Topology and Geometry Seminar

Characteristic Classes (I)

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$\sim$ § Vector Bundles § $\sim$

Locally Trivial Fibre Bundles

- For a map $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ and $U \subseteq B$, denote $E|_U$ the preimage of U.
- A map $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ is called **locally trivial fibre bundle** if

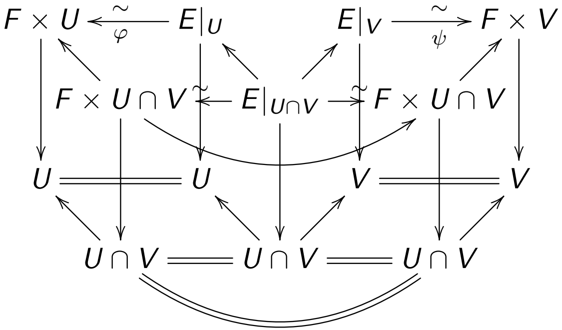
  For any $x \in B$, there is a neighborhood U, such that $\begin{bmatrix} E|_U \\ \downarrow \\ U \end{bmatrix} = \begin{bmatrix} F \times U \\ \downarrow \\ U \end{bmatrix}$

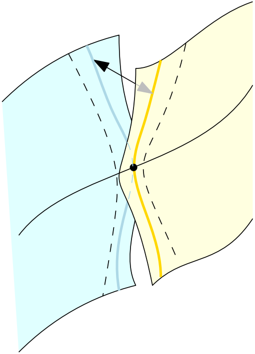
$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & F \times U \\ \downarrow & & \downarrow \text{proj} \\ U & = & U \end{array}$$

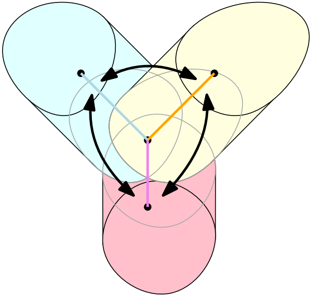
The isomorphism $\varphi : E|_U \to F \times U$ is called a **trivialization** or **local coordinate** of U.

Structure Group

- For a locally trivial fibre bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, two localizations define a automorphism over intersection. Namely, if φ and ψ are the localization of U and V respectively, then there is a map $U \cap V \to \mathrm{Aut}(F, F)$, such that







Vector Bundles

- Denote $G = \mathrm{GL}_n(\mathbb{F})$, and $F = \mathbb{F}^n$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.
- A rank n **vector bundle** is a local trivial fibre bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, with giving local coordinates of an open covering of B, say, $\{(U_i, \varphi_i) : i \in I\}$ with the map $\varphi_{ji} : U_i \cap U_j \to \mathrm{Aut}(F, F)$ continuous and taking value in G continuously.
- Formally, we need to assume $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$ whenever defined.

Properties

- Philosophically, vector bundle is a local trivial fibre bundle of fibre a vector space, but we can distinguish stuff in the vector space what GL can distinguish (see below).
- A (global) section of a fibre $\xi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ is a map $\sigma : B \to E$ with value of $x \in B$ taking value in its fibre, say $\xi \circ \sigma = \mathrm{id}_B$.
- There exists a section for a vector bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, just assign **the** 0 in each fibre at for each $x \in B$, (but generally not for fibre bundles of fibre vector space).

Operators

- The bundle $\mathbb{1} = \begin{bmatrix} F \times B \\ \downarrow \\ B \end{bmatrix}$ is called the **trivial bundle**.
- We take what we can do for vector bundle, say, dual space, tensor product, Hom, symmetric product, exterior product, direct sum. It has all the **natural** isomorphisms like

$$\mathrm{Hom}(\xi, \mathbb{1}) = \xi^*, \quad \mathrm{Hom}(\xi \otimes \eta, \zeta) = \mathrm{Hom}(\xi, \mathrm{Hom}(\eta, \zeta)), \quad \ldots$$

- For example, direct sum (historically called the **Whitney sum**). We firstly take local coordinate, then do direct sum locally, and finally glue them up.
- Generally, kernel, image, cokernel may not be vector bundle except the map is of local constant rank.

