Topology and Geometry Seminar

K-theory (I)

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∼ § Definitions § ∼

Definition

- For further discussion, we will mainly focus on complex vector bundles.
- For a compact CW-complex X, we put the **K-group**

$$K(X) = \frac{\bigoplus_{\xi \in \mathrm{Vec}\, X} \mathbb{Z} \cdot [\xi]}{\left\langle [\xi] = [\xi_1] + [\xi_2] : \begin{array}{c}\text{there exists a short exact sequence} \\ 0 \to \xi_1 \to \xi \to \xi_2 \to 0\end{array} \right\rangle}$$

- For a compact CW-complex X with base point $*$, define $\tilde{K}(X) = \ker(K(X) \to K(*))$, called the **reduced K-group**.
- It is clear that $K(X) = \tilde{K}(X) \rtimes \mathbb{Z}$.

Definitions

- For $X_0 \subseteq X$, we define the **relative K-group**

$$K(X, X_0) = \tilde{K}(X/X_0),$$

the quotient X/X_0 is homotopy quotient rather than set-theoretic, namely, the mapping cone of the inclusion.

- For noncompact but locally compact X, we define the **K-group with compact support**

$$K(X) = \tilde{K}(X_{\cup\infty}),$$

with X_∞ the one-point-compactification with base point the infinity point. Note that it is compatible with the case when X is compact.

Motivation

Theorem

For connected finite CW-complex X,

$$K(X) = \pi(X, BGL \times \mathbb{Z}), \qquad \tilde{K}(X) = \pi(X, BGL)$$

where $BGL = \bigcup BGL_n$.

The proof (sketch)

The proof is in principle easy.

- Firstly, we can pick unitary inner product making the short exact sequence split; secondly, any vector bundle is included in some trivial bundle of finite rank.
- So, any element of $K(X)$ is presented by $[\xi] - \mathbb{1}^n$. Assume ξ is classified $\varphi : B \to BGL_N$, then define

$$\hat{\varphi} : X \longrightarrow BGL \times \mathbb{Z} \qquad x \longmapsto (\varphi(x), \operatorname{rank} \xi - n).$$

- Conversely, since X is compact, the image must lies in BGL_N for some N. So the construction is invertible.

Product structure

- For finite CW-complex X, $K(X)$ forms a commutative ring under tensor product and direct sum. We also have product $K(X, X_0) \times K(X, X_1) \to K(X, X_0 \cup X_1)$.
- Each element of $\tilde{K}(X)$ is presented by a virtual vector bundle ξ. Two $[\xi]$ and $[\eta]$ are equal in $\tilde{K}(X)$ if $\xi \oplus \mathbb{1}^n \cong \eta \oplus \mathbb{1}^m$. So $\tilde{K}(X)$ is also called **stable K-group**.

$\gg$ *Questions?* $\ll$

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Bott Periodicity and Chern Character

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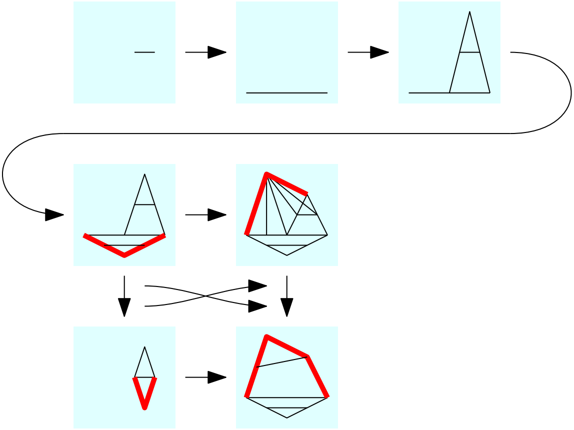
Motivations

- A general fact in homotopy theory is that we have the following exact sequence

$$\pi(X/X_0,*) \to \pi(X,*) \to \pi(X_0,*)$$

this is more or less a repetition of definition of mapping cone.

- Can we extend them? We can! Note that $X \to X/X_0$ is also an inclusion. What is the homotopy quotient would be?



