Sheaf Theory (I)

Xiong Rui

November 28, 2020

§ Sheaves §

Presheaves

- Let X be a topological space. A **presheaf** over X is an assignment for each open set an abelian group

$$\mathcal{F} : U \mapsto \mathcal{F}(U),$$

and **restriction map**

$$\mathcal{F}(U) \to \mathcal{F}(V) \qquad s \mapsto s|_V$$

for each $V \subseteq U$ with the following properties
  - For empty set, $\mathcal{F}(\varnothing) = 0$;
  - For open subset U, $[\mathcal{F}(U) \to \mathcal{F}(U)] = \mathrm{id}$;
  - For $W \subseteq V \subseteq U$, $[\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)] = [\mathcal{F}(U) \to \mathcal{F}(W)]$.

Stalks

- Let X be a topological space, and $\mathcal{F}$ be a presheaf.
- We will call an element of $\mathcal{F}(U)$ a **section** of $\mathcal{F}$ over U. A global section refers the case $U = X$. We use the notation

$$\Gamma(U; \mathcal{F}) = H^0(U; \mathcal{F}) = \mathcal{F}(U).$$

- For each point $x \in X$, we define the **stalk** at x

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

each element is presented by a section over some neighborhood of x, and two of them are equal if and only if they coincide in a smaller neighborhood.

Support

- For any $s \in \mathcal{F}(U)$, denote

$$\operatorname{supp} s = \{x \in U : \text{the image of } s \text{ in } \mathcal{F}_x \text{ does not vanish}\}.$$

- For $\mathcal{F}$, denote

$$\operatorname{supp} \mathcal{F} = \{x \in U : \mathcal{F}_x \neq 0\}.$$

Sheaves

- Let X be a topological space. A **sheaf** is a presheaf $\mathcal{F}$ with
  - For an open covering $\mathcal{U}$ of an open subset U_0, and $s \in \mathcal{F}(U_0)$, then

$$s = 0 \iff \forall U \in \mathcal{U}, s|_U = 0.$$

  - For an open covering $\mathcal{U}$ of an open subset U_0, and $\{s_U \in \mathcal{F}(U) : U \in \mathcal{U}\}$ then

$$\text{There exists } s \in \mathcal{F}(U_0) \text{ with } s|_U = s_U \text{ for all } U \in \mathcal{U} \iff \forall_{U,V \in \mathcal{U},} s_U|_{U\cap V} = s_V|_{U\cap V}$$

    We will say s is glued/patched by $\{s_U\}$.

Associated Sheaves

Let X be a topological space, and $\mathcal{F}$ be a presheaf.

- It would be better to think sheaf as a "fibre". Say, consider

$$\mathbf{F} = \bigsqcup_{x \in X} \mathcal{F}_x \xrightarrow{\pi} X.$$

  Then any $s \in \mathcal{F}(U)$ defines a section of π over U. That is, a map $s : U \to \mathbf{F}$ with $\pi \circ s = \mathrm{id}$. Then we define the weakest topology over $\mathbf{F}$ with all such s continuous.

- Remind the weakest topology,

$$? \subseteq \mathbf{F} \text{ is open } \iff s^{-1}(?) \text{ is open for all section } s.$$

Associated Sheaves

- We can define a sheaf associated with $\mathcal{F}$ by adding the formally glued section. Say, simply the continuous section of above $\mathbf{F} \xrightarrow{\pi} X$,

$$\mathcal{F}^{\dagger}(U) = \{\text{continuous } s : U \to \mathbf{F} : \pi \circ s = \text{id}\}.$$

- Note that such $s : U \to \mathbf{F}$ is continuous, if and only if for any section $t \in \mathcal{F}(V)$,

$$\{x \in U : \text{image of } t \text{ and image of } s \text{ in } \mathcal{F}_x \text{ coincides}\}$$

is open. That is, for any $x \in U$, there is a neighborhood V, s coincides with t in $\mathcal{F}_y$ for all point of $y \in V$.

