Spectral Sequences (II)

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∼ § Remind § ∼

Remind

Theorem

Each filtered (cochain) complex $(C, \mathcal{F})$ *determines a spectral sequence* E *with*

$$\begin{aligned} E_0^{pq} &= \mathcal{F}^p C^{p+q} / \mathcal{F}^{p+1} C^{p+q} \\ E_1^{pq} &= H^{p+q}(\mathcal{F}^p C / \mathcal{F}^{p+1} C). \end{aligned}$$

If the filtration $\mathcal{F}$ *over* C *is lower bounded and upper exhaustive then* E *converges to* $H^\bullet(C)$. *More exactly,*

$$E_\infty^{pq} \cong \mathcal{F}^p H^{p+q}(C, d) / \mathcal{F}^{p+1} H^{p+q}(C, d),$$

where $\mathcal{F}$ *is lower bounded and exhaustive filtration over* $H^\bullet(C, d)$.

$\gg$ *Questions?* $\ll$

∼ § Double complexes § ∼

Double complexes

- Recall the notion of **double complex**.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \to & & C^{a-1,b+1} & \to & C^{a,b+1} & \to & C^{a+1,b+1} & & \to \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \to & & C^{a-1,b} & \to & C^{ab} & \to & C^{a+1,b} & & \to \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \to & & C^{a-1,b-1} & \to & C^{a,b-1} & \to & C^{a+1,b-1} & & \to \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

each square **anticommutes**, that is, $d_{\to}d_{\uparrow} + d_{\uparrow}d_{\to} = 0$.

Double complex

- We can take total of the double complex to be a complex

$$\operatorname{Tot} C = \begin{cases} \operatorname{Tot}^{n+1} C = \bigoplus_{i+j=n+1} C^{ij} \\ \qquad\qquad \uparrow \qquad\qquad\qquad d : d_{\uparrow} + d_{\rightarrow} \\ \operatorname{Tot}^{n} C = \bigoplus_{i+j=n} C^{ij} \end{cases}$$

Denote $H^{\bullet}(C) = H^{\bullet}(\operatorname{Tot}^{\bullet} C)$ the cohomology group of a double complex.

SS for double complexes

Theorem

Each double complex $(C^{\bullet\bullet}, d^{\rightarrow}, d^{\uparrow})$*, determines two spectral sequences* ${}^{\mathrm{I}}E$ *and* ${}^{\mathrm{II}}E$ *with*

$$
\begin{aligned}
{}^{\mathrm{I}}E_1^{pq} &= H^q(C^{p\bullet}, d^{\uparrow}) & \qquad {}^{\mathrm{II}}E_1^{qp} &= H^p(C^{\bullet q}, d^{\rightarrow}) \\
{}^{\mathrm{I}}E_2^{pq} &= H^p(H^q(C, d^{\uparrow}), d^{\rightarrow}). & \qquad {}^{\mathrm{II}}E_2^{qp} &= H^q(H^p(C, d^{\rightarrow}), d^{\uparrow}).
\end{aligned}
$$

If the double complex C *lies in the first quadrant, or the third quadrant, then* ${}^{\mathrm{I}}E$ *and* ${}^{\mathrm{II}}E$ *converge to* $H^{\bullet}(\mathrm{Tot}^{\bullet}(C))$*.*

Sketch of the proof

- Let us only do the first one.

$$\mathcal{F}^p \mathrm{Tot}^\bullet C = \mathrm{Tot}^\bullet C_{\mathrm{I} \geq p},$$

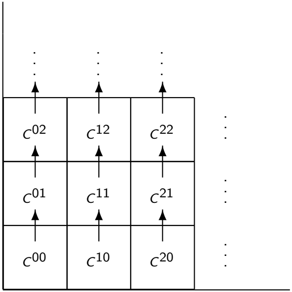
  by truncating the double complexes. Say $C^{ij}_{\mathrm{I}\geq p} = C^{ij}$ when $i \geq p$ and zero otherwise.