Definitions

- Let us define lower K-group,

$$\tilde{K}^{-n}(X) = \tilde{K}(S^n X), \qquad K^{-n}(X) = \tilde{K}^{-n}(S^n X_{\cup\infty})$$
$$K^{-n}(X, X_0) = K(S^n(X/X_0)),$$

and admit the convention $K^0(X) = K(X)$.

Long Exact Sequences

- In this case, we get a long exact sequence of K-groups

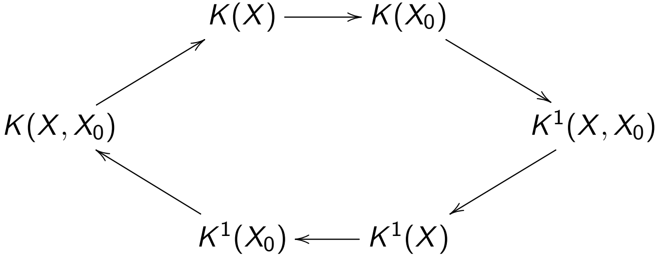
$$\cdots \to K^{-2}(X, X_0) \longrightarrow K^{-2}(X) \longrightarrow K^{-2}(X_0) \longrightarrow$$

$$\longrightarrow K^{-1}(X, X_0) \longrightarrow K^{-1}(X) \longrightarrow K^{-1}(X_0) \longrightarrow$$

$$\longrightarrow K^{0}(X, X_0) \longrightarrow K^{0}(X) \longrightarrow K^{0}(X_0)$$

Motivation

- Can we define the the K-group for positive degree?
  The answer is yes, actually, $K^n(X) = K^{n+2}(X)$ (in complex case), known as **Bott Periodicity**. So it naturally extended to positive case.
- But to be systematic, it is better to keep this in mind, and develop more powerful result getting this result as a corollary.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$K(X_0)$
$K(X, X_0)$
$K^1(X, X_0)$
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$K^1(X)$

Chern character

- Chern class is nice, but it has different rule for direct sum and tensor product with respect to K-group So we need **Chern character**.
- For a vector bundle ξ, assume its Chern class splits

$$c(\xi) = (1 + x_1) \cdots (1 + x_n) \in H^*(X).$$

Then we define the **Chern character** to be

$$\mathrm{ch}(\xi) = e^{x_1} + \cdots + e^{x_n} = \sum_{k=0}^{\infty} \frac{1}{k!}(x_1^k + \cdots x_n^k) \in \prod_{i \geq 0} H^i(X; \mathbb{Q}).$$

Chern character

- As a result, we have a ring homomorphism

$$\mathrm{ch} : K^{-n}(X) \to H^*(X;\mathbb{Q}), \qquad \mathrm{ch} : K^{-n}(X, X_0) \to H^*(X, X_0;\mathbb{Q}).$$

Note that $H^{*+n}(S^n X) = H^*(X)$.

Theorem

For finite CW-complex X, the following map is an isomorphism

$$\mathrm{ch} : K(X) \otimes \mathbb{Q} \to H^{2*}(X;\mathbb{Q}).$$

- The proof is standard by 5-lemma.

K-group for S^1 and S^2

- Note that $\tilde{K}(S^1) = \pi_1(BGL) = \pi_0(GL_\infty) = 1$, so $K(S^1) = \mathbb{Z}$, given by rank.
- Note that $\tilde{K}(S^2) = \pi(S^2, BGL) = \pi_2(BGL) = \pi_1(GL_\infty) = \mathbb{Z}$. As a result, $K(S^2) = \mathbb{Z} \times \mathbb{Z}$.
- Let $\zeta = \mathcal{O}(1)$ be the dual of tautological bundle over $S^2 = \mathbb{C}P^1$, by Chern class, ζ is the generator of $\tilde{K}(S^2)$. By Chern character, $\mathrm{ch}(\zeta) = 1 + c_2(\zeta)/2$, so as ring

$$K(S^2) = \mathbb{Z}[\zeta]/(\zeta - 1)^2.$$

First Formulation of Bott Periodicity

Theorem

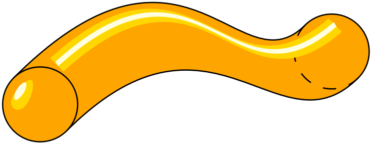
For any CW-complex X,

$$K(X \times S^2) = K(X) \otimes K(S^2).$$

given by product of vector bundles. In particular, $\tilde{K}(X \times S^2) = K(X)$.

