Spectral Sequences (I)

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∼ § Introduction § ∼

Introduction

- **Spectral sequence** is a strong tool to understand (co)homology group (especially in computation) created by Leray During the second world war. This is so powerful so that it becomes the standard notion recently.
- It simplifies computations, constructions and proofs heavily, and our subsequent lectures rely on this tool more or less. Actually, one can find the material which is spectral-sequence-free, but it will be very long and tedious, also hard to understand.
- Roughly speaking, spectral sequence is a collection of all useful exact sequences, so it also helps understand homological algebra.

∼ § <u>Filtered complexes</u> § ∼

Filtered complexes

- Let $C^\bullet$ be a (cochain) complex in some module category

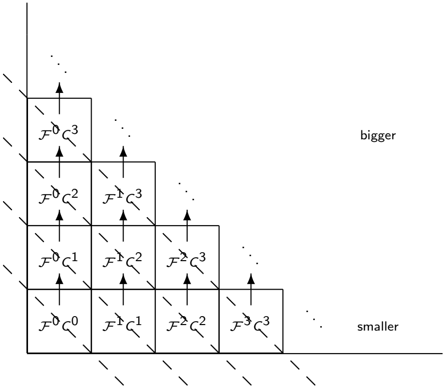
$$\cdots \to C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \to \cdots .$$

- Assume it is equipped with a (cochain) filtration $\mathcal{F}^*$ over the $C^\bullet$. (Note: the index bigger, the module finer)

$$\mathcal{F}: \quad \cdots \subseteq \mathcal{F}^{p+1} C^n \subseteq \mathcal{F}^p C^n \subseteq \mathcal{F}^{p-1} C^n \subseteq \cdots ,$$

and compatible with d

$$\mathcal{F}^p C^\bullet: \quad \cdots \to \mathcal{F}^p C^{n-1} \xrightarrow{d} \mathcal{F}^p C^n \xrightarrow{d} \mathcal{F}^p C^{n+1} \to \cdots .$$


$\mathcal{F}^0C^3$
$\mathcal{F}^0C^2$
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$\mathcal{F}^2C^3$
$\mathcal{F}^0C^0$
$\mathcal{F}^1C^1$
$\mathcal{F}^2C^2$
$\mathcal{F}^3C^3$
bigger
smaller

Filtered complexes

- The complex (formally, the graded complex with respect to the filtration)

$$\cdots \to \frac{\mathcal{F}^p C^{n-1}}{\mathcal{F}^{p+1} C^{n-1}} \to \frac{\mathcal{F}^p C^n}{\mathcal{F}^{p+1} C^n} \to \frac{\mathcal{F}^p C^{n+1}}{\mathcal{F}^{p+1} C^{n+1}} \to \cdots$$

  is easy to be understood.

- Toy example: consider a CW topological space X. The chain of singular homology complex $S^\bullet(X) = S_{-\bullet}(X)$ has a filtration (indices to suit our convention)

$$\mathcal{F}^{-p} S^\bullet = S_{-\bullet}(X_{\mathrm{dim} \leq p})$$

  Then $\frac{\mathcal{F}^p C^{n-1}}{\mathcal{F}^{p+1} C^{n-1}} = S_\bullet(X_{\mathrm{dim} \leq p}, X_{\mathrm{dim} \leq p-1})$.

Toy example

- We then have a long exact sequence

$$\cdots \to H_\bullet(X_{<p}) \to H_\bullet(X_{\le p}) \xrightarrow{\pi} H_\bullet(X_{\le p}, X_{<p}) \xrightarrow{\delta} H_{\bullet-1}(X_{\dim<p}) \to \cdots$$

We get the cellular complex

$$\cdots \to H_p(X_{\le p}, X_{<p}) \xrightarrow{\pi\circ\delta} H_{p-1}(X_{\le p-1}, X_{<p-1}) \to \cdots$$

- We know $H_\bullet(X_{\dim\le p}, X_{\dim<p}) = \begin{cases}\sum_{\text{cell } C \text{ of dim } p} \mathbb{Z}\cdot[C], \bullet=p \\ 0, \text{otherwise.}\end{cases}$, so above complex computes the cohomology of $H_\bullet(X)$ (standard topology).
- The chain of singular cohomology complex $S^\bullet(X)$ has a filtration

$$\mathcal{F}^n S^\bullet = \ker[S^\bullet(X) \to S^\bullet(X_{\dim\le n})]$$

has the similar property.

