Characteristic Classes (II)

Xiong Rui

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Preface

- As suggested by Wang Liao that, the definition of Chern class in algebraic geometry can be found in Fulton's intersection theory with less assumption.
- Still suggested by the same audience, the computation of Chow ring of $\mathbb{P}(\mathcal{E})$ is no that easy.

∼ § Axioms of Chern Classes § ∼

Remind

- For a CW-complex B, the rank n vector bundles is classified by the homotopy class of $B \to \mathcal{G}r(n, \infty)$. Namely,

$$\mathrm{Vec}^n B \xleftrightarrow{1:1} \pi(B, \mathcal{G}r(n, \infty)).$$

- For a vector bundle ξ over B, which is classified by φ, we define the Chern classes

$$c(\xi) = 1 + c_2(\xi) + c_4(\xi) + \cdots + c_{2n}(\xi), \qquad c_{2k} = \varphi^*(e_{2k})$$

with $e_{2k} \in H^{2k}(\mathcal{G}r(n, \infty))$, the description, see last lecture.

Properties (1) — Functorial Property

- For $f : X \to Y$, and ξ a vector bundle over Y, we define $f^*\xi$ the pull back (see the diagram below).
- The Chern classes are functorial.

$$f^*c(\xi) = c(f^*\xi). \quad \left| \begin{array}{ccc} E(f^*\xi) & \to & E(\xi) \\ \downarrow & \text{pull back} & \downarrow \xi \\ X & \to & Y \end{array} \right.$$

This property is indicated from the definition.

Properties (2) — Whitney Sum

- Chern class acts well for direct sum

$$c(\xi \oplus \eta) = c(\xi)c(\eta).$$

  This follows from our computation of e_{2k}.

$$\mathbb{C}P^\infty \times \overset{n}{\cdots} \times \mathbb{C}P^\infty \to \mathcal{G}r(n,\infty).$$

- By factoring through $\mathcal{F}\ell(n,\infty)$ one can also show $c(\xi) = c(\eta)c(\xi/\eta)$, for any sub-vector bundle $\eta \subseteq \xi$.

Properties (3) — Special Value

- $c_0 = 1$.
- For tautological bundle τ over $\mathbb{C}P^\infty$,

$$-c_2(\tau) = \text{the canonic generator of } H^*(\mathbb{C}P^\infty),$$

  the Poincaré dual of hyperplane class. Equivalently,

$$c_2(\tau) = e(\tau) = \text{the Euler class of } \tau.$$

- We can change ∞ by each N.
- Actually, generally, the top Chern class is the Euler class in general.

Axioms of Chern classes

Theorem

The following axioms characterize the total Chern classes as the assignment $\operatorname{Vec} B \to H^*(B)$ *for each CW-complex* B.

- $f^*c(\xi) = c(f^*\xi)$. *Functorial Property*
- $c(\xi \oplus \eta) = c(\xi)c(\eta)$. *Whitney Sum*
- $c(\tau) = 1 + e(\tau)$. *recover Euler Class for projective space*

The proof is clear.

$\gg$ ***Questions?*** $\ll$

∼ § CC in Differential Geomtry § ∼

The de Rham Cohomology

- Let us fix some (not standard) notation, denote the fibre bundle and the space of global sections of differential forms,

$$\Lambda^* M, \qquad \Omega^*(M).$$

- Recall de Rham cohomology complex

$$0 \to \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0.$$

It computes $H^*(M; \mathbb{R})$.

Connections

- For a (smooth) vector bundle E over a smooth manifold M. We consider the vector bundle E but with coefficient differential forms, and its global section

$$\Lambda^* M \otimes E, \qquad \Omega^*(M; E).$$

- Define the **connection** ∇ over E to be a map

$$\Omega^*(M; E) \to \Omega^{*+1}(M; E)$$

such that for $\alpha \in \Omega^*(M)$, and $s \in \Omega^*(M; E)$,

$$\nabla(\alpha \wedge s) = d\alpha \otimes s + (-1)^{\deg \alpha} \alpha \wedge \nabla(s).$$

Connections (continued)

- For any
$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$
satisfying
$$\nabla(fs) = df \otimes s + f\nabla s$$
uniquely extends to a connection.
- Note that ∇ is generally not a tensor, that is, not an element in
$$\Gamma(M; \mathrm{Hom}(\Lambda^* M \otimes E, \Lambda^{*+1} M \otimes E)).$$

