Computations

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October 30, 2020

Trivial Bundles

Theorem

If ξ is the real trivial bundle, then the Stiefel-Whitney classes $\mathrm{sw}(\xi) = 1$. If ξ is the complex trivial bundle, then the Chern classes $c(\xi) = 1$.

- Note that trivial bundle is classified by $X \to \mathrm{pt} \to \mathcal{G}r(n, \infty)$.
- Alternatively, $\xi = n\mathbb{1}$, and $c(\mathbb{1}) = 1$, so $c(\xi) = c(\mathbb{1})^n = 1$.

Trivial Bundles

- If ξ has a nowhere vanishing frame, then ξ is trivial. More exactly, if $v_1, \ldots, v_n$ is, this assignment can be extended to a map $n\mathbb{1} \to \xi$, which is isomorphic at each fibre. By a set-point topology, one can check the converse is continuous.
- In particular, the normal bundle (the highest degree exterior algebra) is trivial if and only if the manifold is orientable.
- For a contractible CW-complex X, then any vector bundle is trivial. Due to homotopy invariance, it suffices to consider $X = \text{pt}$. Then a vector bundle over pt is nothing but a vector space.

Spheres

- Let S^n be the n-dimensional spheres. Of course,

$$H^k(S^n) = \begin{cases} \mathbb{Z}, & k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

- The tangent bundle of S^n is generally not trivial (hairy ball theorem).
- The classification of real or complex vector bundles over S^1, S^2, S^3, see Lecture 4.

Spheres

Theorem

The tangent bundle of S^n has trivial Stiefel-Whitney classes.

- Note that for S^n

$$\tau \oplus \nu = \text{(tangent bundle)} \oplus \text{(normal bundle)} = \text{(total space)} = \mathbb{1}^{n+1}$$

But ν is also trivial. So $w(\tau) = w(\tau)w(\nu) = w(\mathbb{1})^n = 1$.

Projective Spaces

- Note that as ring

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[t]/(t^{n+1}), \qquad \deg t = 2.$$

- It is clear from the definition for the invertible sheaf $\mathcal{O}(n)$,

$$c(\mathcal{O}(n)) = 1 + nt.$$

- Next, we shall consider tangent space.

Projective Spaces

Theorem

For $\mathbb{C}P^n$, the tangent bundle $\mathcal{T}$ satisfies the following exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to \mathcal{T} \to 0.$$

In particular,

$$c(\mathcal{T}) = (1+t)^{n+1}.$$

- Roughly, tangent space is the infinitesimal movement. Such can be written as $\mathcal{H}om(\mathcal{O}(-1), \mathbb{1}^{n+1})$, but moving along the line oneself should not be counted, so it is $\mathcal{H}om(\mathcal{O}(-1), \mathbb{1}^{n+1}/\mathcal{O}(-1)) = \mathcal{O}(1)^{n+1}/\mathcal{O}$.

Determinant

Theorem

For an n-dimensional real vector bundle ξ,

$$\mathrm{sw}(\Lambda^k \xi) = \mathrm{sw}(\xi)_{\leq n+1-k} = 1 + \mathrm{sw}_1(\xi) + \cdots + \mathrm{sw}_k(\xi)$$

For an n-dimensional complex vector bundle ξ,

$$c(\Lambda^k \xi) = c(\xi)_{\leq 2(n+1-k)} = 1 + c_1(\xi) + \cdots + c_k(\xi)$$

- By splitting lemma, assume $\xi = \xi_1 \oplus \cdots \oplus \xi_n$,

$$\Lambda^k \xi = \bigoplus_{i_1 < \ldots < i_k} \xi_{i_1} \otimes \cdots \otimes \xi_{i_k}$$

Orientation

Theorem

The manifold M is orientable if the first SW class of tangent bundle is zero.