Spectral Sequences

A **spectral sequence** is the following

- a family of modules of $\mathcal{C}$

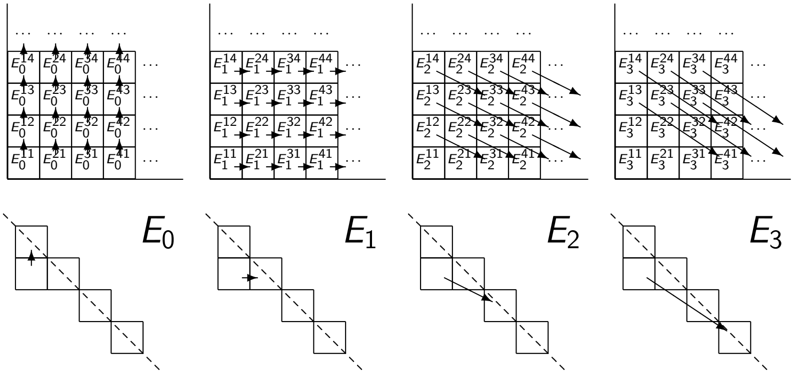
$$E = \{E_r^{pq} : p, q \in \mathbb{Z}, r \gg 0\};$$

- a family of differentials $d : E \to E$ with d^r of degree $(r, -r+1)$ for each r, that is

$$d = \{d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}\}, \quad d^r \circ d^r = 0;$$

- a family of isomorphisms of

$$E_{r+1}^{pq} \cong H^{pq}(E_r) = \frac{\ker[E_r^{pq} \xrightarrow{d} \cdots]}{\operatorname{im}[\cdots \xrightarrow{d} E_r^{pq}]} = \ker d_r^{pq} \,/\, \operatorname{im} d_r^{p-r,q+r-1}.$$


E_0
E_1
E_2
E_3

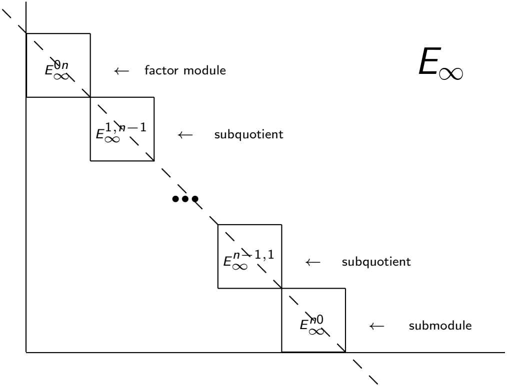
Convergence

- Under certain finiteness condition, for each p, q, $E_r^{pq} = E_{r+1}^{pq}$ for $r \gg 0$. So we can write E_∞^{pq}.
- Let E be a spectral sequence. Given a family of objects $\{H^n : n \in \mathbb{Z}\}$. We say that $\{E_r^{pq}\}$ **converges to** $\{H^n\}$, if there exists a filtration of each H^n

$$0 = \mathcal{F}^s H^n \subseteq \ldots \subseteq \mathcal{F}^{p+1} H^n \subseteq \mathcal{F}^p H^n \subseteq \ldots$$

such that $\bigcup_p F_p H^n = H_n$ and $E_\infty^{pq} \cong \mathcal{F}^p H^{p+q} / \mathcal{F}^{p+1} H^{p+q}$. We write

$$E_r^{pq} \Longrightarrow H^{p+q}.$$


E_∞^{0n}
← factor module
$E_\infty^{1,n-1}$
← subquotient
$E_\infty^{n-1,1}$
← subquotient
E_∞^{n0}
← submodule
E_∞

SS for filtered complexes

Theorem

Each filtered (cochain) complex $(C, \mathcal{F})$ determines a spectral sequence E with

$$\begin{aligned} E_0^{pq} &= \mathcal{F}^p C^{p+q} / \mathcal{F}^{p+1} C^{p+q} \\ E_1^{pq} &= H^{p+q}(\mathcal{F}^p C / \mathcal{F}^{p+1} C). \end{aligned}$$

If the filtration $\mathcal{F}$ over C is lower bounded and upper exhaustive then E converges to $H^\bullet(C)$. More exactly,

$$E_\infty^{pq} \cong \mathcal{F}^p H^{p+q}(C, d) / \mathcal{F}^{p+1} H^{p+q}(C, d),$$

where $\mathcal{F}$ is lower bounded and exhaustive filtration over $H^\bullet(C, d)$.

