K-theory (II)

Xiong Rui

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~ § K in Algebra § ~

Algebraic K_0

- Let R be a ring, and denote $\operatorname{proj} R$ be the full subcategory of finitely generated projective modules. We can define the K_0 of R to be

$$K_0(R) = \frac{\bigoplus_{M \in \operatorname{proj} R} \mathbb{Z} \cdot [M]}{\left\langle [M] = [M_1] + [M_2] : \begin{array}{c}\text{there exists a short exact sequence} \\ 0 \to M_1 \to M \to M_2 \to 0\end{array} \right\rangle}.$$

- Since $[M_1] + [M_2] = [M_1 \oplus M_2]$, each element of $K_0(R)$ is presented by a difference of two f.g. projective modules. Therefore, $[M] - [R^m]$.
- Two elements $[M] - [R^m], [N] - [R^n]$ present the same element in $K_0(R)$, if and only if $M \oplus R^{n+N} \cong N \oplus R^{m+N}$ for some N. Note that we cannot simply cancel N as topological case.

Examples

- For a field F, $K_0(F) = \mathbb{Z}$, the map is given by dim.
- For a PID R, $K_0(R) = \mathbb{Z}$, the map is given by rank.
- For a semisimple ring R, $K_0(R)$ is the free abelian group generated by the classes of simple modules. This follows from the Jordan–Hölder theorem.
- In particular, for a group algebra $R = k[G]$ for finite group G over characteristic zero field k, $K_0(R)$ is the representation ring (it is a ring due to the Hopf structure).

Examples (continued)

- Generally, for an artinnian ring R, K_0 is the free abelian group generated by the classes of indecomposable projective modules.
  This follows from the Krull–Schmidts theorem.
- For a Dedekind domian R, $K_0(R) = \mathrm{Cl}(R) \oplus \mathbb{Z}$.
  This follows from the theorem that any finitely generated projective module of R is a direct sum of (fractional) ideals; $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{ab} \oplus R$. See Milnor P9.
- For a compact Hausdorff space X, denote $R = \mathcal{C}(X)$ the Banach algebra of complex continuous functions, then $K_0(R) = K^0(X)$ the topological K-theory.
  Since there is a category equivalence between $\operatorname{proj} R$ and $\mathrm{Vec}_{\mathbb{C}} X$ (Swan theorem).

Milnor Patching

- Assume we have the following ring pull back

$$R = \ker[R_1 \oplus R_2 \xrightarrow{\text{difference}} R_0] \quad \begin{array}{|ccc} R & \to & R_1 \\ \downarrow & & \downarrow \\ R_2 & \to & R_0 \end{array}$$

Given two modules P_1, P_2 over R_1, R_2 respectively with $R_0 \otimes_{R_1} P_1 \cong R_0 \otimes_{R_2} P_2 \cong P_0$. We can construct pull back

$$P = \ker[P_1 \oplus P_2 \xrightarrow{\text{difference}} P_0] \quad \begin{array}{|ccc} P & \to & P_1 \\ \downarrow & & \downarrow \\ P_2 & \to & P_0 \end{array}$$

- When $R_1 \to R_0$ is surjective, then this construction gives all f.g. projective modules over R. (see Milnor P19, Weibel P15 2.7)

For ring without unit

- Let I be a ring without unit, and any ring R acting I both sides, for example $\mathbb{Z}$. We can define

$$K_0(I) = \ker[K_0(I \rtimes R) \to K_0(R)].$$

This is well-defined by Milnor patching $\begin{bmatrix} I \rtimes \mathbb{Z} & \to & I \rtimes R \\ \downarrow & & \downarrow \\ \mathbb{Z} & \to & R \end{bmatrix}$; in which case, we can lift the patching.

- We have the exact sequence in the case I is an ideal of R,

$$K_0(I) \to K_0(R) \to K_0(R/I)$$

still by Milnor patching $\begin{bmatrix} I \rtimes \mathbb{Z} & \to & R \\ \downarrow & & \downarrow \\ \mathbb{Z} & \to & R/I \end{bmatrix}$.

For ring without unit

- This is an analogue of excision. Let $R = \mathcal{C}(X)$ the complex functions over compact X, and I be the functions vanishing at closed subset Y. Then

$$\begin{aligned} R/I &= \text{functions over } Y \\ I &= \text{functions over } X \setminus Y \text{ vanishing at infinity.} \\ I \rtimes \mathbb{C} &= \text{functions over one point compactification of } X \setminus Y. \end{aligned}$$

So, $K_0(I) = K_c^0(X \setminus Y)$.