The proof

- The main step is to decompose any vector bundle ξ over $X \times S^2$ into sum of $\alpha \otimes \mathbb{1}$ and $\alpha \otimes \zeta$.
- We cut S^2 into two pieces of discs $D_1 \cup D_2$, and we restrict ξ over $X \times D_1$ and $X \times D_2$. Assume they are presented by α as vector bundle over X.
- So to give ξ, it is equivalent to give such α, and the "clutch data" a global section of automorphism u of $\alpha \times \mathbb{1}$ over $X \times S^1$, where $S^1 = D_1 \cap D_2$.
- We will prove that we can chose u to be of certain specific form.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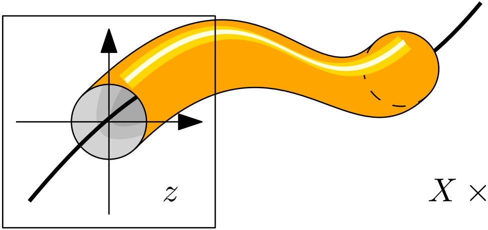
z
$X \times S^1$

The proof (continued)

- We can expand u as Fourier series index-wise, say $u = \sum_n u_n(x)z^n$, with u_n automorphism of α over X.
- Since small deformation does not change the result, we can replace u by a partial sum, say $u = \sum_{-N \leq n \leq N} u_n(x)z^n$.
- The product of z over u corresponds to tensoring $\mathbb{1}_X \otimes \zeta$ over $X \times S^2$, so to get a proof, is harmless to show the case u is a polynomial.

The proof (continued)

- Then we replace α by $\alpha \oplus \mathbb{1}^N$, and extend u by 1, this is also harmless.
- Now, u is of the form

$$\begin{pmatrix} u & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \underset{\text{digging hole}}{\simeq} \underbrace{\begin{pmatrix} u_0 & u_1 & \cdots & u_n \\ -z & 1 & & \\ & \ddots & \ddots & \\ & & -z & 1 \end{pmatrix}}_{=a(x)+zb(x)}$$

Finally, we reduce to the case u is linear in z.

The proof (continued)

- We can split α into $\alpha = \beta \oplus \gamma = \overline{\beta} \oplus \overline{\gamma}$, with

$$\begin{cases} \lambda a(x) + \mu b(x) : \beta \to \overline{\beta} & \text{and is iso when } |\lambda| \geq |\mu| \\ \lambda a(x) + \mu b(x) : \gamma \to \overline{\gamma} & \text{and is iso when } |\mu| \geq |\lambda| \end{cases}$$

  This is linear algebra from the fact that $\ker a \cap \ker b = 0$.
- In particular, $a(x)$ is isomorphic over β, and $b(x)$ is isomorphic over γ.

The proof (continued)

- Next,

$$\alpha = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \overset{\lambda a(x)+\mu b(x)}{\rightarrow} \begin{pmatrix} \overline{\beta} \\ \overline{\gamma} \end{pmatrix} \overset{\binom{a^{-1}(x)}{b^{-1}(x)}}{\rightarrow} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \alpha.$$

  When $\{^{\lambda=1}_{\mu=z}$, this is u. Over β, $a(x) + zb(x)$ is homotopy (inside auto) to $a(x)$, and over γ, $a(x) + z(b)$ is homotopy (inside auto) to $zb(x)$.
- Finally,

$$\alpha = (\beta \otimes 1) + (\gamma \otimes \zeta).$$

  The proof is complete.

Formulation of Bott Periodicity

Theorem (Bott Periodicity)
For any CW-complex X, $K^{n+2}(X) = K^n(X)$.

