

# Half of Advanced Algebra (With Hints)

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# 半本高代习题集

(带提示的那种)

## Preface

► 翻译: 前言

For some reasons, the linear algebra is a course with serial difficult problems, especially the part afterwards involving similarity and Jordan cardinal form. But actually, such problems are limited, some classic conclusions can solve a large branch of problems. However, it seems to be less convenient to find a book collecting them, so I make one. Besides the classic conclusions, I also add some problems of profound background from algebra or analysis.

► 翻译: 由于某些原因, 线性代数是一个有一系列难题的课程, 尤其是后面部分涉及相似以及 Jordan 标准型. 但是实际上, 这样的问题是有限的, 一些经典的结论能够解决一系列问题. 但是, 似乎并没有那么方便地找到一本收集了这些问题的书, 所以我写了一本. 除了经典结论, 我还加入了一些有深入背景的问题, 这些背景来自于代数和分析.

In my book, only the problems without any difficulties do not have a hint. Most of problems have detailed guidance. The exercises are divided into three types, exercises, problems and fuxercises (Feeling-Upset exercises). Exercises are not hard. Problems are not easy and important. Fuxercises are really hard but not of importance for linear algebra, so do not disturb yourself if not figure them out.

► 翻译: 在本书中, 只有没有任何难度的问题没有提示. 大部分问题都有详实的指导. 习题被分为三类, 习题, 问题和刁题. 习题比较容易, 问题并不容易, 且很重要. 刁题很刁, 但对线性代数并不重要, 所以没有做出来也不必担心.

I wish you could enjoy these problems.

► 翻译: 但愿你能享受这些问题.

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► 翻译:

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# 1 Week 1 (25 Feb - 3 Mar)

► 翻译: 第一周 (2月25日 - 3月3日)

**Exercise 1** Let  $[a, b]$  be a closed interval in  $\mathbb{R}$ , and  $\mathcal{C}[a, b]$  be all the continuous function from  $[a, b]$  to  $\mathbb{R}$ .

► 翻译: 令  $[a, b]$  是实数上的闭区间, 令  $\mathcal{C}[a, b]$  是所有  $[a, b] \rightarrow \mathbb{R}$  的连续函数.

(0) Check that  $\mathcal{C}[a, b]$  is an  $\mathbb{R}$ -linear space under the natural addition and scalar product.

► 翻译: (0) 验证  $\mathcal{C}[a, b]$  对自然的加法和数乘构成一个  $\mathbb{R}$ -线性空间.

(1) Show that  $\mathcal{C}[a, b]$  cannot be finite dimensional. Hint Show that it has arbitrarily many linearly independent vectors. For example, without loss of generality, take  $[a, b] = [0, 1]$ , one can take some function only “living” in  $[\frac{i}{n}, \frac{i+1}{n}]$ . Such method can show that  $\mathcal{C}^\infty[a, b]$  cannot be finite dimensional if you know ‘bump function’.

► 翻译: (1) 证明  $\mathcal{C}[a, b]$  不能是有限维.

(2) Prove that  $f_1, \dots, f_n \in \mathcal{C}[a, b]$  is linearly independent iff

$$\exists x_1, \dots, x_n \in [a, b], \text{ such that } \det(f_i(x_j))_{ij} \neq 0$$

Hint Using induction on  $n$ . More exactly, firstly,  $\Leftarrow$  part is easy. For the inverse direction, one can find  $x_1, \dots, x_{n-1}$  such that  $\det(f_i(x_j))_{1 \leq i, j \leq n-1} \neq 0$ , then consider  $\det(f_i(x_j))_{1 \leq i, j \leq n}$  as a function of  $x_n$ , it can not be zero, otherwise,  $f_1, \dots, f_n$  is linearly dependent.

► 翻译: (2) 证明  $f_1, \dots, f_n \in \mathcal{C}[a, b]$  是线性无关的当且仅当

$$\exists x_1, \dots, x_n \in [a, b], \text{ 使得 } \det(f_i(x_j))_{ij} \neq 0$$

(3) Prove that  $f_1, \dots, f_n \in \mathcal{C}[a, b]$  has rank  $r$  then there exists  $r$   $f_i$ 's, say  $f_{i_1}, \dots, f_{i_r}$ , and  $x_1, \dots, x_r \in [a, b]$  such that  $\det(f_{i_k}(x_j))_{kj} \neq 0$ . Hint This is clear.

► 翻译: (3) 证明  $f_1, \dots, f_n \in \mathcal{C}[a, b]$  的秩是  $r$  那么存在  $r$  个  $f_i$ , 设为  $f_{i_1}, \dots, f_{i_r}$ , 和  $x_1, \dots, x_r \in [a, b]$  使得  $\det(f_{i_k}(x_j))_{k,j} \neq 0$ .