- Then (let us only do the first one)

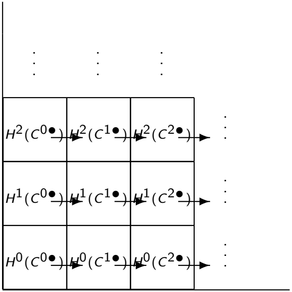
$$E_0^{p\bullet} = \frac{\mathcal{F}_{\mathrm{I}}^p \mathrm{Tot}^\bullet C}{\mathcal{F}_{\mathrm{I}}^{p+1} \mathrm{Tot}^\bullet C} = \frac{\mathrm{Tot}^\bullet C_{\geq p}}{\mathrm{Tot}^\bullet C_{\geq p+1}} \cong \mathrm{Tot}^\bullet C_{\mathrm{I}=p} = C^{p\bullet},$$

  the p-th column.

- Check d^1 carefully, it is exactly induced by $d^{\rightarrow}$.


C^{02} C^{12} C^{22}
C^{01} C^{11} C^{21}
C^{00} C^{10} C^{20}

E_0


$H^2(C^{0\bullet})$ $H^2(C^{1\bullet})$ $H^2(C^{2\bullet})$
$H^1(C^{0\bullet})$ $H^1(C^{1\bullet})$ $H^1(C^{2\bullet})$
$H^0(C^{0\bullet})$ $H^0(C^{1\bullet})$ $H^0(C^{2\bullet})$

E_1

Remarks

- One should be careful with the direction of the second spectral sequence. The best way is to using the transpose. If one insists, there will be a steep arrow ($\nwarrow$ rather than $\nwarrow$).
- It also has homology version (similarly).

$\gg$ *Questions?* $\ll$

∼ § <u>Čech cohomology</u> § ∼

Čech cohomology

- It is very useful to use Čech cohomology to compute cohomology (especially in sheaf theory).
- Let $\mathcal{U} = \{U_i : i \in I\}$ be an open covering of X. Denote

$$U_{\mathbf{i}} = U_{i_0,\dots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Assume I is totally ordered, and $S^\bullet(-)$ be the complex of singular cohomology (of sheaves). Denote $q \geq 0, p \geq 0$,

$$C^{pq} = \prod_{\mathbf{i}=i_0<\cdots<i_p} S^q(U_{\mathbf{i}}).$$

Čech cohomology (continued)

- For each fixed p,

$$d^{\uparrow}:\prod_{\mathbf{i}} S^q(U_{\mathbf{i}}) \to \prod_{\mathbf{i}} S^{q+1}(U_{\mathbf{i}})$$

is defined to be the differential of singular cohomology summand-wise, say $\prod_{\mathbf{i}} d$.

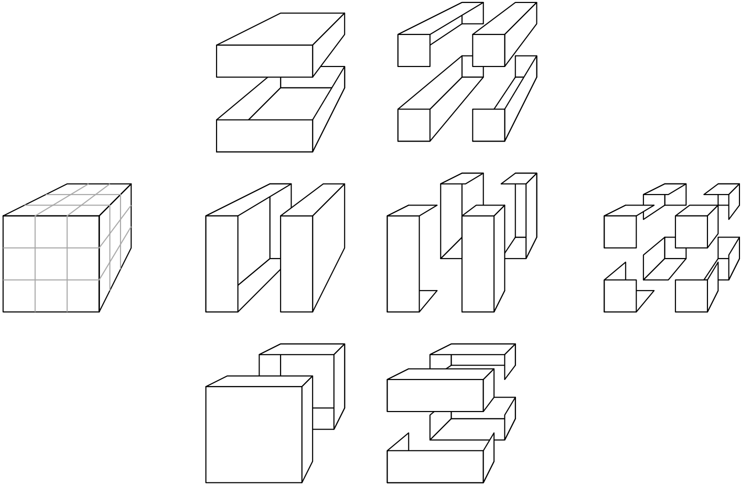
- For each fixed q,

$$d^{\rightarrow}:\prod_{\mathbf{i}:i_0<\cdots<i_p} S^q(U_{\mathbf{i}}) \to \prod_{\mathbf{j}:j_0<\cdots<j_{p+1}} S^q(U_{\mathbf{j}})$$

is defined by

$$(d\alpha)_{\mathbf{j}} = \sum_{k=0}^{p}(-1)^k \alpha_{j_0<\cdots<\widehat{j_k}<\cdots<j_{p+1}}\big|_{U_{\mathbf{j}}}.$$

- By switching the signs, C forms a double complex.