Curvature Tensor

- But its square $\nabla^2 : \Omega^0(M;E) \to \Omega^2(M;E)$ is, known as the **curvature tensor** K

$$\Gamma(M; \mathrm{Hom}(\Lambda^0 M \otimes E, \Lambda^2 M \otimes E)) = \Omega^2(M; \mathrm{End}(E)).$$

- To check one thing is a tensor, it suffices to show it is linear with respect to $C^\infty(M)$. So

$$\begin{aligned} \nabla\nabla(fs) &= \nabla(df \otimes s + f\nabla s) \\ &= d(df) \otimes s - df \otimes \nabla s + df \otimes \nabla s + f\nabla\nabla s \\ &= f\nabla\nabla s. \end{aligned}$$

Curvature Tensor

- It is useful to see how it locally looks like. Denote $\xi_i = \frac{\partial}{\partial x^i}$ the local tangent fields with respect to local coordinate $\{x_i\}$.

$$\nabla \xi_i = \sum \theta^{ij} \xi_j, \qquad \theta^{ij} \in \Omega^1(M).$$

- Then

$$\begin{aligned} \nabla\nabla\xi_i &= \nabla\big(\sum \theta^{ij}\xi_j\big) \\ &= \sum d\theta^{ij} \otimes \xi_j - \sum \theta^{ij} \wedge \theta^{jk} \xi_k \end{aligned}$$

So $K = d\theta - \theta \wedge \theta$.

Chern Classes

- Then consider the curvature tensor K as matrix with coefficients in $\Omega^2(M)$. The index-wise d,

$$\begin{aligned} dK &= d(d\theta - \theta \wedge \theta) \\ &= -d\theta \wedge \theta + \theta \wedge d\theta \\ &= -K \wedge \theta + \theta \wedge K = [\theta, K]. \end{aligned}$$

- An genius observation is that any symmetric polynomial in eigenvalues is closed, since the following genius observation

$$\begin{aligned} d\,\mathrm{tr}\,K^k &= \sum \mathrm{tr}[K \cdots dK \cdots K] \\ &= \sum \mathrm{tr}[K \cdots [\theta, K] \cdots K] = 0. \end{aligned}$$

Chern Classes

- We can define the Chern classes $c_{2k}(E;\nabla) \in H^2(M;\mathbb{C})$ to be

$$\frac{1}{(2\pi i)^k}(\text{k-th elementary symmetric polynomial in eigenvalues of } K)$$

  which appears as coefficients of characteristic polynomial.

- The Chern class is actually homotopy invariant. Consider the map "integral over $[0,1]$"

$$Q : \Omega^*(M \times [0,1]) \to \Omega^{*-1}(M).$$

  It satisfies $dQ - Qd = i_1^* - i_0^*$. Apply this formula on the curvature tensor over $M \times [0,1]$, we get the homotopy invariance.

- In particular, the class does not rely on the choice of ∇.

Chern Classes

Theorem

The Chern classes defined above is the same as we defined for topological space after complexification and tensoring over $\mathbb{C}$ (actually over $\mathbb{R}$).

- To prove this, simply check the axioms.
- To make it is functoral, one can define the pull back of connection.
- By direct sum of connections, it is easy to check it has the Whitney sum property.
- To work better in the category of smooth manifold, we should consider $\mathbb{C}P^N$ for all N.

- In terms of Riemannian Geometry, and $E = TM$, we defined the Levi–Civita conenction $\nabla_X Y$, which turns out to be a tensor in X, so it defines a connection $\nabla : \mathfrak{X}(M) = \Omega^0(M; TM) \to \Omega^1(M; TM)$. Namely by

$$\langle \nabla Y, X \otimes 1 \rangle = \nabla_X Y.$$

- Recall the **Riemannian curvature tensor**

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

I claim this is exactly the tensor we just defined.

The proof

- Let us denote the pairing $\langle X \wedge Y \otimes \mathrm{id}, s\rangle = s(X, Y)$ for $s \in \Omega^2(M; TM)$ for short, the similar for Ω^1.
- Since ∇ is torsionfree, there is a formula for $s \in \Omega^1(M; TM)$,

$$(\nabla s)(X, Y) = \nabla_X\big(s(Y)\big) - \nabla_Y\big(s(X)\big) - s([X, Y])$$