Reduce to Compact Group

- For a CW-complex B (or general paracompact space), we can find a Riemmanian metric over any real vector bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$. Say, there is a positive definite section of $\mathrm{Hom}(\xi \otimes \xi, \mathbb{1})$ (by partition of unity).
- In particular, it is harmless to change $G = \mathrm{O}_n$.
- As a result, $\xi \cong \xi^*$ for real vector bundle.
- For a CW-complex B, we can find a unitary metric over any complex vector bundle $\xi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$.
- In particular, it is harmless to change $G = \mathrm{U}_n$.
- Note, there is no natural isomorphism $\xi \cong \xi^*$ for unitary space.

$\gg$ <u>*Questions?*</u> $\ll$

∼ § Classifying Vector Bundles § ∼

Homotopy-invariance

Theorem

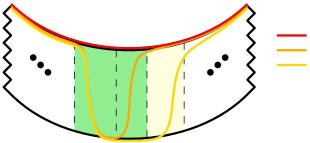
For a CW-complex X, for any vector bundle ξ over $X \times I$, then

$$\xi|_{X\times 0} \cong \xi|_{X\times 1}.$$

$$\begin{array}{ccccccc} & E|_{X\times 0} & \rightarrow & E & \leftarrow & E|_{X\times 1} & \\ & \downarrow & & \downarrow & & \downarrow & \\ X = & X\times 0 & \rightarrow & X & \leftarrow & X\times 1 & = X \end{array}$$

Proof

- We can cover $B \times I$ by countable open sets of the form $U \times (a, b)$ such that ξ is trivial over any of them.
- Since along I, we can easily glue the trivialization by "direction" $0 \to 1$, refining U if necessary, we can actually assume them to be $U \times I$.
- Denote $\xi_1 = \xi|_{X \times 1}$ and $\xi_0 = \xi|_{X \times 0}$. We can exchange ξ_1 to ξ_0 over U_i one by one, by restriction on the graph of partial sum of partition of unity.



Classifying map

- Denote the **Grassmannian** manifold

$$\mathcal{G}r(k,n) = \{\text{k-dimensional subspace } V \subseteq \mathbb{F}^n\}.$$

By $\mathbb{F}^n \hookrightarrow \mathbb{F}^{n+1}$, it defines

$$\mathcal{G}r(k,\infty) = \varinjlim_{n\to\infty} \mathcal{G}r(k,n) = \bigcup_{n\geq 0} \mathcal{G}r(k,n).$$

Theorem

Let B be a CW-complex (still, or general paracompact space). There is a natural bijection between

$$\mathrm{Vect}^k_{\mathbb{F}} B \leftrightarrow \pi(B, \mathcal{G}r(k,\infty)),$$

where the left side is the set of equivalent classes of rank k vector bundles and the right hand side stands for the homotopic class of map $B \to \mathcal{G}r(k,\infty)$.

Tautological Bundle

- Consider the **tautological bundle**

$$\tau = \begin{bmatrix} \{(V,x) \in \mathcal{G}r(k,\infty) \times \mathbb{F}^\infty : x \in V\} \\ \downarrow \\ \mathcal{G}r(k,\infty) \end{bmatrix}$$

That is, the fibre at V is $V \subseteq \mathbb{C}^\infty$ itself.

- Note that

$$\tau \subseteq \begin{bmatrix} \mathcal{G}r(k,\infty) \times \mathbb{F}^\infty \\ \downarrow \\ \mathcal{G}r(k,\infty) \end{bmatrix}$$

the trivial "infinite vector bundle". The coordinate map $x_i : \mathcal{G}r \times \mathbb{C}^\infty \to \mathbb{C}^\infty \to \mathbb{C}$ restricts to τ, which defines a set of global sections $\{x_i\}$ of τ^*.