- Actually, we have the short exact sequence

$$0 \to \tilde{K}^{-n}(X \wedge Y) \to \tilde{K}^{-n}(X \times Y) \stackrel{*}{\to} \tilde{K}^{-n}(X \vee Y) \to 0$$

  simply because $\tilde{K}^{-n}(X \vee Y) = \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y)$, and the sum of projection $X \times Y \to X$ and $X \times Y \to Y$ forms the splitting of $*$.

The proof (continued)

- In particular, for $Y = S^2$,

$$\begin{array}{ccccccc}
0 \to \tilde{K}(S^2X) \to & \tilde{K}(X \times S^2) & \to & \tilde{K}(X \vee S^2) & \to 0 \\
\| & \| & & \| & \\
0 \to \tilde{K}^{-2}(X) \to & \tilde{K}(X) \otimes K(S^2) & \to & \tilde{K}(X) \oplus \tilde{K}(S^2) & \to 0
\end{array}$$

So $\tilde{K} = \tilde{K}^{-2}$, in particular, $K = K^{-2}$.

K-group for point

Theorem

For point pt,

$$K^n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases}$$

K-group for Spheres

Theorem

$$\tilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases}, \qquad K(S^n) = \tilde{K}(S^n) \oplus \mathbb{Z}.$$

The ring structure can be computed by Chern character.

K-groups for $\mathbb{C}P^n$

Theorem

For projective space, as ring

$$K^*(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}[\zeta]/(\zeta - 1)^{n+1}, & * \text{ is even,} \\ 0, & * \text{ is odd.} \end{cases}$$

where $\zeta = \mathcal{O}(1)$ the dual of tautological bundle.

The proof

- Note that $\mathbb{C}P^n/\mathbb{C}P^{n-1} = S^n$, so the long exact sequence

$$\begin{array}{ccc} \mathbb{Z} & \to K(\mathbb{C}P^n) \to & K(\mathbb{C}P^{n-1}) \\ \uparrow & & \downarrow \\ K^1(\mathbb{C}P^{n-1}) & \leftarrow K^1(\mathbb{C}P^n) \leftarrow & 0 \end{array}$$

- $\mathrm{ch}(\zeta) = e^{c_2(\zeta)}$, it definitely satisfies the relation in $H^*(\mathbb{C}P^n)$.

≫ *Questions?* ≪

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Thom isomorphisms and AtiyahHirzebruch SS

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Projective Bundle Theorem

Theorem (Projective Bundle Theorem)

Let ξ be a vector bundle over finite CW-complex X. we have as ring

$$K(\mathbb{P}(\xi)) = K(X)[\zeta] \Big/ (1 - \xi \cdot \zeta + \cdots + (-1)^n (\wedge^n \xi) \zeta^n),$$

and $K^1(\mathbb{P}(\xi)) = K^1(X)$.

The proof

- Firstly, that $K(\mathbb{P}(\xi))$ freely generated over $K(X)$ follows from 5-lemma. But the relation of them is a little subtle which cannot been proven in this way.
- Note that actually, we proved the splitting principle for K-theory. So we can assume ξ splits into line bundles $\xi_1 \oplus \cdots \oplus \xi_n$. Now the relation reduces to

$$(\xi_1\zeta - 1)\cdots(\xi_n\zeta - 1).$$

The proof

- Picking a nonzero section s_i of each ξ_i, such that s_i not vanish simultaneously.
- Now $\zeta^* \cong \xi_i$ over $D_i = \{s_i \neq 0\}$ thus $\xi_i\zeta - 1$ defines over $K(\mathbb{P}(\xi), D_i)$,
- Then $(\xi_1\zeta - 1)\cdots(\xi_n\zeta - 1)$ defines over

$$K(\mathbb{P}(\xi), \bigcup D_i) = K(\mathbb{P}(\xi), \mathbb{P}(\xi)) = 0.$$

Thom isomorphisms

Theorem (Thom)

For a vector bundle $\xi : E \to B$, $K(E) \cong K(B)$.