  Actually, this can be understood as a generalization of the formula $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$.
- In particular, when $s = \nabla Z$, a fortiori $s(A) = \nabla_A Z$,

$$(\nabla^2 Z)(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X, Y)Z.$$

$\gg$ *Questions?* $\ll$

∼ § CC in Algebraic Geometry § ∼

Chow Groups

- We can define for a variety X over algebraic closed field $\Bbbk$ a Chow group by
$$\mathrm{Ch}^p(X) = \frac{\sum_{\substack{\text{subvarity } Y \\ \text{codim } Y = p}} \mathbb{Z} \cdot [Y]}{\text{Rational equivalence}}.$$
- Two subvarieties Y_0, Y_1 are said to be rational equivalent if there is an codimensional $p+1$ irreducible $Z \subseteq \mathbb{P}^1_X$ which is not "vertical" with
$$Y_0 = Z \cap (X \times \{0\}), \qquad Y_1 = Z \cap (X \times \{1\}),$$
Here vertical means the projection of Z to $\mathbb{P}^1_{\Bbbk}$ is not a point (say, Z not live over one point).

Chow Rings

- When X is smooth, then Chow group is equipped with a ring structure (in which case, it will be called Chow ring) with the product by "transversal intersection".
- For smooth (regular) variety X, $\mathrm{Ch}^1(X)$ is exactly the class group $\mathrm{Cl}(X)$. The same story appears, so we hope to define Chern class for algebraic vector bundle (locally trivial sheaf).
- Tips: there is no 2!

Chern Classes

- The construction is due to Grothendieck.
- For a locally trivial sheaf $\mathcal{E}$ of rank n over X, we can define the associated projective bundle $\mathbb{P}(\mathcal{E}) \to X$.
- Then $\mathcal{O}(1)$ is defined over $\mathbb{P}(\mathcal{E})$, which defines an element $\zeta \in \mathrm{Ch}^1(\mathbb{P}(\mathcal{E}))$. By more effort, one can show that $\mathrm{Ch}(\mathbb{P}(\mathcal{E}))$ is freely generated by $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ over $\mathrm{Ch}(X)$ by excision property of Chow ring.
- As a result, there is a relation

$$\zeta^n + c_1\zeta^{n-1} + \cdots + c_n = 0 \in \mathrm{Ch}(\mathbb{P}(\mathcal{E})), \qquad c_k \in \mathrm{Ch}(X).$$

Then we simply define the Chern class $c_k(\mathcal{E}) = c_k$.

The Axioms of Chern Classes

Theorem

The following axioms characterize the total Chern classes as the assignment $\operatorname{LocTri} X \to \operatorname{Ch}(X)$ *for each smooth varieties* X.

- *For any morphism of variety* *Functorial Property*

$$f^* c(\mathcal{E}) = c(f^* \mathcal{E}).$$

- *For short exact sequence* $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ *Whitney Sum*

$$c(\mathcal{E}) = c(\mathcal{F}) c(\mathcal{G}).$$

- *The first Chern class is constant* $1 = [X]$. *The second Chern class gives the isomorphism* $\operatorname{Cl}(X) \to \operatorname{Ch}^1(X)$ *we stated before.*

The proof

- The theorem should be proven simultaneously with splitting principle. But we cannot have an "algebraic" splitting, but a filtration. Say, consider $\mathcal{F}\ell(\mathcal{E}) \to X$, with the fibre of $x \in X$ to be $\mathcal{F}\ell(\mathcal{E} \otimes k(x))$.
- The the tautological flag forms a filtration of pull back of E.
- This is essentially the same to what we did last time, since $\mathcal{F}\ell(\mathcal{E})$ can be built by set-by-set projective bundles.
- Repeat what we do for projective bundles, we can conclude that $\mathrm{Ch}(X) \to \mathrm{Ch}(\mathcal{F}\ell(\mathcal{E}))$ is injective.

The proof

- Then we can assume that we have

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{E},$$

  Denote $\mathcal{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$. We can also assume $\mathcal{F}_i/\mathcal{F}_{i-1}^*$ has nonzero global section from the construction.
- By picking a nonzero section over $(\mathcal{F}_i/\mathcal{F}_{i-1})^*$, we can talk about "coordinate".
- Put $\pi : \mathbb{P}(\mathcal{E}) \to X$. Now $\mathcal{O}(-1) \cong \pi^*\mathcal{Q}_i$ over the open set of the i-th coordinate none zero.
- So $\zeta + c(Q_i)$ can be chosen to be combination of subvarities out of this region. But these regions intersect empty. The proves $(\zeta + c(Q_1)) \cdots (\zeta + c(Q_n)) = 0$.