First proof

- Let $\xi : \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ be a vector bundle, we can find a set of global sections $\{s_i\}$ such that at each point s_i spans the fibre (by partition of unity), then

  we hope to define the map ϕ and Φ of fibre bundles (see right) such that $x_i \circ \varphi = \Phi \circ s_i$.

  $$\begin{array}{ccc} E & \xrightarrow{\Phi} & E(\tau^*) \\ \downarrow & & \downarrow \\ B & \xrightarrow[\phi]{} & \mathcal{G}r \end{array}$$

- If so, then this is a pull back map, thus E is determined by ϕ.

First proof (continued)

- For $b \in B$, and $v \in E_b$, assume v is write as $\sum c_i s_i(b)$. We can define

$$\Phi(v) = \sum c_i x_i = \sum c_i x_i(\varphi(b))$$

and

$$\phi(b) = \{(x_1, \ldots) \in \mathbb{C}^\infty : {}^{a_1 s_1(b)+\cdots=0}_{\Rightarrow a_1 x_1+\cdots=0}\} \in \mathcal{G}r(k, \infty).$$

- This is well-defined, and proves our first claim.

First proof (continued)

- Actually, conversely, any pull back τ^* has such global sections.
- So for one fixed vector space, different choice of map $B \to \mathcal{G}r(k, \infty)$ corresponds to the different choice of $\{s_i\}$. Then the homotopy of global sections $\{(1-t)s_i\} \cup \{ts_i'\}$ (still countable), gives rise a homotopy of $B \to \mathcal{G}r(k, \infty)$.
- There are two gaps, we should show changing order of $\{s_i\}$ and removing zeros in $\{s_i\}$ are harmless. But this is easy to check.

Second proof

- Let $\xi : \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ be a vector bundle, we hope to include ξ in some trivial bundle (maybe infinite),

$$\begin{array}{ccccc} \mathbb{F}^\infty \times B & \supseteq & E & \xrightarrow{\Psi} & E(\tau) \\ & \searrow & \downarrow & & \downarrow \\ & & B & \xrightarrow[\psi]{} & \mathcal{G}r \end{array}$$

  Then we simply map $b \in B$ to its fibre $E_b \subseteq \mathbb{F}^\infty$.
- If so, then this is a pull back map, thus E is determined by ψ.

Second proof (continued)

- We can find a countable open covering $\mathcal{U} = \{U_i\}$ with partition of unity $\{\iota_i\}$, and $E|_{U_i}$ is trivial for all U_i, say by the trivialization by $\varphi_i : E|_{U_i} \to \mathbb{F}^n \times B$ for each i.
- Since $(\mathbb{F}^n)^\infty = \mathbb{F}^\infty$, we can define $E \to \mathbb{F}^\infty \times B$ by

$$(\underbrace{\iota_1\varphi_1}_{n\text{-tuple}}, \underbrace{\iota_2\varphi_2}_{n\text{-tuple}}, \ldots) \in \mathbb{F}^\infty.$$

As we desired.

Second proof (continued)

- Still, any pull back of τ gives an inclusion $E \subseteq \mathbb{F}^\infty \times B$.
- So for one fixed vector space, different choice of map $B \to \mathcal{G}r(k, \infty)$ corresponds to the different embedding $\mathbb{F}^\infty \times B$.
- By $\mathbb{F}^{2\infty} \oplus \mathbb{F}^{2\infty+1} = \mathbb{F}^\infty$, the obvious homotopy

$$\begin{aligned}(*, *, *, \cdots) \quad &\simeq (*, 0, *, 0, *, 0, \cdots) \\ &\simeq (0, *, 0, *, 0, *, \cdots) \\ &\simeq (*, *, *, \cdots).\end{aligned}$$

preserves the property of being k-dimension of E. As a result, it induces a homotopy of $B \to \mathcal{G}r(k, \infty)$.

Remark

- It is equivalent to say $\text{Vect}^k B$ is representable.
- Two proofs use different bundles τ^* and τ. The first proof maybe of more algebraic geometry, the latter is more topological geometry.
- There is also an analogue in algebraic geometry where there are two defects. Firstly, there is not $\mathcal{G}r(k,\infty)$ but only $\mathcal{G}r(k,n)$; secondly, we may not have enough global sections. Adding more restriction, we can get a so-called "universal property of Grassmannian".