Restriction

- Let $Y \subseteq X$ be a subset. For a sheaf $\mathcal{F}$ we define its restriction

$$\mathcal{F}(Y) := \{\text{continuous } s : Y \to \mathbf{F} : \pi \circ s = \mathrm{id}\}.$$

- By definition, $s : Y \to \mathbf{F}$ is a choice of stalk which glued to a global section over a neighborhood of Y, so

$$\mathcal{F}(Y) = \varinjlim_{V \supseteq Y} \mathcal{F}(V).$$

$\gg$ *Questions?* $\ll$

§ Morphisms §

Morphisms

- Let $\mathcal{F}, \mathcal{G}$ be two presheaves. A morphism between them $\mathcal{F} \to \mathcal{G}$ is an assignment of homomorphism of abelian groups of $\mathcal{F}(U) \to \mathcal{G}(U)$ commuting with restriction

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

- As a result, it induces $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ for each point $x \in X$.
- Morphism of sheaves is the same to presheaves'.

Universal Property Associated Sheaves

- The associated sheaf $[\mathcal{F} \mapsto \mathcal{F}^\dagger]$ has the following universal property

  For any sheaf $\mathcal{G}$, and a morphism of $\mathcal{F} \to \mathcal{G}$, there exists a unique $\mathcal{F}^\dagger \to \mathcal{G}$ with the right diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & \to & \mathcal{F}^\dagger \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

- Equivalently,

$$\mathrm{Hom}_{X\text{-Presheaf}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{X\text{-Sheaf}}(\mathcal{F}^\dagger, \mathcal{G}).$$

Kernel and Cokernel

- Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then we define its **kernel**, **cokernel**, **image**

$$\begin{aligned} \ker \varphi &:= \big[U \mapsto \ker[\mathcal{F}(U) \to \mathcal{G}(U)]\big], \\ \operatorname{cok} \varphi &:= \big[U \mapsto \operatorname{cok}[\mathcal{F}(U) \to \mathcal{G}(U)]\big]^{\dagger}, \\ \operatorname{im} \varphi &:= \big[U \mapsto \operatorname{im}[\mathcal{F}(U) \to \mathcal{G}(U)]\big]^{\dagger}. \end{aligned}$$

  They are all sheaves.
- Since the injective limit defining stalk is filtered, or by direct check, we have

$$\begin{aligned} (\ker \varphi)_x &= \ker[\mathcal{F}_x \to \mathcal{G}_x], \\ (\operatorname{cok} \varphi)_x &= \operatorname{cok}[\mathcal{F}_x \to \mathcal{G}_x], \\ (\operatorname{im} \varphi)_x &= \operatorname{im}[\mathcal{F}_x \to \mathcal{G}_x]. \end{aligned}$$

Exact sequences

- Then we can define subsheaves, quotient sheaves, and exact sequence. To summarize, we can write all of them in term of stalks.
- For a sheaf $\mathcal{F}$, $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ if and only if $\mathcal{G}_x$ is subablian group of $\mathcal{F}_x$. We can define the quotient $\mathcal{F}/\mathcal{G}$ by the cokernel of inclusion morphism.
- For a sequence of morphisms between sheaves

$$\cdots \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \cdots$$

is exact if and only if at each point, the stalk is exact

$$\cdots \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to \cdots$$

- In one word, the sheaves over X forms an abelian category.

Theorem

The sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

to be exact if and only if the following sequence is exact

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

for any open subset U.

- The "if" part follows from the fact that limit defining stalk is directed, so it commutes with kernel.
- If $s \in \mathcal{F}(U)$ is mapped to $0 \in \mathcal{G}(U)$, then it is zero at each $\mathcal{G}_x$, so s is zero at each $\mathcal{F}_x$.
- If $s \in \mathcal{G}(U)$ is mapped to $0 \in \mathcal{H}(U)$, then it comes from some $s \in \mathcal{F}(U)$, this glued up to a section of $\mathcal{F}$.