(4) Point out which fact (3) is a general form of. Hint If a matrix has rank  $r$ , then it has a submatrix of size  $r \times r$  whose determinant non-vanishes.

► 翻译: 指出 (3) 是哪个事实的推广.

**Exercise 2** Given a vector space  $V$ .

► 翻译: 给定一个线性空间  $V$ .

(1) Prove that if two subspaces  $U, W \subseteq V$ , such that  $U \cup W$  is still a subspace, then  $U \subseteq W$  or  $W \subseteq U$ .

► 翻译: (1) 证明两个子空间  $U, W \subseteq V$  如果使得  $U \cup W$  仍然是线性子空间, 那么  $U \subseteq W$  或  $W \subseteq U$ .

(2) Can you give the general case of (1) for more subspace? Hint For  $n$  subspace  $U_1, \dots, U_n \subseteq V$ , if  $U_1 \cup \dots \cup U_n$  is still a subspace, then there exists some  $U_k$  contains all members of  $\{U_i\}$ . Merely because finite proper subspaces cannot cover whole space.

► 翻译: (2) 你能给出 (1) 的一般情况吗?

**Exercise 3** Prove the following ‘modular property’.

► 翻译: 证明下面的“模性”.

Let  $A, B, C$  be subspace of some bigger linear space. If  $C \subseteq A$ , then

$$(A \cap B) + C = A \cap (B + C)$$

And point out by giving example that if releases the condition  $C \subseteq A$ , the above statement breaks.

► 翻译: 令  $A, B, C$  是某个更大线性空间的子空间, 如果  $C \subseteq A$ , 那么

$$(A \cap B) + C = A \cap (B + C)$$

并且用例子指出如果去掉条件  $C \subseteq A$ , 那么上述命题失效.

Notes that there is no ‘distribution law’ for linear space, that is

$$(U + V) \cap W \text{ not in general equals to } (U \cap W) + (V \cap W)$$

$$(U \cap V) + W \text{ not in general equals to } (U + W) \cap (V + W)$$

► 翻译: 注意到线性空间没有分配律, 即

$$(U + V) \cap W \text{ 一般不等于 } (U \cap W) + (V \cap W)$$

$$(U \cap V) + W \text{ 一般不等于 } (U + W) \cap (V + W)$$

**Exercise 4** Show that for any matrix  $A$  of size  $n \times n$ , there exists polynomial  $f \neq 0$  such that  $f(A) = 0$ . And one can assume more  $\deg f \leq n^2$ . Hint  
Prove that  $\{I, A, A^2, \dots, A^{n^2-1}\}$  is linearly dependent.

► 翻译: 证明任何  $n$  阶方阵  $A$ , 都存在非零多项式  $f$  使得  $f(A) = 0$ . 且可以假设  $\deg f < n^2$ .

And if  $A$  is invertible, prove that we can assume the constant term of  $f$  can be taken to be nonzero.

► 翻译: 如果  $A$  可逆, 证明  $f$  的常数项可以取得非零.

**Exercise 5** Calculate the integral for  $m, n \in \mathbb{Z}_{\geq 0}$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \cos m\theta d\theta \quad \frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \sin m\theta d\theta \quad \frac{1}{2\pi} \int_0^{2\pi} \sin n\theta \sin m\theta d\theta$$

to show  $\{\cos nx : n \in \mathbb{Z}_{\geq 0}\} \cup \{\sin nx : n \in \mathbb{Z}_{\geq 0}\}$  is linearly independent.

► 翻译: 对  $m, n \in \mathbb{Z}_{\geq 0}$ , 计算下列积分

$$\frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \cos m\theta d\theta \quad \frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \sin m\theta d\theta \quad \frac{1}{2\pi} \int_0^{2\pi} \sin n\theta \sin m\theta d\theta$$

来导出  $\{\cos nx : n \in \mathbb{Z}_{\geq 0}\} \cup \{\sin nx : n \in \mathbb{Z}_{\geq 0}\}$  是线性无关的.

**Problem 6 (Subspace avoidance)** In this problem, we will give several proof of the fact ‘finite proper subspaces cannot cover whole space’. More precisely, given a linear space  $V$  over, for example  $\mathbb{Q}$ ,  $U_1, \dots, U_n \subsetneq V$  are proper subspaces, then

$$U_1 \cup \dots \cup U_n \subsetneq V$$

► 翻译: (子空间回避) 在本习题中, 我们将要给出各种对于“有限真子空间不能覆盖整个空间”的证明. 更准确地说, 对于, 例如  $\mathbb{Q}$  上的, 线性空间  $V$ ,  $U_1, \dots, U_n \subsetneq V$  是真子空间, 那么

$$U_1 \cup \dots \cup U_n \subsetneq V$$

The first proof is by using induction.