$\gg$ *Questions?* $\ll$

∼ § Chern Classes § ∼

Line bundles

- Just a notation, **line bundle** is defined to rank 1 vector bundle, and the set of equivalent classes is denoted by $\mathrm{Cl}(X) = \mathrm{Vec}^1(X)$ called the **class group**.
- For $k = 1$, $\mathcal{G}r(1, \infty) = \mathbb{P}(\mathbb{F}^\infty)$, the projective space. We know

$$\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1), \qquad \mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

the **Eilenberg-MacLane space**. As a result,

$$\mathrm{Cl}^1_{\mathbb{R}} B = \pi(B, \mathbb{R}P^\infty) = H^1(X, \mathbb{Z}/2),$$
$$\mathrm{Cl}^1_{\mathbb{C}} B = \pi(B, \mathbb{C}P^\infty) = H^2(X, \mathbb{Z}).$$

Euler class again

- Assume B is a compact oriented smooth manifold. Assume we work in $\mathbb{C}$.
- Let ξ be a vector bundle over B, then it is classified by $\varphi : B \to \mathbb{C}P^n$ for n big (since B is compact).
- By definition, the cohomology class in $H^2(X, \mathbb{Z})$ is given by $\varphi^*(h)$ where h is chosen to be the Poincaré duality of homotopy class of any hyperplane $H \subseteq \mathbb{C}P^n$.
- We can change φ by a smooth map transverse to this hyperplane H. Then $\varphi^*(h)$ is the Poincaré duality of homologic class of $\varphi^{-1}(H) \subseteq B$.

Euler class again (continued)

- Let us fix the bundle over $\mathbb{C}P^n$ to be τ^*, that is, we have the following pull back

$$\begin{array}{ccc} E & \rightarrow & E(\tau^*) \\ \downarrow & & \downarrow \\ B & \underset{\varphi}{\rightarrow} & \mathbb{C}P^n. \end{array}$$

  Note that H is the zero locus of a linear map (say, x_1), that is, a section of τ^*. So, $\varphi^*(h)$ is exactly the zero locus of a general section of ξ.
- This is exactly the Euler class of E!
- Remark: If the geometric meaning is correct, then it can be proven by identities of dual/cup/cap product.

Motivate Chern classes

- For general rank n vector bundle ξ, it locally looks like $\mathbb{C}^n \times B \to B$, so a section can be viewed as n functions. The **Chern classes** is roughly defined in analogy to the Euler class by

$$c_{2k} = \text{Poincaré dual to } \{k \text{ of the functions vanish}\} \in H^{2k}(B).$$

- If ξ is a direct sum of n line bundles, say $\xi_1 \oplus \cdots \oplus \xi_n$, then the Chern class should be defined to be

$$1 + c_2(\xi) + \cdots = (1 + e(\xi_1)) \cdots (1 + e(\xi_n))$$

where $e = c_2$ the Euler class.

Construction

- Consider
$$\mathbb{C}P^\infty \times \overset{n}{\cdots} \times \mathbb{C}P^\infty \to \mathcal{G}r(n,\infty)$$
induced by some choice of $\mathbb{C}^\infty \times \overset{n}{\cdots} \times \mathbb{C}^\infty = \mathbb{C}^\infty$. It is clear, pull back of tautological bundle is still tautological.
- To see this induced an injection in cohomology, we need the intermediary of flag manifold. Consider the Flag manifold $\mathcal{F}\ell(n,\infty)$ the set of flags of length n in $\mathbb{C}^\infty$.

Construction

- Then we have the following

$$\mathbb{C}P^\infty \times \overset{n}{\cdots} \times \mathbb{C}P^\infty \to \mathcal{F}\ell(n,\infty) \to \mathcal{G}r(n,\infty)$$

- By the Serre–Leray spectral sequence or the Leray-Hirsch theorem, the cohomology map induced by the first map is isomorphic, say $H^*(\mathcal{F}\ell(n,\infty)) = \mathbb{Z}[x_1, \ldots, x_n]$; the cohomology map induced by $\mathcal{F}\ell(n,\infty) \to \mathcal{G}r(n,\infty)$ is injective, and the image is the symmetric polynomials.
- So we simply define c_{2k} to be the k-th elementary polynomial in x_i.