- lower bounded $\iff$ for any n, $\mathcal{F}^p C^n = 0$ for some p.
- exhaustive $\iff$ $\bigcup_p \mathcal{F}^p C^n = C^n$.

Sketch of the Proof

- What we want is the kernel and image of $C^\bullet$.
- hat we can compute is the kernel and image of $\mathcal{F}^p C^\bullet / \mathcal{F}^{p+1} C^\bullet$. That is,

$$\mathcal{F}^* C^* + d^{-1}(\mathcal{F}^* C^*) \cap \mathcal{F}^* C^*, \qquad \mathcal{F}^* C^* + d(\mathcal{F}^* C^*) \cap \mathcal{F}^* C^*$$

  where $*$ are some indices.
- So we have a filtration over $\operatorname{im} d$ by $d(\mathcal{F}^* C^*)$ and a filtration over $\ker d$ by $d^{-1}(\mathcal{F}^* C^*)$. So we can use the Zassenhaus' butterfly lemma to analyse.
- Carefully dealing with them, we will get the result.
- Ref: my paper `https://arxiv.org/abs/2002.06394`.

For chain of complexes

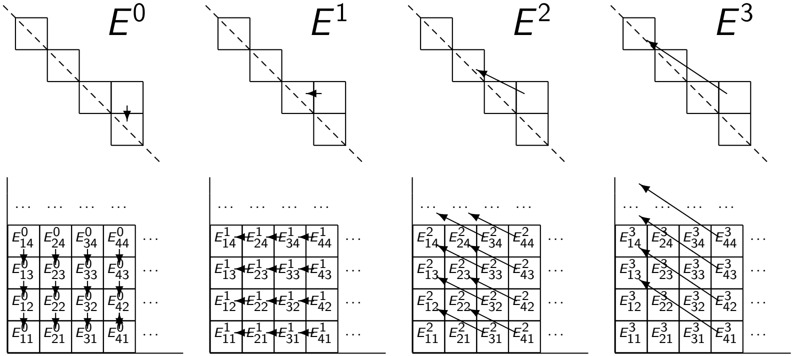
- The same result holds for chain of complexes.
- Everything works by simply $E^r_{pq} = E^r_{-p,-q}$, but of course, we should put it in the first quadratic. By convention, we always draw arrows of gentle slope.
- Formally,

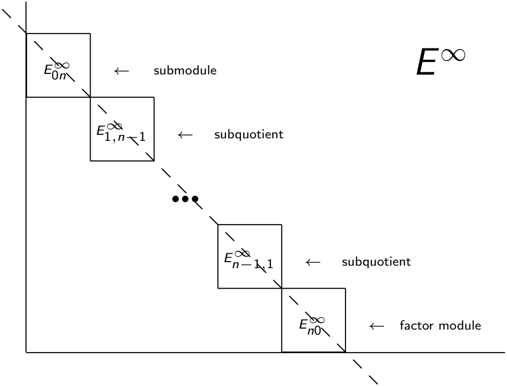
$$\cdots \to C_{n-1} \to C_n \to C_{n-1} \to \cdots$$

The filtration is defined to be (Note: the index bigger, the module bigger)

$$\cdots \subseteq \mathcal{F}_{p-1} C_\bullet \subseteq \mathcal{F}_p C_\bullet \subseteq \mathcal{F}_{p+1} C_\bullet \subseteq \cdots .$$

The filtration of the convergence also takes this order.


E^0
E^1
E^2
E^3



E^{∞}_{0n}
← submodule
$E^{\infty}_{1,n-1}$
← subquotient
$E^{\infty}_{n-1,1}$
← subquotient
E^{∞}_{n0}
← factor module
E^{∞}

∼ § Leray–Serre Spectral Sequences § ∼

A short discussion of fibre bundle

- For convenience, let us use only (local trivial) **fibre bundle** $\pi = \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ of fibre F, say, for each point $x \in B$, there is a neighborhood U of x, such that
$$\begin{array}{ccc} \pi^{-1}(U) & \cong & U \times F \\ \downarrow & & \downarrow \text{proj} \\ U & = & U \end{array}$$
- Toy example: $E = B \times F$. Then the cohomology can be computed by the Künneth formula. For example, if the coefficient ring is a field, then $H^*(E) = H^*(B) \otimes H^*(F)$.
- Toy example: vector bundles (we will face it several times in lectures after) always with $F = \mathbb{C}^n$. For example, tangent bundles over manifolds.

