)^i \operatorname{tr}(t; H^i(X,\xi)) = \sum_{x \in X^T} \frac{\operatorname{tr}(t;\xi_x)}{\det(1 - t|_{T_x^*})},$$

where T_x^* is the cotangent bundle.

Proof of the second assertion

- Let V be a representation of T. Then as an equivariant vector bundle of pt, then $\sum(-1)^i[\Lambda^i V]$ is presented by

$$\sum(-1)^i \operatorname{tr}(t; \Lambda^i V) = \det(1 - [V \xrightarrow{t} V]),$$

by linear algebra.

- Denote $\pi : X \to \mathrm{pt}$. Then the left hand side is $\pi_*(\xi)$. The right hand side is $(\pi \circ i)_* i^* (i_* i^*)^{-1}(\xi)$.

$$\begin{array}{ccccccccc}
K_T(X^T) & \xrightarrow{i_*} & K_T(X) & \xrightarrow{i^*} & K_T(X^T) & \xrightarrow{\frac{1}{\sum(\cdots)}} & K_T(X^T) & \xrightarrow{i_*} & K_T(X) \\
& \searrow & \downarrow \pi_* & & & & & \searrow & \downarrow \pi_* \\
& & K_T(\mathrm{pt}) & = & = & = & = & = & K_T(\mathrm{pt})
\end{array}$$

Weyl Character formula

- For G/B, and λ a weight, denote $\mathcal{L}_\lambda = G \times_B \mathbb{C}\lambda$.

Theorem (Borel–Weil)

For λ negative,

$$H^0(\mathcal{L}_\lambda; \mathbb{C})$$

is the dual of irreducible representation of G of highest weight λ and its higher cohomology groups vanish.

- We can reprove Weyl character formula.

Theorem (Weyl)

For λ positive, let V be the irreducible representation of G of highest weight λ

$$\mathrm{tr}(t; V) = \sum_{w \in W} (-1)^w \frac{e^{w(\lambda+\rho)}}{\Delta},$$

where W the Weyl group, ρ the half of sum of positive roots, and $\Delta = \prod_{\lambda \in \Phi^+} (e^{\lambda/2} - e^{-\lambda/2})$ the discriminant with Φ^+ the set of positive roots.

- let $\mathfrak{b}$ be Lie algebra of B, and $\mathfrak{n}$ be its nilpotent radical.
- At the fixed point $w \cdot B/B$, its tangent bundle bundle is isomorphism to $\mathfrak{g}/\operatorname{ad}_w \mathfrak{b}$, thus cotangent bundle is $\operatorname{ad}_w \mathfrak{n}$ by Killing form. So

$$\det(1 - [T_x^* \xrightarrow{t} T_x^*]) = \prod_{\lambda \in \Phi^+} (1 - e^{w(\lambda)}) = (-1)^{\ell(w)} e^{w\rho} (-1)^{\ell(w_0)} \Delta.$$

- For λ a weight, $\mathcal{L}_\lambda$ is an equivariant T-vector bundle,

$$\operatorname{tr}(t; (\mathcal{L}_\lambda)_w) = e^{w\lambda}.$$

- So from Atyiah–Bott localization theorem,

$$\operatorname{tr}(t; H^0(\mathcal{L}_\lambda; \mathbb{C})) = \sum_{w \in W} (-1)^{\ell(w)} \frac{e^{w(\lambda - \rho)}}{(-1)^{\ell(w_0)} \Delta}.$$

Exchanging t to $-t$, we get the Weyl character formula.

$\gg$ ***Questions?*** $\ll$

§ Localization Theorem (V) §

GKM theory

- A more combinatorial of localization theorem is discovered by Goresky, Kottwitz, and MacPherson.
- The description would be a little long. Let X be a smooth projective variety over $\mathbb{C}$ equipped with an algebraic action of torus $T = (\mathbb{C}^\times)^n$. Assume X has finite fixed points and finitely one-dimensional orbit.
- The closure of one-dimensional orbit is a copy of $\mathbb{C}P^1$ with two fixed points.

Theorem (Goresky, Kottwitz, Macpherson, 1998)

The image of the localization map

$$H_T^*(X;\mathbb{Q}) \longrightarrow \bigoplus_{x\in X^T} H_T^*(x;\mathbb{Q})$$

is

$$\left\{(\alpha_x) : \begin{array}{c} \forall\ \mathbb{C}P^1 \textit{ connecting } x \xrightarrow{p} y, \\ \alpha_x|_{\mathfrak{t}_p} = \alpha_y|_{\mathfrak{t}_p} \end{array} \right\}$$

where $\mathfrak{t}_p$ *is the Lie algebra of stablizer of any point of* p*.*

Example

- The best example is G/B.
- All T-orbit is in some B-orbit, thus in some Schubert cells. By analysis of T-action on Schubert cells, it gives

$$H_T(G/B)^* = \left\{ (\lambda_w)_{w\in W} \in \bigoplus_{w\in W} H_T^*(\mathrm{pt}) : \begin{array}{l} \forall \alpha_i \in \Phi^+, w \in W, \\ \alpha_i \mid \lambda_{s_i w} - \lambda_w \end{array} \right\}.$$

See Jantzen 1.13 for details (One can use exponential map to do the same work, but it is not "suitable" for a fact holding for algebraic group).

$\gg$ *Questions?* $\ll$

§ <u>THANKS</u> §

References

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Next Time

- Language of Sheaf Theory.
- Examples.