- This follows from the fact that the first SW class determines the line bundle.
- For projective space $\mathbb{R}P^k$, we see that $(1+t)^{k+1} = 1 + (k+1)t + \cdot$. So $\mathbb{R}P^k$ is orientable if and only if k is odd.
- For complex manifold, the SW classes of the underlying real vector space is the same to Chern classes taking coefficients in $\mathbb{Z}/2$. So complex manifold is always oritentable.

Curves

- It is known that for nonsingular curve X over algebraic closed field

$$K(X) = \mathrm{Pic}(X) \oplus \mathbb{Z}$$

  by exterior product and rank, in both algebraic and topological senses.
- Algebraic sense see Hartshorne II.ex6.11.
- Topological sense due to the fact that when the $\mathrm{rank}_{\mathbb{C}}\,\xi \geq \dim X/2$, we can find a nonwhere vanishing section since the Euler class is zero. This also follows from the computation that $\pi_n(BGL_m) \to \pi_n(BGL_{m+1})$ is isomorphic for $m > n/2$.

Elliptic Curves

- For a complex elliptic curve X, the underlying space is a torus (very famous, parameterized by the **Weiestrass function**).
- The topological K-group is

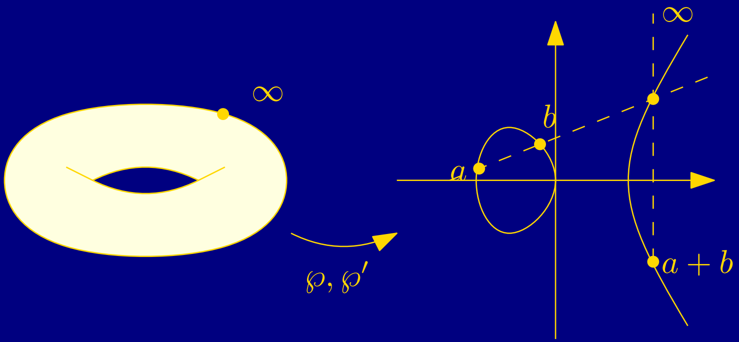
$$K(X) = \mathrm{Pic}(X) \oplus \mathbb{Z} = H^2(X) \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}.$$

- The algebraic K-group is

$$K(X) = \mathrm{Pic}(X) \oplus \mathbb{Z} = \mathrm{Cl}(X) \oplus \mathbb{Z} = X \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

  They are different. (see Hartshorne II.6.10.2)
- The main difference is $\mathrm{CH}^1(X) \neq H^2(X)$, or, actually, any two points are not rational equivalent.
- If there are two points not rational equivalent, it will be a rational curve $\mathbb{C}P^1$ the Riemann sphere, so this difference exists for all irrational curves.

Elliptic Curves



Classifying spaces

- We know for $G = \mathbb{C}^\times$, $B_G = \mathbb{C}P^\infty$, $E_G = \mathbb{C}^\infty \setminus 0$. So
$$H^*(B_G) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[t], \qquad \deg t = 2.$$
- Then consider the fibre bundle $\mathbb{C}\rho = \begin{bmatrix} E_G \times_G \mathbb{C}\rho \\ \downarrow \\ B_G \end{bmatrix}$ with the action of G on $\mathbb{C}\rho = \mathbb{C}$ by ρ. Actually, $-c_2(\mathbb{C}\rho) = \rho$.
- Warning: This minus is due to the fact that of $E_G \times_G \mathbb{C}$ is the tautological bundle.
- Warning: For a character ρ, following the classic notation $[z \mapsto z^n] = nt \in \mathbb{Z} \cdot t$.

Lie groups

- Let K be a compact group, and T_K be its maximal torus. We call K/T_K the **flag manifold**.
- Let $G = K_{\mathbb{C}}$ be a reductive group, T maximal torus and B its Borel subgroup, by the Iwasawa decomposition $G/B \cong K/T_K$. Note that this equips K/T_K with a complex structure.
- Denote the **Weyl group** $W = N_K(T_K)/T = N_G(T)/T$.