$\gg$ ***Questions?*** $\ll$

∼ § K in Analysis § ∼

K-theory of Banach Algebras

- Let $\mathcal{A}$ be a C^*-algebra, that is a complex Banach space with $\mathbb{C}$-algebra (with unit) structure and involution $*$ compatible with norm. Denote the set of equivalent class of Hermitian projections

$$\operatorname{proj}\mathcal{H} = \frac{\{p \in \mathbb{M}_\infty(\mathcal{A}) : p^2 = p = p^*\}}{p \sim q \iff_{\substack{\exists v, \text{such that} \\ p = v^* v, q = v^* v}}}.$$

where $*$ is the transposition and the $*$-involution. Here $\mathbb{M}_\infty = \bigcup_{n \geq 0} \mathbb{M}_n$ by adding infinite many 1's in the diagonal.

For algebra without unit

- By $\mathbb{M}_{2\infty}(\mathcal{A}) \times \mathbb{M}_{2\infty+1}(\mathcal{A}) \to \mathbb{M}_{\infty}(\mathcal{A})$, we can define the sum of two projections which makes $\operatorname{proj}\mathcal{A}$ a monoid. Then, we can define

$$K_0(\mathcal{A}) = \text{group-ization of } \operatorname{proj}\mathcal{A}.$$

- Let $\mathcal{A}$ be a C^*-algebra without unit, then we define

$$K_0(\mathcal{A}) = \ker[K_0(\mathbb{A} \rtimes \mathbb{C}) \to \mathbb{Z}],$$

as we expected.

Examples

- For $\mathcal{C}(X)$, for a compact space X, it is the topological K-theory $K_0(\mathcal{C}(X)) = K^0(X)$. Generally, for a local compact space X, $K_0(\mathcal{C}(X)) = K_c^0(X)$.
  This follows from the fact that any bundle can be equipped with a unitary inner product.
- For any finite dimensional $\mathbb{C}$-algebra $\mathcal{A}$, with $*$-involution, $K_0(\mathcal{A})$ coincides with the algebraic K-theory. But it is not interesting, since the finite dimensional C^*-algebra are semisimple, i.e. product of matrix algebra.

Examples

- For $\mathbb{K}$ the compact operators over ℓ^2, $K_0(\mathbb{K}) = \mathbb{Z}$.
  Since the compact operator which is a projection is of finite rank.
- For $\mathbb{B}$ the bounded operators over ℓ^2, $K_0(\mathbb{B}) = 0$. The same to the Calkin algebra $\mathbb{B}/\mathbb{K}$.
  Since there is an infinite dimension projection P, such that $p \oplus P \sim P$.

Definition of K_1

- We can define for a unitary C^*-algebra $\mathcal{A}$

$$K_1(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A}) \big/ \mathrm{GL}_\infty(\mathcal{A})^\circ,$$

  where $\mathrm{GL}(\mathcal{A})^\circ$ is the component of the identity. Here $\mathrm{GL}_\infty = \bigcup_{n \geq 0} \mathrm{GL}_n$ by adding infinite many 1's in the diagonal.
- For algebra without unit, we define $K_1(\mathcal{A}) = K_1(\mathcal{A} \rtimes \mathbb{C})$.
- Note that K_1 is equipped with a commutative multiplication, since

$$\begin{pmatrix} uv & \\ & 1 \end{pmatrix} \equiv \begin{pmatrix} u & \\ & v \end{pmatrix} \equiv \begin{pmatrix} v & \\ & u \end{pmatrix} \equiv \begin{pmatrix} vu & \\ & 1 \end{pmatrix}.$$

- What is important, we still have Bott periodicity!

Examples

- Still, for $\mathcal{C}(X)$ with X compact, $K_1(\mathcal{C}(X)) = K^{-1}(X)$ the topological K-theory. Since,

$$\begin{aligned} K^{-1}(X) &= \pi(SX_{\cup\infty}, \mathrm{BGL}) \\ &= \pi(X_{\cup\infty}, \mathrm{GL}) \\ &= \mathrm{GL}_\infty(\mathcal{C}(X))/\,\mathrm{GL}_\infty(\mathcal{C}(X))^0 \\ &= K_1(X). \end{aligned}$$

Exmaples

- $K_1(\mathbb{C}) = 0$, since $\mathrm{GL}(\mathbb{C})$ is connected.
- $K_1(\mathbb{K}) = 0$, since the spectra of compact operator is discrete.
- $K_1(\mathbb{B}) = 0$, since $\mathrm{GL}(\mathbb{B}) = \mathrm{GL}_1(\mathbb{B}) = \mathbb{B}^\times$ is connected.
- $K_1(\mathbb{B}/\mathbb{K}) = K_0(\mathbb{K}) = \mathbb{Z}$, by Bott periodicity and exact sequence. Actually, $\mathrm{GL}(\mathbb{B}/\mathbb{K}) = \mathrm{GL}_1(\mathbb{B}/\mathbb{K}) = (\mathbb{B}/\mathbb{K})^\times$, and the map is given by the **Fredholm index**.