Hom and $\mathcal{H}$om

- For a sheave $\mathcal{F}$ over X, and an open subset $Y \subseteq X$, define
$$\mathcal{F}|_Y := [V \mapsto \mathcal{F}(V)].$$
- We define
$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) := [U \mapsto \mathrm{Hom}_{U\text{-Sheaf}}(\mathcal{F}|_U, \mathcal{G}|_U)]$$
It is a sheaves. In particular,
$$\Gamma(X; \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{X\text{-Sheaf}}(\mathcal{F}, \mathcal{G}).$$
- However, note that, in general MSE16203
$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})_x \neq \mathrm{Hom}_{\mathrm{Abel}}(\mathcal{F}_x, \mathcal{G}_x).$$

$\gg$ ***Questions?*** $\ll$

§ Examples §

Vector Bundles

- For space X, we define

$$\mathcal{O}(U) = \{\text{continuous function } U \to \mathbb{R}\}.$$

- For a vector bundle $\pi = \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$, we can define

$$\mathcal{E}(U) = \{\text{continuous } s : U \to E : s \circ \pi = \mathrm{id}_U\}.$$

- For example, when E is trivial bundle of rank one, $\mathcal{E} = \mathcal{O}$.
- Since we assume π is locally trivial, so $\mathcal{E}$ locally isomorphism to $\mathcal{O}^{\oplus \operatorname{rank} E}$.

Vector Bundles

- We can exchange the terminology to work in different kind of geometry

continuous	vector bundle
smooth	smooth vector bundle
analytic	analytic vector bundle
algebra(regular)	locally free sheaf

See Hartshorne Ex II. 5.18.

- Actually, we have an equivalence

$$\begin{pmatrix} \text{Vector bundles} \\ \text{maps of vector bundles} \end{pmatrix} \cong \begin{pmatrix} \text{locally free sheaves} \\ \mathcal{O}\text{-module morphism} \end{pmatrix},$$

see Lecture 6.

Coherent Sheaves

- As we stated before, vector bundle does not closed under taking kernel and cokernel. We say a sheaf is **coherent**, if it is a cokernel of morphism over between two vector bundles.
- In algebraic case, when it is a Noetherian scheme, they form an abelian category.
- But in smooth case, it turns out to be of less usage, so this notion is seldom used. But alternatively, the complex of vector bundles are more used.

Morphisms are different

- For smooth manifold M, the constant sheaf $\mathbb{R}$ is resoluted by de Rham complex

$$0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \to \cdots$$

  Note that d is not a morphism of vector bundles, but a morphism of sheaves.
- But Koszul complex is.

No longer Projective

- Note that for a vector bundle $\mathcal{E}$, it is in general not the projective object in coherent sheaves, even for $\mathcal{O}$.
- We know that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{F}) = \Gamma(X; \mathcal{F})$$

  which is not an exact functor. Recall, $\mathcal{H}$om does not commute with localization.
- But it is flat, since tensor product commutes with localization.

Constant Sheaves

- For an abelian group G, we define

$$G := [U \mapsto G]^{\dagger}$$

the constant sheaf. Note that when X is connected, then $[U \mapsto G]$ is already a sheaf.

Local Systems

- For a topological space X, a **local system** is an assignment of an abelian group L_x for each point $x \in X$, and a homomorphism from $L_x \to L_y$ for each homotopy class of path connecting x and y such that we have

$$[L_x \xrightarrow{p} L_y \xrightarrow{q} L_z] = [L_x \xrightarrow{pq} L_z],$$

  where pq is path obtained by the connecting p and q.
- That is, it is a representation of the groupoid of X.
- One example is $x \mapsto H_*(M, M \setminus x)$ the orientation bundle.

Local Systems

Given a local system is equivalent to give a $\pi_1(X, x)$ representation on L_x when X is path-connected.