Bruhat decomposition

- It is known as the **Bruhat decomposition** that G/B decomposes in to cells $\bigsqcup_{w\in W} BwB/B$, with each $BwB/B \cong \mathbb{C}^{\ell(w)}$ with ℓ the length function.
- So $H^*(G/B)$ is of only even dimensions, and free abelian of rank $|W|$.
- Actually, $H^*(G/N_G(T);\mathbb{Q}) = H^*(\mathrm{pt};\mathbb{Q})$. Since $G/T \to G/N_G(T)$ is a W-covering, so

$$H^*(G/N(T);\mathbb{Q}) = H^*(G/T;\mathbb{Q})^W$$

has only even dimensions. Then Euler characteristic forces $H^*(G/N(T);\mathbb{Q})$ has only one dimension.

Classifying spaces

Theorem

$$H^*(B_G;\mathbb{Q}) = H^*(B_T;\mathbb{Q})^W$$

Note that $B_T = (\mathbb{C}P^\infty)^n$, $H^*(B_T;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}]$, *a polynomial ring in* rank $G =$ dim T *variables.*

- Firstly, we take a contractible $E = E_G$ such that G acts freely, then for any subgroup H, $B_H = E/H$, by the Milnor construction.
- Now $B_G \to B_{N_G(T)}$ is a fibre bundle with fibre $G/N_G(T)$ which is $\mathbb{Q}$-acyclic, and $B_{N(G)} \to B_T$ is an W-covering. So the proof is complete.

Flag manifolds

- Consider the space of all flags $\mathcal{F}\ell(V)$ in the vector space V, where a flag means a chain of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$$

  with $\dim V_i = i$.
- Let $G = \mathrm{GL}_n$, and B the group of upper triangular matrices. By assigning the flag with i-th space the space spanned by the first column i-vectors, we can find a bijection $G/B \cong \mathcal{F}\ell(n)$.
- Denote $K = \mathrm{U}_n$ and T the diagonal matrices, we can also find $K/T \cong \mathcal{F}\ell(n)$.
- In this case, the Weyl group is the symmetric group $\mathfrak{S}_n$.

Flag manifolds

Theorem (Borel)

$$H^*(G/T;\mathbb{Q}) = H^*(B_T;\mathbb{Q}) \underset{H^*(B_G;\mathbb{Q})}{\otimes} \mathbb{Q}$$

- Consider the fibre bundle $B_T \to B_G$ whose fibre is G/T. By the Serre–Leray spectral sequence, since all of cohomology of the spaces are of only even dimensions, so there is no nonzero differentials. So

$$H^*(B_T;\mathbb{Q}) = H^*(B_G;\mathbb{Q}) \otimes H^*(G/T;\mathbb{Q})$$

  as $H^*(B_G;\mathbb{Q})$-module.
- So the restriction of $H^*(B_T;\mathbb{Q}) \to H^*(G/T;\mathbb{Q})$ factors through the right hand side.

Geometric meaning

- For any character ρ of T, denote the line bundle $\mathbb{C}\rho = \left[\begin{array}{c} G \times_T \mathbb{C}\rho \\ \downarrow \\ G/T \end{array}\right]$ where T acts on $\mathbb{C}\rho = \mathbb{C}$ by ρ. Then $-c_2(\mathbb{C}\rho)$ is presented by ρ.
- This follows easily from the fact that fibre bundle $G \to G/T$ is classified by the inclusion $G/T \to BT$.
- Warning: the minus also comes from the fact that $E_T \times_T \mathbb{C}$ is the tautological bundle.

Examples

- Consider the case $G = \mathrm{GL}_n$, then

$$\begin{aligned} H^*(B_G;\mathbb{Q}) &= H^*(B_T;\mathbb{Q})^{\mathfrak{S}_n} \\ &= \mathbb{Q}[x_1,\dots,x_n]^{\mathfrak{S}_n}. \end{aligned}$$

$$\begin{aligned} H^*(\mathcal{F}\ell(n),\mathbb{Q}) &= \mathbb{Q}[x_1,\dots,x_n]\otimes_{\mathbb{Q}[x_1,\dots,x_n]^{\mathfrak{S}_n}}\mathbb{Q} \\ &= \frac{\mathbb{Q}[x_1,\dots,x_n]}{\langle e_1,\dots,e_n\rangle}, \end{aligned}$$

  where $e_1,\dots,e_n$ are elementary symmetric polynomials.
- In the case $n = 2$, $\mathcal{F}\ell(2) = \mathbb{C}P^1$, the $x_1 \in H^*(G/T)$ is the Chern class of $\mathcal{O}(1)$.