Statement

Theorem (Leray–Serre)

Assume we have a fibration $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ *with fibre* F*, then there is a spectral sequence* E *with*

$$E_2^{pq} = H^p(B; \mathcal{H}^q(F))$$

converging to $H^\bullet(E; R)$.

- Here $\mathcal{H}^q(F)$ stands for the local coefficient system of cohomology of fibre. If in particular when B is simply connected, it will be simply $H^q(F)$.

Sketch of the proof

- We can assume B is a CW complex (by approximation). Then we can filtered X by the preimage of $B_{\mathrm{dim}\leq \bullet}$. Let us, an abuse of notation, simply denote $X_{\leq \bullet}$.
- Then, as we stated $S^\bullet(X)$ has a filtration by

$$\mathcal{F}^p S^\bullet = \ker(S^\bullet(X) \to S^\bullet(X_{\leq p})),$$

and then

$$\begin{aligned} E_1^{pq} &= H^{p+q}(X_{\leq p}, X_{<p}) \\ &= \textstyle\sum_{i+j=p+q} H^i(B_{\leq p}, B_{<p}) \otimes H^j(F) \\ &= H^p(B_{\leq p}, B_{<p}) \otimes H^q(F) \end{aligned} .$$

- Analysing d_1 carefully, we get it is exactly the cellular cohomology chain to compute $H^p(B; \mathcal{H}^q(F; R))$.

Remarks

- We have the same result for homology

$$E^2_{pq} = H_p(B; \mathcal{H}_q(F))$$

  and it converges to $H_{p+q}(E)$.
- The cup product is well-defined, say

$$E_r^{pq} \times E_r^{p'q'} \to E_r^{p+p',q+q'}$$

  compatible with taking cohomology, and compatible with the subquotient $H^{p+q}(E)$. The same for cap produce.

Remarks (continued)

- There is also a relative version, say,

$$H_p(B, B_0; \mathcal{H}_q(F, R))$$

converges to $H_{p+q}(E, E_0)$, where $E_0 = \pi^{-1}(B)$;

$$H_p(B; \mathcal{H}_q(F; F_0, R))$$

converges to $H_{p+q}(E, E_0)$, where E_0 is a subbundle of F with fibre F_0.

- As a result, it also has the version for cohomology of compact support (more generally the Borel–Moore (co)homology).

Remarks (continued)

- This spectral sequences constructed is also functorial, since the map can be approximated by cellular map.
- As a result, the map $\begin{bmatrix} F \\ \downarrow \\ pt \end{bmatrix} \to \begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, shows that the map $H^*(E) \to E_\infty^{*0} \subseteq H^*(F)$ coincides with the induced map.
- As another result, the map $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix} \to \begin{bmatrix} B \\ \downarrow \\ B \end{bmatrix}$, shows that the map $H^*(B) \twoheadrightarrow E_\infty^{0*} \to H^*(E)$ coincides with the induced map.
- Actually, we can say more, the map $E_n^{0,n-1} \to E_n^{n,0}$ is described, called the **transgression**.

Remarks (continued)

- Assume the fibre F to be orientable, compact and smooth of dimension d. Now, E_r^{pq} vanishes for $q > d$, and if B is orientable, then $\mathcal{H}^d(F) = H^d(F)$. Now $E_\infty^{\bullet d}$ is the a factor module, so

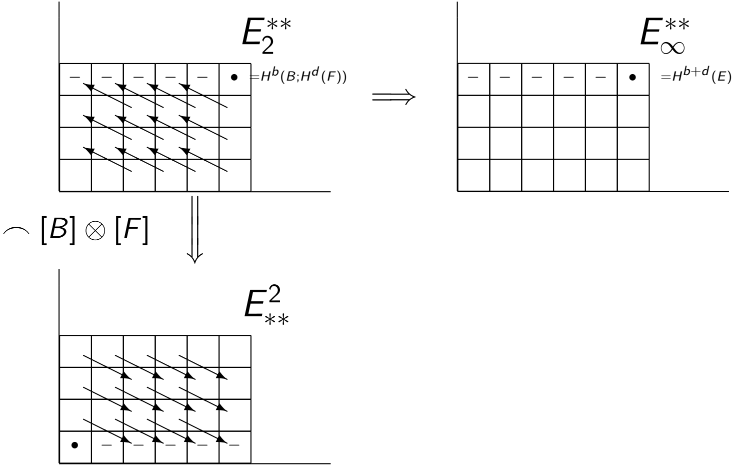
$$H^{\bullet+d}(E) \to E_\infty^{\bullet d} \subseteq E_2^{\bullet d} = H^\bullet(B; \mathcal{H}^d(F)) = H^\bullet(B).$$