- Firstly choose a path p_y connect x and y in advavce.
- Secondly assign L_x at each point; for any path p from y to z.
- Lastly, assign the homomorphism corresponding to $x \xrightarrow{p_y} y \xrightarrow{p} z \xrightarrow{p_z^{-1}} x$.

As a result, for path-connected and simply connected space, there is only trivial local system up to isomorphism.

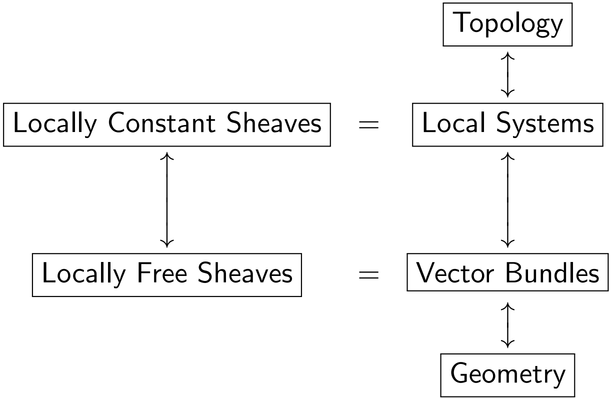
Local Systems

- Assume X is locally path-connected and locally simply path-connected, then a local system is equivalent to give a local constant sheaf.
- For U open in X, we define

$$\mathcal{L}(U) = \left\{ s : U \to \bigcup_x L_x : \begin{array}{l} \text{We have } s(x) \in L_x; \\ \text{For any path } p \text{ connecting} \\ x \text{ and } y, \text{ the map } L_x \to L_y \\ \text{send } s(x) \text{ to } s(y). \end{array} \right\}$$

- If X is locally simply path-connected, then $\mathcal{L}$ is a local constant.

Locally Constant Versus Locally Trivial



- Even the locally constant sheaves and coherent sheaves are distant, but there would be some connection, say **connection**.

$\gg$ *Questions?* $\ll$

§ Functors §

Preimage

- Let $X \to Y$ be a continuous map, and a sheaf $\mathcal{G}$ over Y, we define the **preimage** of $\mathcal{G}$ a sheaf over X by

$$f^*\mathcal{G} := \left[U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \right]^\dagger .$$

- Note that, in Harstorne, it is denoted by f^{-1}.
- Then by definition,

$$\begin{aligned} f^*\mathcal{G}_x &= \varinjlim_{U \ni x} \left(\varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \right) \\ &= \varinjlim_{V \ni f(x)} \mathcal{G}(V) \\ &= \mathcal{G}_{f(x)} \end{aligned}$$

This is what we expected "pull back", in particular, f^* is exact.

Direct image

- Let $X \to Y$ be a continuous map, and a sheaf $\mathcal{F}$ over X, we define the **direct image** of $\mathcal{F}$ a sheaf over Y by

$$f_*\mathcal{F} := \left[V \mapsto \mathcal{F}(f^{-1}(V)) \right].$$

  This definition is simple, but not exact.

- Actually, it is left exact by the theorem of being exact.
- Note that

$$f_*\mathcal{F}_y = \varinjlim_{V \ni y} \mathcal{F}(f^{-1}(V)) = \varinjlim_{f^{-1}(V) \supseteq f^{-1}(y)} \mathcal{F}(f^{-1}(V)).$$

  So f_* is "taking section around each fibre of f".

Proper image

- Simultaneously, we define the **proper image** of $\mathcal{F}$ a sheaf over Y by

$$f_!\mathcal{F}(V) = \{s \in \mathcal{F}(f^{-1}(V)) : \operatorname{supp} s \to V \text{ is proper}\}.$$

  Proper = the preimage of compact subset is still compact.
- This is also left exact, since s the support does not change under inclusion.
- We will show

$$f_!\mathcal{F}_y = \Gamma_c(f^{-1}(x); \mathcal{F}).$$

  So $f_!$ is "taking compact-supported section along each fibre of f".