Cells

- So how to express the cohomology class of a cell BwB/B?
- The answer is the **Schubert polynomials** (cf. Lascoux and Schützenberger's unaccessible paper).

$$[BwB/B] = \mathfrak{S}_w(x).$$

where $\mathfrak{S}_{w_0} = x_1^{n-1} \cdots x_{n-1}$, and

$$\ell(ws_i) - 1 = \ell(w) \quad \Longrightarrow \quad \mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i},$$

with ∂_i the **Demazure operator**

$$\partial_i f(x) = \frac{f(\cdots, x_i, x_{i+1}, \cdots) - f(\cdots, x_{i+1}, x_i, \cdots)}{x_i - x_{i+1}}.$$

Grassmannians

- Consider the space of all k-dimensional spaces $\mathcal{G}r(k, V)$ in the vector space V.
- Let $G = \mathrm{GL}_{n+k}$, and $P = \begin{pmatrix} \mathrm{GL}_k & * \\ & \mathrm{GL}_n \end{pmatrix}$. By assigning the space spanned by the first column k-vectors, we can find a bijection $G/P \cong \mathcal{G}r(k, n+k) = \mathcal{G}r(k, \mathbb{C}^{n+k})$.
- We can also find $\mathrm{U}_{n+k} \,/\, \mathrm{U}_k \times \mathrm{U}_n \cong \mathcal{G}r(k, n+k)$.

Grassmannians

Theorem

$$H^*(\mathcal{G}\mathrm{r}(k, n+k); \mathbb{Q}) = H^*(BGL_k \times BGL_n; \mathbb{Q}) \bigotimes_{H^*(BGL_{n+k};\mathbb{Q})} \mathbb{Q}.$$

- Consider the fibre bundle $BGL_n \times BGL_k \to BGL_{n+k}$ whose fibre is homotopy equivalent to $\mathcal{G}\mathrm{r}(k, n+k)$. By the Serre–Leray spectral sequence, since all of cohomology of the spaces are of only even dimensions, so there is no nonzero differentials. So the desired expression.

Grassmannians

- We can compute

$$H^*(\mathcal{G}\mathrm{r}(k,n+k);\mathbb{Q}) = H^*(BGL_k \times BGL_n;\mathbb{Q}) \bigotimes_{H^*(BGL_{n+k};\mathbb{Q})} \mathbb{Q}$$
$$= \frac{\mathbb{Q}[e_1(x),\dots,e_k(x),e_1(y),\dots,e_n(y)]}{\langle e_1(x,y),\dots,e_{n+k}(x,y)\rangle}$$

  Since any

$$e_n(y) = e_n(x,y) - e_{n-1}(y)(\cdots) - \cdots,$$

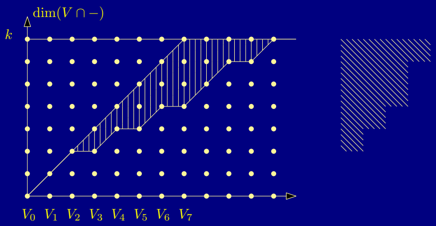
  $H^*(\mathcal{G}\mathrm{r}(k,n);\mathbb{Q})$ is generated by $e_1(x),\dots,e_k(x)$. That is, a quotient ring of the symmetric polynomials.
- Actually, the total Chern class of the dual of the tautological bundle of $\mathcal{G}\mathrm{r}(k,n)$ is exactly $1+e_1(x)+\ldots+e_k(x)$.