Chern Classes

- So we can define the **Chern class** for a rank n vector bundle ξ over a CW complex X to be

$$c(\xi) = 1 + c_2(\xi) + \cdots + c_n(\xi), \qquad c_{2k} = \varphi^*(c_{2k}),$$

with $\varphi : X \to \mathcal{G}r(n, \infty)$ the classifying map i.e. the pull back of τ^* is ξ. Call c_{2k} the ***k*-th Chern class**.

Properties of Chern Classes

- From definition, we can see the **Whitney** property

$$c(\xi \oplus \eta) = c(\xi)c(\eta),$$

since

$$\begin{array}{ccc} (\mathbb{C}P^\infty)^n \times (\mathbb{C}P^\infty)^m & = & (\mathbb{C}P^\infty)^{m+n} \\ \downarrow & & \downarrow \\ \mathcal{G}r(m+n,\infty) & \leftarrow & \mathcal{G}r(m,\infty) \times \mathcal{G}r(n,\infty) \end{array}$$

Splitting principle

Theorem

For any vector bundle ξ over some CW complex X, there always exists $Y \xrightarrow{f} X$ such that $f^* : H^*(X) \to H^*(Y)$ is injective, and the pull back of ξ splits into line bundles.

Proof

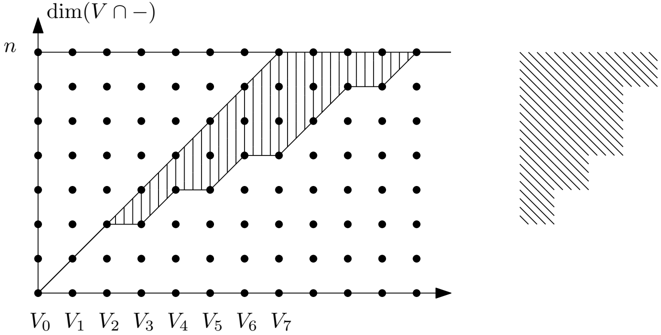
- We consider the associated projective bundle $\mathbb{P}(\xi)$ of ξ, namely, the fibre of $x \in X$ is the projective space of the fibre of x.
- By a Mayer-Vietoris argument, $H^*(\mathbb{P}(\xi))$ is free over $H^*(X)$.
- Now the pull back of ξ has a sub-line-bundle, say the dual of tautological bundle τ (the fibre at the line in fibre at x is the line itself).
- By picking a unitary product, the pull back splits into small dimension. The general facts follows from induction.

Remark

- There is a cellular structure over $\mathcal{G}r(n,\infty)$, say Schubert cells, for a length n partition $\lambda_1 \geq \cdots \geq \lambda_n$,

$$\Sigma_\lambda(F) = \{V \in \mathcal{G}r(n,\infty) : {}^{\forall i=1,\dots,n,}_{\dim(V \cap V_{\lambda_i+i}) \geq i}\},$$

  where V_k is the first k-coordinate. Actually, $\Sigma_\lambda \cong \mathbb{C}^{|\lambda|}$.
- Then the cohomology class of the cell $\Sigma_{1\geq \overset{k}{\cdots} \geq 1 \geq 0}$ is exactly c_{2k}.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V_0 V_1 V_2 V_3 V_4 V_5 V_6 V_7

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Next Time

- Axioms of Chern classes.
- Gallery of characteristic classes.
  - General Classifying bundles
  - Stiefel-Whitney Classes
  - Pontryagin Classes
  - Euler Class
  - Characteristic classes in differential geometry.
  - Characteristic classes in algebraic geometry.

$\gg$ *Questions?* $\ll$

$\sim$ § <u>THANKS</u> § $\sim$