$\gg$ *Questions?* $\ll$

∼ § K in Algebraic Geometry § ∼

K-theory in AG

- We can define Grothendieck group for a scheme X to be

$$K(X) = \frac{\bigoplus_{\mathcal{F}\in\mathcal{C}\mathrm{oh}\,X} \mathbb{Z}\cdot[\mathcal{F}]}{\left\langle [\mathcal{F}] = [\mathcal{F}_1] + [\mathcal{F}_2] : \substack{\text{there exists a short exact sequence} \\ 0\to\mathcal{F}_1\to\mathcal{F}\to\mathcal{F}_2\to 0} \right\rangle},$$

where $\mathcal{C}\mathrm{oh}\,X$ the category of coherent sheaves.
- The construction based on the locally trivial sheaves should also be considered. It follows from the **Hilbert's syzygy theorem**, each coherent sheaf admits a finite resolution of locally trivial bundle over smooth variety over fields. In this case, the two constructions coincide.

Examples

- For affine space $\mathbb{A}^n$, $K(\mathbb{A}^n) = \mathbb{Z}$. This is a part of Hilbert's syzygy theorem, any finitely generated module over $\Bbbk[x_1, \ldots, x_n]$ admits a finite free resolution. Actually, By Serre–Quillen–Suslin theorem, any finitely generated projective module is free.
- For projective space $\mathbb{P}^k$, we can compute by excision

$$K(\mathbb{P}^{k-1}) \to K(\mathbb{P}^k) \to K(\mathbb{A}^k) \to 0.$$

Consider the image of basis of $K(\mathbb{P}^{k-1})$, one can conclude it is also left exact. So the result is the same to the topological one.

K-theory in AG

- Now we can define pull back and push forward sheaf-theoretically. Consider a morphism of noetherian schemes $f : X \to Y$ over field.
- If f is projective, then the push forward

$$[\mathcal{F}] \longmapsto \sum (-1)^i [Rf_*^i \mathcal{F}]$$

makes sense (cf Hartshorne III.8.8)

- If f is flat, then the pull back f^* is exact, so

$$[\mathcal{G}] \longmapsto [f^*\mathcal{G}]$$

makes sense (cf Hartshorne III.9). Generally if f has finite torsion-dimension, then we can also define, for example the closed immersion.

K-theory in AG

- In algebraic geometry level, we can also define the Chern character

$$\mathrm{ch} : K(X) \to \mathrm{Ch}(X) \otimes \mathbb{Q},$$

this commutes with pull back.

- But for push forward, we need a correction, the Todd class of the tangent bundle.

Theorem (Grothendieck–Riemann–Roch)

Consider a proper morphism $f : X \to Y$ between smooth quasi-projective schemes,

$$\mathrm{Td}(X)\cdot\mathrm{ch}-\begin{array}{ccc} K(X) & \xrightarrow{f_*} & K(Y) \\ \downarrow & & \downarrow \\ \mathrm{Ch}(X)\otimes\mathbb{Q} & \xrightarrow[f_*]{} & \mathrm{Ch}(Y)\otimes\mathbb{Q} \end{array} \mathrm{Td}(Y)\cdot\mathrm{ch}-$$

$\gg$ *Questions?* $\ll$

∼ § Higher K § ∼

Nerves

- Let $\mathcal{C}$ be a category, put the n-dimensional simplexes

$$\Delta^n = \{\text{length } n+1 \text{ chain} : X_0 \to \cdots \to X_n\},$$

and define the i-th face $\partial_i(X_0 \to \cdots \to X_n)$ to be

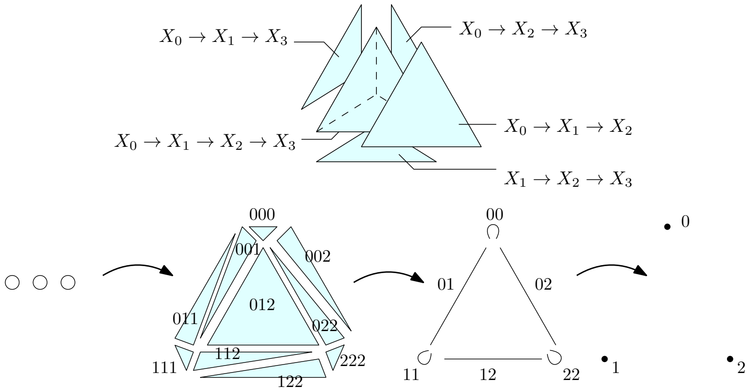
$$X_0 \to \cdots \to X_{i-1} \xrightarrow{\text{composition}} X_{i+1} \to \cdots \to X_n$$

and i-th degeneracy $d(X_0 \to \cdots \to X_n)$ to be

$$X_0 \to \cdots \to X_i \xrightarrow{\text{identity}} X_i \to \cdots \to X_n.$$

We can realize it geometrically, known as $B\mathcal{C}$.