Cells

- Like Flag manifolds, Grassmannians also admit cellular structure. Denote the Schubert cells for λ with Young diagram inside $n \times k$ boxes.

$$\Sigma_\lambda(F) = \{V \in \mathcal{G}r(k, n+k) : {}^{\forall i=1,\dots,k,}_{\dim(V \cap V_{k-i+\lambda_i}) \geq k-i}\},$$

where $0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ is some flag.



Cells

- So how to express the cohomology class of a cell Σ_λ?
- The answer is the **Schur polynomials**.

$$\Sigma_\lambda = s_\lambda(x) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ \vdots & \ddots & \vdots \\ x_1^{\lambda_n+n-n} & \cdots & x_n^{\lambda_n+n-n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ \vdots & \ddots & \vdots \\ x_1^{0} & \cdots & x_n^{0} \end{vmatrix}}.$$

Digression

- The famous question, how many lines lie on a smooth cubic hyperplane in $\mathbb{C}P^3$, can be answered after introduction of the Schubert cells. (27)
- The sets of all lines in $\mathbb{C}P^3$ is exactly $\mathcal{G}r(2,4)$.
- Note that a line L lies on the cubic hyperplane $\{f = 0\}$ if any only if $f|_L = 0$. So the number is the Euler class of $S^3\mathcal{T}^*$ where $\mathcal{T}$ is the tautological bundle of $\mathcal{G}r(2,4)$.

Digression

$$c(\mathcal{T}^*) = 1 + e_1 + e_2 = (1 + x_1)(1 + x_2).$$

$$c(S^3(\mathcal{T}^*)) = (1 + 3x_1)(1 + 2x_1 + x_2)(1 + x_1 + 2x_2)(1 + 3x_2).$$

the coefficient in front of $s_{\yng(2,2)}$ in f

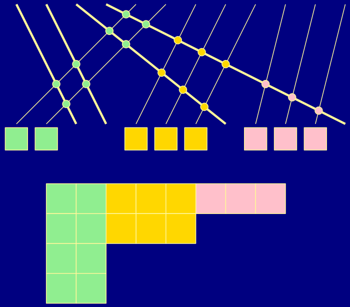
$=$ the coefficient in front of $x_1^{2+1}x_2^2$ in $f\Delta$

the coefficient in front of $x_1^3 x_2^2$ in $3x_1(2x_1 + x_2)(x_1 + 2x_2)(3x_2)(x_1 - x_2)$.

$$= 9 \cdot (-2 + 4 + 1) = 27.$$

Cells

- There is a natural map $\mathcal{F}\ell(n+k) \to \mathcal{G}r(k, n+k)$ which assign the k-th space of flag. This map can be proven to be cellular, so the Schur polynomials can be computed as a special case of Schubert polynomials.



Classifying spaces

- For G a compact Lie group, for any finite dimensional representation V, we can consider
$$\begin{bmatrix} EG\times_G V \\ \downarrow \\ BG \end{bmatrix},$$
so we get a map $R(G)\to K(BG)$.
- Note that, in this case, $K(BG)=\pi(BG, BGL\times\mathbb{Z})$. Actually, any representation $G\to \mathrm{GL}_n$ induces a map $BG\to BGL$.
- Atiyah and Segal:
$$K(BG)=\widehat{R(G)}, \qquad K^1(BG)=0,$$
where $\hat{*}$ is the completion with respect to the augment ideal $\ker[R(G)\to\mathbb{Z}]$. (proof, see Atiyah and Segal, equivariant K-theory and completion)

Flag manifolds

- On one hand, using Atyiah–Hirzebruch Spectral Sequence,

$$H^p(G/T; K^q(pt)) \Longrightarrow K^{p+q}(G/T),$$

  has only even dimensional stuff, so $K^1(G/T) = 0$, and $K(G/T)$ is free abelian of order $|W|$.
- The algebraic K-theory also gives the same answer, say the push forward of $\mathcal{O}_{BwB/B}$ to G/B forms a basis (use a little higher K-theory).