- I claim this map coincides with the **push forward** defined as follows,

$$H^{\bullet+d}(E) \xrightarrow[\text{Poincaré duality}]{} H_*(E) \xrightarrow{\pi_*} H_*(B) \xleftarrow[\text{Poincaré duality}]{} H^d(B)$$

  If both B, E is orientable, compact and smooth of dimension $b, b+d$.

- Since formally Poincaré duality is taking cap product with fundamental class. We know $H_b(B; H_d(F)) = H_{b+d}(E)$, it suffices to show that $[B] \otimes [F]$ corresponds to $[E]$.


E_2^{**}
$=H^b(B;H^d(F))$
E_∞^{**}
$=H^{b+d}(E)$
$\frown [B] \otimes [F]$
E^2_{**}

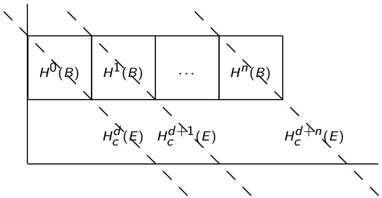
$\sim$ § Applications § $\sim$

Thom isomorphism

- Let $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ be a vector bundle of rank d with B orientable, compact and smooth. Since $H_c^\bullet(\mathbb{R}^d)$ has only one nonvanishing dimension, and when B is orientable, $\mathcal{H}_c^\bullet(\mathbb{R}^d) = H^\bullet(\mathbb{R}^d)$ and $H^\bullet(B) = H_c^\bullet(B)$, so

$$H^n(B) \cong H_c^{n+d}(E),$$

known as the **Thom isomorphism**.



Thom classes

- Let us call the image of $1 \in H^0(B)$ under this isomorphism by the **Thom class**. It is originally defined by the image of the fundamental class of $[B]$ under

$$H_n(B) \xrightarrow[\text{zero section}]{} H_n(E) \xrightarrow[\text{Poincaré duality}]{} H_c^d(X)$$

- The Thom isomorphism can be written by

$$H^n(B) \cong H^n(E) \xrightarrow{\smile\tau} H_c^{n+d}(E),$$

where $\tau \in H_c^d(E)$ is the Thom class.

- The proof is easy, note that the isomorphism given by the spectral sequence is simply due to $H_*(B) \to H_*(E)$.

Remark

- Equivalently, we can consider the unit disc at each point (first choose a Riemannian metric), say W, and apply the compactification at each fibre, denoting the resulting fibre be $\begin{bmatrix} X \\ \downarrow \\ B \end{bmatrix}$,

$$H_c^{n+d}(X) = H^{n+d}(W, \partial W) = H^{n+d}(X; B_\infty),$$

  where B_∞ the infinite section.

- Thom class is useful to define the "cycle". Let $N \to M$ be an embedding, consider its normal bundle as a "tube" W, then it induces

$$H^n(N) \to H_c^n(W, \partial W) \to H^{n+d}(M, M \setminus N) \to H^{n+d}(M)$$

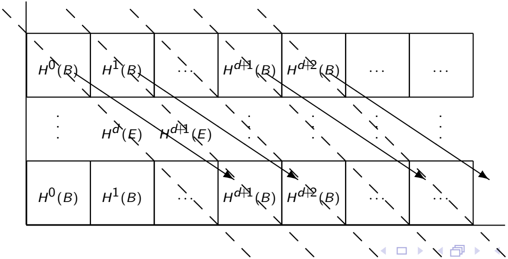
  which is dual to $H_n(N) \to H_n(M)$. This is the strict way to explain the intersection product.