- An element in $f_!\mathcal{F}_x$ is by definition presented by a section $s \in \mathcal{F}(f^{-1}(V))$ with V some neighborhood of x and $\operatorname{supp} s \to V$ proper. If s is zero over $f^{-1}(x)$, then $f(\operatorname{supp} s)$ will disjoint with x, so shrink V if necessary, $s = 0$.
- Conversely, if we have a global section over $f^{-1}(x)$ with compact support, then it is presented by some $s \in \mathcal{F}(V)$ of compact support for a neighborhood of $f^{-1}(x)$. Then we can assume this neighborhood to be of the form $f^{-1}(V)$ due to compactness.

Properties

Theorem

For proper map $f : X \to Y$, $f_* = f_!$.

- Actually, for any section $s \in \mathcal{F}(f^{-1}(U))$, $\operatorname{supp} s \to U$ is proper, since the preimage is simply the preimage intersects with $\operatorname{supp} s$. But it is clear, intersection compact set with closed set is still compact.

Theorem

All $f^*, f_*, f_!$ are functors. Say,

$$(f \circ g)^* = g^* \circ f^*, \qquad \begin{array}{c} (f \circ g)_* = f_* \circ g_* \\ (f \circ g)_! = f_! \circ g_! \end{array}$$

Properties

Theorem

For continuous $X \xrightarrow{f} Y$; sheaves $\mathcal{F}$ and $\mathcal{G}$ over X and Y respectively

$$\mathrm{Hom}_{X\text{-Sheaf}}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Y\text{-Sheaf}}(\mathcal{G}, f_*\mathcal{F}).$$

- By definition

$$\begin{aligned}
&\mathrm{Hom}_{X\text{-Sheaf}}(f^*\mathcal{G}, \mathcal{F}) \\
= &\varprojlim_{U \text{ open in } X} \mathrm{Hom}(f^*\mathcal{G}(U), \mathcal{F}(U)) \\
= &\varprojlim_{U \text{ open in } X} \varinjlim_{f^{-1}(V) \supseteq U} \mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U)) \\
= &\varprojlim_{V \text{ open in } Y} \varprojlim_{U \subseteq f^{-1}(V)} \mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U)) \\
= &\varprojlim_{V \text{ open in } Y} \mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V))) \\
= &\mathrm{Hom}_{Y\text{-Sheaf}}(\mathcal{G}, f_*\mathcal{F})
\end{aligned}$$

Properties

Theorem

For a Cartesian square $\begin{array}{ccc} W & \xrightarrow{F} & Y \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$, *then* $f^* \circ g_! = G_! \circ F^*$.

- It would not be hard to construct a map

$$\begin{array}{ccccc} & & f^* \circ g_* \circ F_* \circ F^* = f^* \circ f_* \circ F^* \circ F^* & & \\ & \nearrow & & \searrow & \\ f^* \circ g_* & & & & G_* \circ F^* \\ & \searrow & & \nearrow & \\ & & G_* \circ G^* \circ f^* \circ g_* = G_* \circ F^* \circ g^* \circ g_* & & \end{array}$$

  Then we get a map $f^* \circ g_! \to G_! \circ F^*$.
- Actually,

$$\begin{aligned} G_! F^* \mathcal{F}_z &= \Gamma_c(G^{-1}(z); F^*\mathcal{F}) = \Gamma_c(g^{-1}(f(z)); \mathcal{F}) \\ &= g_! \mathcal{F}_{f(z)} = f^* g_! \mathcal{F}_z \end{aligned}$$

$\gg$ *Questions?* $\ll$

§ Examples §

For a point

Consider $\pi : X \to \mathrm{pt}$.

- Note that for a point, the sheaf is nothing but a pure abelian group.
- For an abelian group G,

$$\pi^* G = [U \mapsto G]^{\dagger}$$

the constant sheaf with respect to V.

- For a sheaf $\mathcal{F}$,

$$\pi_* \mathcal{F} = \mathcal{F}(\pi^{-1}(\mathrm{pt})) = \Gamma(X; \mathcal{F})$$

the global section. The same reason,

$$\pi_! \mathcal{F} = \Gamma_c(X; \mathcal{F})$$

the global section of compact support.