Flag manifolds

- On the other hand, using the fibre bundle $BT \to BG$,

$$\begin{array}{ccc} H^p(BG; K^q(G/T)) & \Longrightarrow & K^{p+q}(BT) \\ \uparrow & & \uparrow \\ H^p(BG; K^q(pt)) & \Longrightarrow & K^{p+q}(BG) \end{array}$$

Since $K(G/T)$ is finite dimensional,

$$K(G/T) = K(BT) \underset{K(BG)}{\otimes} \mathbb{Q} = R(T) \otimes_{R(G)} \mathbb{Q}.$$

Examples

- For the case $G = \mathrm{U}_n$,

$$R(T) = \mathbb{Z}[e^{x_1}, \ldots, e^{x_n}]$$
$$R(G) = \mathbb{Z}[e^{x_1}, \ldots, e^{x_n}]^{\mathfrak{S}_n}$$

Note that e^{x_1} stands for the dual of the representation of character e^{x_1}, so that in the case $T = \mathbb{C}^\times$, the generator is $\mathcal{O}(1)$.

$$R(G) \to \mathbb{Z} \qquad e^{x_i} \mapsto 1.$$

$$\begin{aligned} K(G/T) &= R(T) \otimes_{R(G)} \mathbb{Q} \\ &= \frac{\mathbb{Q}[e^{x_1}, \ldots, e^{x_n}]}{\langle f \in \mathbb{Q}[e^{x_1}, \ldots, e^{x_n}]^{\mathfrak{S}_n} : f(e^0, \ldots, e^0) = 0 \rangle}. \end{aligned}$$

It is suggested to use $X_i = 1 - e^{-x_i}$, in this case,

$$K(G/T) = \frac{\mathbb{Z}[X_1, \ldots, X_n]}{\langle e_i(X) : i = 1, \ldots, n \rangle}$$

with e_i the i-th elementary symmetric polynomial.

Cells

- So how to express the class of a cell $\mathcal{O}_{BwB/B}$?
- The answer is the **Grothendieck polynomials** (still cf. the unaccessible paper).

$$[Bw_0wB/B] = \mathfrak{G}_w(X).$$

where $\mathfrak{G}_{w_0} = X_1^{n-1} \cdots X_{n-1}$, and

$$\ell(ws_i) - 1 = \ell(w) \implies \mathfrak{S}_w = \pi_i \mathfrak{S}_{ws_i},$$

with π_i the **isobaric Demazure operator**

$$\pi_i f(X) = \frac{(1 - X_{i+1}) f(\cdots, X_i, X_{i+1}, \cdots) - (1 - X_i) f(\cdots, X_{i+1}, X_i, \cdots)}{X_i - X_{i+1}}$$

$$\pi_i f(e^x) = \frac{e^{x_i} f(\cdots, e^{x_i}, e^{x_{i+1}}, \cdots) - e^{x_{i+1}} f(\cdots, e^{x_{i+1}}, e^{x_i}, \cdots)}{e^{x_i} - e^{x_{i+1}}}.$$

Grassmannians

- For Grassmannians, on one hand, push forward of $\mathcal{O}_{\Sigma_\lambda}$ forms a basis, and on the other hand, we can compute by spectral sequences. Completely the same with the cohomology version.
- For the push forward of $\mathcal{O}_{\Sigma_\lambda}$, it is known as "symmetric Grothendieck polynomials".

References

- May. A concise introduction to algebraic topology.
- Harris, Eisenbud. 3264 and all that. (homework: find out the statement of tangent bundle of Grassmannians, and as a result, when the real Grassmannian $\mathcal{G}r(k, n)$ is orientable)
- Fulton. Young tableaux with applications in Algebra and Geometry. For an introduction to Schur polynomial.
- For Schubert polynomials and Grothendieck polynomials, good references are by searching the names in arXiv, but the best reference is by computing yourself unfortunately.

Next Time

- Equivariant cohomology.
- Localization theorem.

$\sim$ § THANKS § $\sim$