► 翻译: 第一个证明来自归纳法.

(1) Firstly, prove the case of  $n = 2$ .

► 翻译: (1) 首先, 证明  $n = 2$  的情况.

(2) Secondly, by induction, assume for any  $U_i$ ,  $U_i$  is not contained in  $U_1 \cup \dots \cup \widehat{U_i} \cup \dots \cup U_n$ . Where  $\hat{*}$  means skip. thus one can pick  $x_i$  such that

$$x_i \in U_j \iff i = j$$

Note that  $|\{x_n + \lambda x_i : \lambda \in \mathbb{Q}\}| = \infty$ , thus by pigeonhole principle, for some  $U_j$ ,  $x_n + \lambda x_i, x_n + \lambda' x_i \in U_j$  for  $\lambda \neq \lambda'$ , then  $x_i \in U_j$ , then  $i = j$ , then  $x_n \in U_i$ , a contradiction.

Write down the proof above with more details.

► 翻译: (2) 其次, 根据归纳法, 假设任何  $U_i$ ,  $U_i$  不包含在  $U_1 \cup \dots \cup \widehat{U_i} \cup \dots \cup U_n$  之中, 其中  $\hat{*}$  表示跳过, 于是可以挑选  $x_i$  使得

$$x_i \in U_j \iff i = j$$

注意到  $|\{x_n + \lambda x_i : \lambda \in \mathbb{Q}\}| = \infty$ , 那么根据鸽笼原理, 对某个  $U_j$ ,  $x_n + \lambda x_i, x_n + \lambda' x_i \in U_j$  对  $\lambda \neq \lambda'$ , 于是  $x_i \in U_j$ , 从而  $i = j$ , 进而  $x_n \in U_i$ , 产生矛盾. 将上述证明更详细地写下来.

The second proof is based on Vandermonde determinant.

► 翻译: 第二个证明基于 Vandermonde 行列式.

(3) Firstly, pick a set of basis of  $V$ , say  $e_1, \dots, e_n$ , consider

$$x_\lambda = e_1 + \lambda e_2 + \dots + \lambda^{n-1} e_n$$

show that any  $n$  members of  $\{x_\lambda : \lambda \in \mathbb{Q}\}$  forms basis for  $V$ .

► 翻译: (3) 首先, 选择  $V$  的一组基  $e_1, \dots, e_n$ , 考虑

$$x_\lambda = e_1 + \lambda e_2 + \dots + \lambda^{n-1} e_n$$

证明任意  $n$  个  $\{x_\lambda : \lambda \in \mathbb{Q}\}$  中元素形成  $V$  的一组基.

(4) Prove that  $U_i$  can only has finite many member of  $\{x_\lambda : \lambda \in \mathbb{Q}\}$ . Thus the proof is complete.

► 翻译: (4) 证明  $U_i$  只能含有  $\{x_\lambda : \lambda \in \mathbb{Q}\}$  有限的成员. 证明完毕.

The third proof is based on some idea of algebraic geometry.

► 翻译: 第三个证明基于一些代数几何的想法.

(5) Prove that there is no loss of generality to assume  $U_i$  is of codimension 1 (that is,  $\dim U_i = \dim V - 1$ ).

► 翻译: (5) 证明假设  $U_i$  余维数都是 1 并不损耗一般性.

(6) By assuming  $V = \mathbb{Q}^n$ , show that  $U_i$  is zero of some linear polynomial of  $X_1, \dots, X_n$ . And  $U_1 \cup \dots \cup U_n$  is zero of their product.

► 翻译: (6) 通过假设  $V = \mathbb{Q}^n$ , 证明  $U_i$  是某个线性多项式的零点,  $U_1 \cup \dots \cup U_n$  则是他们乘积的零点.

(7) Complete the proof. Hint The polynomial  $\neq 0$ , thus some élément of  $\mathbb{Q}^n$  such it takes non-zero value.

► 翻译: (7) 完成证明.

**Problem 7** Prove that countable many proper subspaces of  $\mathbb{R}$ -linear space cannot cover the whole space. Hint First prove the case of dimension 1. And using induction, note that the cardinality of proper subspace is beyond countability, and two different proper subspaces intersect a proper subspace of both of them. Of course, if you know Lebsgue measure or Baire category's argument, then you can prove it more directly.

► 翻译: 证明可数个  $\mathbb{R}$ -线性空间的真子空间不能覆盖整个空间.

**Problem 8 (Algebraic number)** In this problem, we will introduce the concept of algebraic number. A number  $\alpha \in \mathbb{C}$  is said to be an algebraic number if there exists nonzero polynomial  $f(X) \in \mathbb{Q}[X]$  such that  $f(\alpha) = 0$ .