It is kind of "fat" simplicial cohomology.

Čech cohomology (continued)

- Define the **Čech cohomology** be the cohomology group of the following complex

$$\check{C}^\bullet : \check{C}^p = \prod_{\mathbf{i}: i_0 < \cdots < i_p} H^0(U_{\mathbf{i}})$$

with

$$d : \prod_{\mathbf{i}: i_0 < \cdots < i_p} H^0(U_{\mathbf{i}}) \to \prod_{\mathbf{j}: j_0 < \cdots < j_{p+1}} H^0(U_{\mathbf{j}})$$

defined by the same formula

$$(d\alpha)_{\mathbf{j}} = \sum_{k=0}^{p} (-1)^k \alpha_{j_0 < \cdots < \widehat{j_k} < \cdots < j_{p+1}} \big|_{U_{\mathbf{j}}}.$$

- The combinatorial description of d is left to readers.

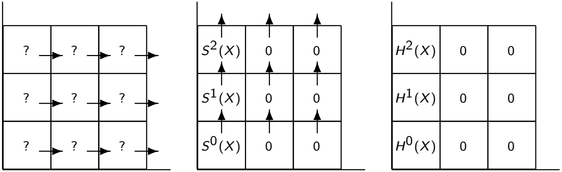
Čech cohomology Theorem

Theorem (Čech)

When all finite intersection of members in $\mathcal{U}$ are acyclic (that is, only has zero cohomology), then Čech cohomology computes the cohomlogy.

The proof

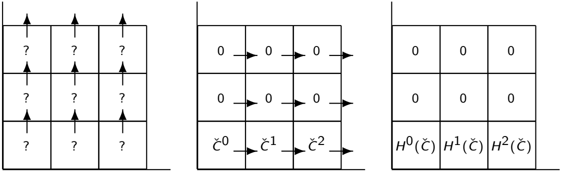
- One can compute, for each q, $d^{\rightarrow}$ is exact except the 0-term. Here we use the assumption that U is acyclic to make the “integral”.
- The zeroth cohomology is homotopic to $S^\bullet(X)$ (recall how we prove the excision theorem) (if it is a sheaf, then just equal). This more or less equivalent to say, we can “glue” a section.



- So $H^\bullet(C) = H^\bullet(\mathrm{Tot}^\bullet(C)) = H^\bullet(X)$.

The proof (continued)

- On the other hand, if each U is acyclic and connected, then



- So $H^\bullet(C) = H^\bullet(\mathrm{Tot}^\bullet(C)) = H^\bullet(\check{C})$.

Remarks

- Čech cohomology is very important. If readers know sheaf theory, the Čech cohomology computes the cohomology both conceptually and efficiently.
- Firstly, if we denote $\check{H}^\bullet(\mathcal{U})$ the Čech cohomology with respect to $\mathcal{U}$, we can write

$$H^\bullet(X) = \varinjlim_{\mathcal{U} \text{ finer}} \check{H}^\bullet(\mathcal{U}),$$

  for locally contractible space X. This is useful when showing two cohomology groups coincide.
- Secondly, if we have a nice decomposition into a contractible space (for example $\mathbb{C}P^n$), then the cohomology can be computed discretely.

$\gg$ ***Questions?*** $\ll$

∼ § Grothendieck SS § ∼

Remind

- Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. Assume $\mathcal{A}$ has enough injective. Then the right derived functor R^iF is defined by
$$R^iF(M) = H^i(F(I^\bullet)),$$
where $0 \to M \to I^\bullet$ is an injective resolution in $\mathcal{A}$.
- If each $I^\bullet$ is F-acyclic, that is, $R^iF(I) = 0$ when $i > 0$, it computes the same result (by the trick of dimension shift).
- We can define the left derived functor as well.
- Examples in algebra like Hom and $\otimes$ are well-discussed in any modern homological algebra. We should focus more on sheaves.