$\gg$ *Questions?* $\ll$

$\sim$ § General Characteristic Classes § $\sim$

Stiefel–Whitney Classes

- For $\mathbb{R}$, as we stated, real line bundle is classifies by $H^1(X;\mathbb{Z}/2)$. So we also has a systematic class to extend this, called the **Stiefel–Whitney Classes**.
- We can do the same computation for real Grassmanian, say the map

$$\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \to \mathcal{G}r(n,\infty)$$

induces an injection in cohomology group

$$H^*(\mathcal{G}r(n,\infty);\mathbb{Z}/2) \to H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty) = \mathbb{Z}/2[x_1,\ldots,x_n]$$

with image exactly the symmetric polynomials. So we can define Stiefel–Whitney class similar to Chern class.

Theorem

The following axioms characterize the total Stiefel–Whitney classes as the assignment $\operatorname{Vec} B \to H^*(B; \mathbb{Z}/2)$ *for each CW-complex* B.

- $f^* c(\xi) = s(f^* \xi)$. *Functorial Property*
- $s(\xi \oplus \eta) = s(\xi) s(\eta)$. *Whitney Sum*
- $s(\tau) = 1 + e(\tau)$. *recover Euler Class for projective space*

Pontryagin Classes

- One can consider the Chern class of the complexification of real bundle. It is well-defined for any coefficient ring with $1/2 \in R$, and only nontrivial terms appear in H^{4*}. This is (up to a subtle sign) called **Pontryagin classes**.
- Actually, we can compute

$$H^*(\mathcal{G}r(n,\infty); R) = R[p_4, \dots, p_{4n}].$$

One can show they coincide, and they also admit an axioms description.

Principal bundle

- We call a locally trivial fibre bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ a G-principal bundle if G acts on E freely, with $B = E/G$, and ξ the natural projection. Equivalently, each fibre is a copy of G, and are glued by left multiplication of G. Here is left because

$$\mathrm{Aut}_{\text{right } G}(G, G) = \text{left multiplication of } G.$$

Principal bundle

- For vector bundle ξ, we can naturally associate to a GL_n-principal bundle by picking invertible element in $F \otimes F^*$.
- Conversely, for a GL_n-principal bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, we can define $E \times_G \mathbb{F}^n$ a vector bundle over B.
- So there is a natural equivalence

$$\mathrm{Vect}^n B \xleftrightarrow{1:1} \mathrm{GL}_n\text{-Prin}\, B.$$

- This allows us to generalize the classifying theorem.

Milnor construction

Theorem (Milnor)

For any topological group G of CW-type then there is a principal bundle $\begin{bmatrix} E_G \\ \downarrow \\ B_G \end{bmatrix}$, such that for any CW-complex B

$$G\text{-Prin}\, B \xleftrightarrow{1:1} \pi(B, B_0).$$

Besides, this E_G is contractible, and any principal bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ satisfies this property if and only if E is contractible.

In some way, to find out a classfying space is simply to find a big enough space such that G acts freely and contractible.

Examples

- For discrete group G, $B_G = K(G, 1)$ the Eilenberg-MacLane space, E_G its universal covering.
- For $G = \mathrm{GL}_n$, then $B_G = \mathcal{G}r(n, \infty)$, by the equivalence what we stated, and E_G corresponding to the tautological bundle.
- For $G = \mathrm{O}_n$, then B_G is exactly real Grassmannian, by picking Riemannian metric, E_G the Stiefel variety (say, the fibre at L is the set of choice of orthogonal basis L).
- For $G = \mathrm{U}_n$, then B_G is exactly complex Grassmannian, by picking unitary metric.
- For $G = \mathrm{Sp}_{2n}$, then B_G is the exactly Quaternionic Grassmannians.

General Characteristic Classes

- Actually,

Any computation of $H^*(B_G)$ gives a theory of characteristic classes

  .

  As we did for vector bundle.

- For example Euler class appear in the

$$H^*(BSL_n(\mathbb{R}); R) = H^*(BSO_n(\mathbb{R}); R)$$

for $1/2 \in R$. Note that BSO_n is a two fold covering of BO_n.

General Characteristic Classes

- The same story holds for manifold — characteristic classes can be computed by connection and curvature. But in this case curvature should be Lie-algebra-valued.
- Actually, for a compact Lie group or a reductive group, G, we know

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$$

  where G acts on the polynomial over its Lie algebra $\mathfrak{g}$ by adjoint action, $\mathfrak{h}$ the Cartan subalgebra, Lie algebra of torus, and W the Weyl group.
- In the case of $G = GL_n(\mathbb{C})$, the left hand side is the symmetric polynomials in eigenvalues.