Nerves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Examples and Properties

- For the category of $\{0, \ldots, n\}$, with morphism by $\leq$, the $B\mathcal{C}$ is exactly the simplex.
- For the category of a single point, but morphism a group element, then $B\mathcal{C}$ is the same to the classifying space BG with G discrete.
- $\pi_0(B\mathcal{C})$ is the connected component of $\mathcal{C}$.
- Functor $\mathcal{C}_1 \to \mathcal{C}_2$ induces a cellular map $B\mathcal{C}_1 \to B\mathcal{C}_2$.
- Natural transform induces homotopy $B\mathcal{C}_1 \times I \to B\mathcal{C}_2$.

Quillen construction

- Let $\mathcal{C}$ be an exact category, consider the category $Q(\mathcal{C})$ with same objects but morphisms from M to M' of the form
$$M \twoheadleftarrow N \hookrightarrow M'.$$
Say, M appears as a subquotient of M'. The composition is by exchanging indicated in the following diagram
$$M \twoheadleftarrow \underbrace{N \hookrightarrow M' \twoheadleftarrow N'}_{N \twoheadleftarrow N \times_{M'} N' \hookrightarrow N'} \hookrightarrow M''.$$
- Quillen shows that $\pi_1(BQ(\mathcal{C}), 0) = K_0(\mathcal{C})$.

The sketch of the proof

- Generally, let us consider a covering of $B\mathcal{C}$. For a covering $X \to B\mathcal{C}$, we consider the fibre $X(c)$ of $c \in \mathcal{C}$. This defines a functor $\mathcal{C} \to \mathrm{Set}$ with morphisms invertible.
- Conversely, if we are given a functor $F : \mathcal{C} \to \mathrm{Set}$ with morphisms invertible. Then we can construct $F\backslash\mathcal{C}$ to be the category of pairs (c, x) with $x \in F(c)$. Then $B(F\backslash\mathcal{C}) \to B\mathcal{C}$ is the desired covering.

The sketch of the proof

- Now, let us consider the functor $BQ\mathcal{C} \to \mathrm{Set}$ with morphisms invertible corresponding to the universal covering.
- Since each morphism in the image of F is invertible we can normalize F if necessary to assume $F(M) = F(0)$ and
$$F([0 \twoheadleftarrow 0 \hookrightarrow M]) = \mathrm{id}\,.$$
Then
$$\begin{aligned} F(M \twoheadleftarrow N \hookrightarrow M') &= F(M \twoheadleftarrow N \hookrightarrow M') \circ F(0 \twoheadleftarrow 0 \hookrightarrow M) \\ &= F(0 \twoheadleftarrow K \hookrightarrow M') \qquad \text{where } K = \ker[N \to M] \\ &= F(K \twoheadleftarrow K \hookrightarrow M') \circ F(0 \twoheadleftarrow K \hookrightarrow K) \\ &= F(0 \twoheadleftarrow 0 \hookrightarrow M') \circ F(0 \twoheadleftarrow K \hookrightarrow K) \\ &= F(0 \twoheadleftarrow K \hookrightarrow K) \end{aligned}$$

The sketch of the proof

- Let $M \in \mathcal{C}$, consider the map $[M] : F(0) \to F(0)$ defined by
$$F(0 \twoheadleftarrow M \hookrightarrow M).$$
- Then F is a functor if and only if
$$\begin{array}{l}\text{there exists a short exact sequence} \\ 0 \to M_1 \to M \to M_2 \to 0\end{array} \Longrightarrow [M] = [M_1][M_2].$$
(homological algebra)
- As a result,
$$\pi_1(BQ\mathcal{C}, 0) = K_0(\mathcal{C}).$$

The sketch of the proof

- We can define

$$K_i(\mathcal{C}) = \pi_{i+1}(BQ(\mathcal{C}), 0).$$

- For the case $\mathcal{C}$ the category of finitely generated projective modules over R. It turns out the K_i defined here coincides with the Quillen plus construction in particular with K_1 and K_2 by Bass and Milnor.

Warning

- All K-theory have the same resource, but the properties are quite different.
- There are a number of higher K-theory, differently subtly each other but most of them are far from being periodic.
- There is not generally true to have even a comparison map between different K-theory. Most of the known result is based on the computation of small spaces and the way of construction.

$\gg$ *Questions?* $\ll$

∼ § Thanks § ∼

References

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- Weibel. The K-book, an introduction to algebraic K-theory.
- Olsen. K-theory and C^*-algebra, a friendly approach.
- Quillen. Higher algebraic K-theory: I.
- Megurn. An Algebraic Introduction to K-Theory

Next time

- Computations of cohomology and characteristic classes.
- Computations of K-groups.