Grothendieck SS

- Consider now there are two functors with the assumptions

$\mathcal{A}$ and $\mathcal{B}$ have enough injectives; F, G are both left exact; F sends injective objects to G-acyclic objects.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{GF} & \mathcal{C} \\ & F \searrow \quad \nearrow G & \\ & \mathcal{B} & \end{array}$$

Theorem (Grothendieck)

There is a spectral sequence (of functors) E with

$$E_2^{pq} = R^p G \circ R^q F$$

converging to $R^{p+q}(G \circ F)$.

Proof

- The trick is using so-called **Hyper resolution**.
- Let $A \in \mathcal{A}$ be an object. Taking the resolution $0 \to A \to I^\bullet$ in $\mathcal{A}$.
- Then send $I^\bullet$ to $F(I^\bullet)$, we can find a double complex J in $\mathcal{B}$ to "resolve" it. That is,

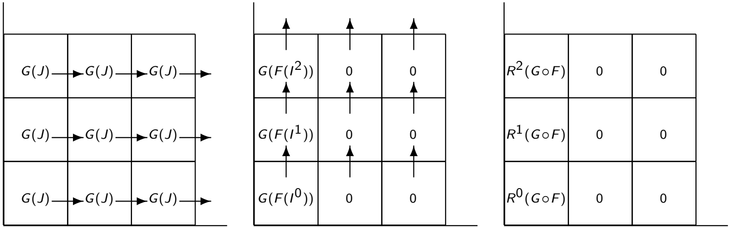
$$F(I^i) \to J^{i\bullet}, \qquad R^iF(A) = H\begin{bmatrix} \uparrow \\ F(I^i) \\ \downarrow \end{bmatrix} \to H\begin{bmatrix} \uparrow \\ J^{i,\bullet} \\ \downarrow \end{bmatrix} := K,$$

$$\text{im}\begin{bmatrix} F(I^{i+1}) \\ \uparrow \\ F(I^i) \end{bmatrix} \to \text{im}\begin{bmatrix} J^{i+1,\bullet} \\ \uparrow \\ J^{i\bullet} \end{bmatrix}, \qquad \ker\begin{bmatrix} F(I^{i+1}) \\ \uparrow \\ F(I^i) \end{bmatrix} \to \ker\begin{bmatrix} J^{i+1,\bullet} \\ \uparrow \\ J^{i\bullet} \end{bmatrix},$$

  are all injective resolutions. Actually, $(J, d^\uparrow)$ is split. This is possible because of horseshoe lemma.
- Then send J to $G(J)$, and do computation.

Proof (continued)

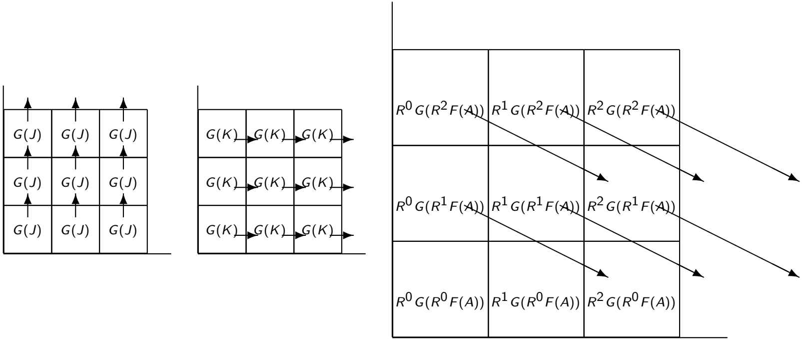
- On one hand, since $R^{\geq 1}G(F(I^\bullet)) = 0$ by assumption



So $H^\bullet(\mathrm{Tot}^\bullet(G(J))) = R^\bullet(G \circ F)$.

Proof (continued)

- On the other hand, since $R^{\geq 1}G(F(I^\bullet)) = 0$ by assumption



As desired.