For closed immersions

Consider closed immersion $i : F \to X$.

- For a sheaf $\mathcal{F}$ over X,

  $$i^*\mathcal{F} = [V \mapsto \mathcal{F}(V)]^\dagger$$

  where the notation of taking any subset in to $\mathcal{F}$ is introduced before.
- Since i is proper, $i_! = i_*$. For a sheaf $\mathcal{F}$ over F,

  $$i_*\mathcal{F} = [U \mapsto \mathcal{F}(U \cap F)].$$

For open immersions

Consider open immersion $j : U \to X$.

- For a sheaf $\mathcal{F}$ over X,

$$j^*\mathcal{F} = [V \mapsto \mathcal{F}(V)]$$

  the restriction $\mathcal{F}|_U$.
- For a sheaf $\mathcal{F}$ over U,

$$j_*\mathcal{F} = [V \mapsto \mathcal{F}(U \cap V)].$$

- For proper direct image, consider extending by zero

$$j_!\mathcal{F} = \left[V \mapsto \begin{cases} \mathcal{F}(V), V \subseteq U \\ 0, \text{otherwise} \end{cases} \right]^\dagger.$$

  Since there is a morphism from right hand side to $j_!\mathcal{F}$, and we can see it is the same by compare the stalk.

For open/closed immersions

Consider closed immersion $i : F \to X$ and open immersion $j : U \to X$ with $F \sqcup U = X$.

- For a sheaf $\mathcal{F}$ over X, there is a natural map

$$0 \to j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \to 0$$

- The first is natural.
- The second is the image of identity under

$$\mathrm{Hom}_{U\text{-Sheaf}}(i^*\mathcal{F}, i^*\mathcal{F}) = \mathrm{Hom}_{X\text{-Sheaf}}(\mathcal{F}, i_* i^* \mathcal{F})$$

- It is easy to see from the stalks, this is a short exact sequence.

For fibre bundles

Let $\pi : X \to B$ be a locally trivial fibre bundle of fibre F.

- For a sheaf $\mathcal{F}$ over B, then

$$\pi^*\mathcal{F}_x = \mathcal{F}_{\pi(x)}$$

  the same along each fibre.

- For a sheaf $\mathcal{F}$ over X, then

$$\pi_*\mathcal{F}_b = \varinjlim \Gamma(f^{-1}(V); \mathcal{F}) \cong \varinjlim \Gamma(V \times F; \mathcal{F})$$

$$\pi_!\mathcal{F}_b = \Gamma_c(f^{-1}(b); \mathcal{F}) \cong \Gamma_c(F; \mathcal{F})$$

For covering

Let $\pi : X \to B$ be a covering with fibre F concrete points.

- Covering is a special case of locally trivial fibre bundle.
- For a local system (local constant sheaf) $\mathcal{L}$ over B. Assume it corresponds to $\pi_1(B) \to \operatorname{Aut} L$, then $\pi^*\mathcal{L}$ corresponds to $\pi_1(X) \subseteq \pi_1(B) \to \operatorname{Aut} L$.
- For a local system $\mathcal{L}$ over X, with correspondent $\pi_1(X)$-module be L, then

$$\pi_*\mathcal{L}_b = \varinjlim \Gamma(V \times F; \mathcal{L}) = L^{\prod F}.$$

$$\pi_!\mathcal{L}_b = \Gamma_c(F; \mathcal{L}) = L^{\oplus F}.$$

They correspond to $\operatorname{Hom}_{\pi_1(X)}(\pi_1(B), L)$ and $\pi_1(B) \otimes_{\pi_1(X)} L$ respectively.

≫ *Questions?* ≪

§ Thanks §

References

- Perhaps... No

Next Time

- Homological Algebra.
- Realize (co)homology groups from sheaves
- Derived categories.
- Reformulate theorems in singular cohomology
- Verdier duality.