- Firstly, a warning,

$$K(E) = \pi(E_{\cup\infty}, BGL) \neq \pi(E, BGL \times \mathbb{Z})$$

which is “K-theory with compact support”, so it does not follow from the “homotomotopy invariance”.

The proof

- As stated in the first lecture, there is a number of realization of Thom space. In this case, the most convenient model is $\mathbb{P}(\xi \oplus \mathbb{1})/\mathbb{P}(\xi)$,

$$K(E) = K(\mathbb{P}(\xi \oplus \mathbb{1}), \mathbb{P}(\xi)) = \tilde{K}(\text{Thom space}).$$

- The proof is generally the same, by the 5-lemma. Note that $\mathcal{O}(1)$ over $\mathbb{P}(\xi \oplus \mathbb{1})$ restrict $\mathbb{P}(\xi)$ to be $\mathcal{O}(1)$.

Explanation

- Remind in the cohomology version of Thom isomorphism, we use push forward (through Poincaré duality).
- Clear after compactification if necessary, we can define pull back for $X \to Y$

$$K(Y) \to K(X), \qquad \tilde{K}(Y) \to \tilde{K}(X),$$

  but there is no natural push forward at this stage.
- Let $i : B \to E$ be the zero section. Note that the sheaf-theoretic push forward $i_* \mathcal{O}$ is not a vector bundle.

- But if there were some well-defined push forward making the map induced by inclusion $pt \to \mathbb{C}^n$

$$K(pt) \to K(\mathbb{C}^n) = \tilde{K}(S^{2n})$$

an isomorphism, then we could gluing together the zero section to get the Thom isomorphism.

- Note that the sheaf-theoretic push forward by zero section $i_*\mathcal{O}$ has a resolution by Kuszul complex

$$0 \to \Omega^n_{E/B} \xrightarrow{i_X} \cdots \xrightarrow{i_X} \Omega^0_{E/B} \to i_*\mathcal{O} \to 0.$$

Actually, $\Omega^*_{E/B} = \Lambda^*\xi^*(\xi^*)$ (the first ξ^* for pull back, the second ξ^* for dual of ξ).

Atiyah–Hirzebruch Spectral Sequence

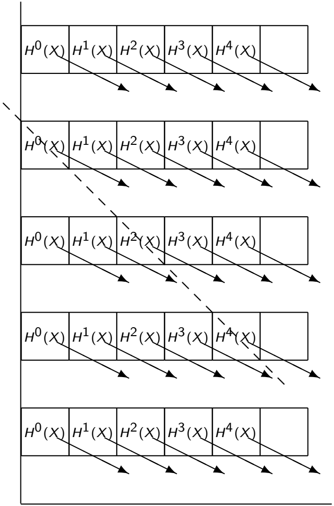
- Following the same principle by when we construct the Serre–Leray spectral sequence, we can obtain the K-theory analogy (but one needs exact couples, since K-group is not computed by a complex).

Theorem (Atiyah–Hirzebruch)

Let $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ *be a fibre bundle with fibre* F*, then there is a spectral sequence* E *such that*

$$E_2^{pq} = H^p(B; \mathcal{K}^q(F))$$

converging to $K^*(E)$*. Where* $\mathcal{K}(F)$ *is the local system of K-theory of the fibre. In particular,* $H^*(X; K^*(pt)) \Longrightarrow K^*(X)$*.*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Remarks

- We can prove Thom isomorphism by spectral sequence (after setting up multiplicative structure). I learnt this from nlab.
- The Atiyah–Hirzebruch spectral sequence firstly appeared in the study of equivariant K-theory which will be discussed late. But actually it works for any general cohomology theory.
- The differential in Atiyah–Hirzebruch spectral sequence is interesting, involving cohomology operators.

References

- Atyiah. K-Theory.
- Benson. Cohomology and Representation, second volume.
- Fomenko, Fuchs. Homotopy Topology. GTM273.
  Be careful, in this case, the K-group is defined to be $\pi(X, BGL)$, which coincides with our definition for connected case.

$\gg$ *Questions?* $\ll$

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