$\gg$ ***Questions?*** $\ll$

$\sim$ § Applications § $\sim$

Remind

- Let $\mathcal{S}h(X)$ be the abelian category of sheaves (of abelian group) over X. The functor of **taking global section**

$$\Gamma : \mathcal{S}h(X) \longrightarrow \mathcal{S}h(\mathrm{pt}) = \mathrm{Ab}, \qquad \mathcal{F} \longmapsto \mathcal{F}(X),$$

is left exact. We call and denote $R^i\Gamma(F) = H^i(X;\mathcal{F})$ the **cohomology group** of $\mathcal{F}$.

- Similarly, the compact global section $\Gamma_c : \mathcal{S}h(X) \to \mathcal{S}h(\mathrm{pt})$ is also left exact. We call $R^i\Gamma_c(F) = H^i_c(X;\mathcal{F})$ the **cohomology group of compact support**.
- We will discuss this in length latter.

Remind (continued)

Theorem

Let $\mathbb{Z}_X$ be the constant sheaf over a CW complex X, then

$$H^i(X; \mathbb{Z}_X) = H^i(X) \qquad H^i_c(X; \mathbb{Z}_X) = H^i_c(X),$$

the singular cohomology group (and with compact support).

- Essentially, $S^\bullet(-)$ is not a sheaf, but $S^\bullet_{\text{locally finite}}(-)$ is.

Leray Sequences

- Let $X \to Y$ be a topological map. The push forward f_* defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sending the flasque sheaves to the flasque sheaves.
- This gives rise to

$$\begin{array}{ccc} \mathcal{S}h(X) & \xrightarrow{\quad\Gamma\quad} & \mathcal{S}h(\mathrm{pt}) = \mathrm{Ab} \\ & {}_{f_*}\searrow \quad \nearrow_{\Gamma} & \\ & \mathcal{S}h(Y) & \end{array}$$

So there is a spectral sequence

$$E_2^{pq} = H^p(Y; R^q f_*\mathcal{F}) \Longrightarrow H^{p+q}(X; \mathcal{F}).$$

Leray Sequences (continued)

- There is a description that $R^q f_* \mathcal{F}$ is the sheafication of

$$U \mapsto H^q(f^{-1}(U); \mathcal{F}|_{f^{-1}(U)}).$$

 (Showing this satisfies the universal property of being the right derived functor of f_*).
- In particular, taking $\mathcal{F} = \mathbb{Z}_X$, and $X \to Y$ a fibre bundle with fibre F, $R^q f_* \mathcal{F}$ is a local constant sheaf, say the local system $\mathcal{H}^q(F)$, so the spectral sequence covers the Leray–Serre spectral sequence.

Remarks

- There is a lot of applications of Grothendieck spectral sequence in algebra, especially in homological algebra, which is easy to find. So I skip.
- We skip the discussion of the **Cartan–Leray spectral sequences**, which are very useful to compute the non-simply connected space.
- We skip the discussion of the **Elienburg–Moore spectral sequneces**, which are very useful to compute the fibre product.
- We skip the discussion of the **Adams spectral sequences**, which are very useful to compute the stable homotopy group.
- We will discuss the **Atiyah–Hirzebruch spectral sequences** later.

References

- Hartshorne, Algebraic Geometry. GTM 52.
  Even though it is an algebraic geometry book, it is also a good introduction of sheaves theory, see III.8 for what I claimed.
- Brown, Cohomology of groups. GTM 87.
  It has a good discussion of cohomology of finite groups and introduce the Cartan–Leray spectral sequences.
- Benson, cohomology and representation theory, II.
  It gives the sketch of construction of the Elienburg–Moore spectral sequneces in a correct way.
- Fomenko, Fuchs, Homotopical Topology. GTM 273.
  Algebraic topology in term of spectral sequences.
- Hatcher, spectral sequences.
- Weibel, an introduction to homological algebra.
  It presents a lot of algebraic application of spectral sequences.

Next time

- The cohomology group of the Grassmannian.
- The topological definition of the Chern classes.
- The differential definition of the Chern classes.

$\gg$ *Questions?* $\ll$

$\sim$ § <u>THANKS</u> § $\sim$