General Characteristic Classes

- Generally, the right hand side $\mathbb{C}[\mathfrak{h}]^W$ can be shown to be $H^*(BG;\mathbb{C})$ (lectures later).
- The **Chern–Weil theory** set up the exact relations. The most famous result is the **Chern–Gauss-Bonnet theorem**

$$\left(\frac{-1}{2\pi}\right)^k \int_M \mathrm{Pf}\, K = \chi(M)$$

where M is a $2k$-dimensional compact Riemannian manifold, Pf the **Pfaffian** and $\chi(M)$ the Euler characteristic.
- More precisely, $\mathrm{Pf} \in \mathbb{C}[\mathfrak{so}_{2k}]^{\mathrm{SO}_{2k}}$ corresponds to the Euler class $e \in H^k(BSO_{2k};\mathbb{C}) = \mathbb{C}[x_1,\ldots,x_k]^{D_n}$.

$\gg$ *Questions?* $\ll$

~ § Vector Bundles over Spheres § ~

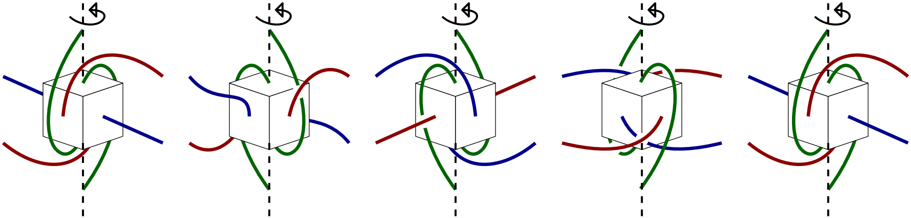
Vector bundles over spheres

- For $B = S^n$, and $\mathbb{F} = \mathbb{R}$,

$$\begin{aligned} \mathrm{Vec}^k\, S^n &= \pi(S^n, BGL) \\ &= \pi_n(BGL_k) && \because \text{GL is connected} \\ &= \pi_{n-1}(\mathrm{GL}_k) && \because \text{long exact sequence} \\ &= \pi_{n-1}(\mathrm{O}_k) \end{aligned}$$

  This is classically proven by the contractibility of the Stiefel varieties.
- So we have the following table

$\mathbb{R}$	Vec^1	Vec^2	Vec^3	Vec^4	Vec^5	$\cdots$
S^1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdots$
S^2	1	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdots$
S^3	1	1	1	1	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$



$$\mathrm{SO}_3 \cong \mathbb{R}P^3.$$

$$n \geq 3 \Longrightarrow \pi_1(\mathrm{SO}_n) = \mathbb{Z}/2.$$

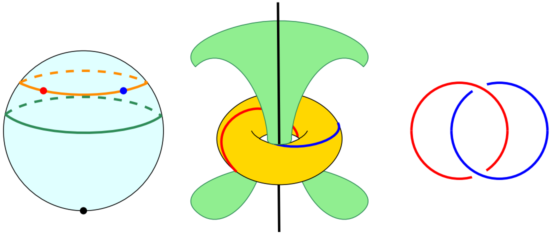
Vector bundles over spheres

- For $B = S^n$, and $\mathbb{F} = \mathbb{C}$,

$$\mathrm{Vec}^k\, S^n \quad = \pi_{n-1}(\mathrm{U}_k)$$

- So we have the following table

$\mathbb{C}$	Vec^1	Vec^2	Vec^3	Vec^4	Vec^5	$\cdots$
S^1	1	1	1	1	1	$\cdots$
S^2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
S^3	1	1	1	1	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$



The Hopf vector bundle (tautological bundle of $S^2 = \mathbb{C}P^1$) is the generator. The unit circle form $\begin{bmatrix} S^3 \\ \downarrow \\ S^2 \end{bmatrix}$ with fibre S^1, the famous Hopf fibration.

The second homotopy groups of Lie groups vanish

- Note that the flag manifold G/T is simply connected, and $H^*(G/T)$ is free abelian. In particular, by Hurewicz theorem, $\pi_2(G/T)$ is free abelian.
- Then consider the long exact sequence

$$\underbrace{\pi_2(T)}_{=0} \to \pi_2(G) \to \pi_2(G/T) \to \pi_1(T) \to \pi_1(G) \to \underbrace{\pi_1(G/T)}_{=0}$$

- The connection map can be computed by the standard trick using SU_2 map. This computes

$$\pi_1(G) = \ker\exp / \langle h_\alpha \in \mathfrak{h} : \alpha \in \text{root} \rangle, \qquad \pi_2(G) = 0.$$

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$\gg$ <u>*Questions?*</u> $\ll$

$\sim$ § THANKS § $\sim$