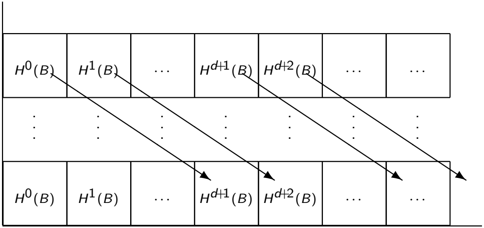
Gysin sequences

- If the fibre $F = S^d$, we will call $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$ a sphere bundle. Then, since $H^*(F) = H^*(S^d)$ has only two nonvanishing dimensions, so there is a long exact sequence

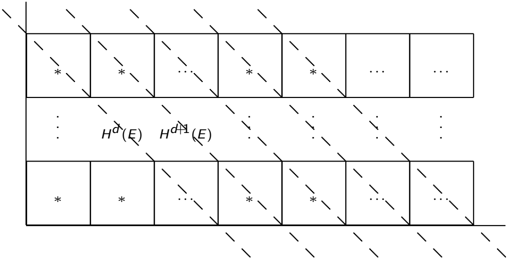
$$\cdots \to H^\bullet(B) \stackrel{\pi^*}{\to} H^\bullet(E) \stackrel{\pi_*}{\to} H^{\bullet-d}(B) \stackrel{d}{\to} H^{\bullet+1}(B) \to \cdots$$

called the **Gysin sequence**.





before achieving E_∞



E_∞

Euler Classes

- The π_* and π^* is well-understood, at least when everything is orientable, compact and smooth.
- Denote the image of $1 \in H^0(B)$ under d, say $e \in H^{d+1}(B)$, and call it the **Euler class**. Since d processes the Lebniz rule with respect to multiplicative, d is simply cup product with e.
- Classically, the Euler class is defined for a vector bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, by restricting the Thom class in $H_c^d(E)$ to $H_c^d(B) = H^d(B)$ by zero section $B \subseteq E$.
- The unit sphere at each point defines an S^{d-1} bundle, say E_S. The coincidence is actually proven by the transgression.

Euler Classes

- The name Euler class is more ancient. The Euler class for tangent bundle is Poincaré dual to the Eular characteristic (alternative sum of Betti numbers). The proof is by the diagonal embedding $M \to M \times M$ whose normal bundle is isomorphic to the tangent bundle of M, then it follows by a computation of the class of diagonal.
- The geometric meaning is really curious. Dualizing everything to the homology group, then Thom class is exactly the cycle of any section, then the Euler class is exactly the zero locus of any section. In particular, this gives a proof of Poincaré–Hopf theorem.

Leray–Hirsch theorem

- There is a classic theorem about the cohomology of fibre bundle by **Leray and Hirsch**. For a fibre bundle $\begin{bmatrix} E \\ \downarrow \\ B \end{bmatrix}$, if each fibre $F_x = \pi^{-1}(x)$ has free cohomology, and there is a set $\{\alpha\} \subseteq H^\bullet(E)$ present the bases restricting each fibre. Then

$$H^*(B) \otimes H^*(F) \longrightarrow H^*(E) \qquad \beta \times i_*\alpha \longrightarrow \pi_*\beta \smile \alpha$$

  is an isomorphism of $H^*(B)$-modules.
- I am not sure whether Leray–Serre spetral sequence implies Leray–Hirsch theorem.

Remarks

- It is classic to use the Leray–Hirsch theorem to compute several cohomology of fibre bundles

$$\mathrm{Sp}(n), \qquad \mathrm{GL}(n;\mathbb{C}) \simeq \mathrm{U}(n), \qquad \mathrm{SL}(n;\mathbb{C}) \simeq \mathrm{SU}(n), \qquad \mathcal{G}r(n,\infty),$$

  which I do not want to present the details here.
- The classic proof of the Leray–Hirsch theorem is very standard trick of algebraic topology, which I want to explain it here. We construct some map which is "natural", and check it is true for the simplest case, and check it is an isomorphism all over all CW complexes by five lemma.

References

- Benson. Cohomology and Representation, second volume.
  This book gives a quick course of topology, and also discusses spectral sequences. The definition and proofs suit better the flavor of algebraists.
- May. A Concise Course in Algebraic Topology.
  This book gives a self-contained and concise (and correct) introduction to topology.
- Broden. Topology and Geometry.
  The classic treatment of Thom class and Euler class.
- Hatcher. Algebraic Topology.
  The classic computation using the Leray–Hirsch theorem.

Next time

- The Spectral Sequences for Double Complex.
- Grothendieck Spectral Sequence.
- Other Spectral Sequences.
- We will not discuss the exact couple, Elienburg–Moore spectral sequences and Adams spectral sequences.

$\sim$ § <u>THANKS</u> § $\sim$