► 翻译: (代数数) 在这个问题中, 我们将要介绍代数数的概念. 一个复数  $\alpha$  被称为是代数数, 如果存在非零多项式  $f(X) \in \mathbb{Q}[X]$  使得  $f(\alpha) = 0$ .

(1) Write the  $\mathbb{Q}$ -linear subspace of  $\mathbb{C}$  expanded by  $1, \alpha, \alpha^2, \dots$  by  $\mathbb{Q}[\alpha]$ . Prove that

$$\dim \mathbb{Q}[\alpha] < \infty \iff \alpha \text{ is an algebraic number}$$

**Hint** The  $\Leftarrow$  is easy by showing any polynomial  $g \in \mathbb{Q}[X]$ , exists  $h \in \mathbb{Q}[X]$  such that  $g(\alpha) = h(\alpha)$  with  $\dim h$  limited. Conversely,  $1, \alpha, \alpha^2, \dots$  is linear dependent, then some polynomial kill  $\alpha$ .

► 翻译: (1) 记  $\mathbb{Q}[\alpha]$  表示  $1, \alpha, \alpha^2, \dots$  在  $\mathbb{C}$  中张成的  $\mathbb{Q}$ -线性子空间. 证明:

$$\dim \mathbb{Q}[\alpha] < \infty \iff \alpha \text{ 是代数数}$$

(2) Show that the sum, the difference, the product and the quotient (with denominator nonzero, of course) of algebraic number are still algebraic number. **Hint** Since  $\mathbb{Q}[\alpha + \beta] \subseteq \mathbb{Q}[\alpha] \cdot \mathbb{Q}[\beta]$ , by picking a basis, one shows that the right side is of finite dimension.

► 翻译: 证明: 代数数的加减乘除 (当然, 要求分母不为零) 依旧是代数数.

**Problem 9** In this problem, we will show the fact that if  $f, g \in \mathbb{R}[T]$  are two polynomials in  $T$ , then there exists nonzero polynomial  $h(X, Y) \in \mathbb{R}[X, Y]$  such that

$$h(f(T), g(T)) = 0$$

It is a special case of theory of transcendant basis. But here it only involves linear algebra.

► 翻译: 在本问题中, 我们将要证明如下事实, 如果  $f, g \in \mathbb{R}[T]$  是两个以  $T$  为不定元的多项式, 那么存在非零多项式  $h(X, Y) \in \mathbb{R}[X, Y]$  使得

$$h(f(T), g(T)) = 0$$

这是超越基理论的特殊情况. 但这里只用到线性代数.

(1) For fixed integers  $m, n, k \geq 0$ , show that the inequality  $mx + ny \leq k$  of  $(x, y)$  has more than  $\frac{(k-m)(k-n)}{2mn}$  solutions, where  $x, y$  are both required to be non-negative integers. **Hint** Draw a figure.

► 翻译: 对固定的  $j \geq 0$ , 证明关于  $(x, y)$  的不等式  $mx + ny \leq k$  有至少  $\frac{(k-m)(k-n)}{2mn}$  个解, 其中  $x, y$  都是非负整数.

(2) Complete the proof. **Hint** Show that  $\{f(T)^x g(T)^y : x, y \in \mathbb{Z}_{\geq 0}\}$  are linearly dependent.

► 翻译: 完成证明.

## 2 Week 2 (4 Mar - 10 Mar)

► 翻译: 第二周 (3月4日 - 3月10日)

**Exercise 10** Given a linear space  $V$ , if  $V = A \oplus B = C \oplus D$  and  $A \subseteq C$ , prove that  $C = A \oplus (B \cap C)$ . Hint Using modular property, you can give a quick proof.

► 翻译: 对于线性空间  $V$ , 若  $V = A \oplus B = C \oplus D$ , 且  $A \subseteq C$ , 求证:  $C = A \oplus (B \cap C)$ .

**Exercise 11** For two fields  $K, F \subseteq \mathbb{C}$ , assume that  $K \subseteq F$ .

► 翻译: 对于两个域  $K, F \subseteq \mathbb{C}$ , 如果  $K \subseteq F$ .

(1) Prove that  $F$  is a  $K$ -linear space.

► 翻译: (1) 证明  $F$  是  $K$ -线性空间.

(2) Prove that any  $F$ -linear space is naturally a  $K$ -linear space.

► 翻译: (2) 证明任何  $F$ -线性空间都自然成为  $K$ -线性空间.

(3) If  $F$  is of  $n$ -dimensional over  $K$ , and  $V$  an  $F$ -linear of dimension  $m$ , show that  $V$  is of dimension  $mn$  as a linear space over  $K$ .

► 翻译: (3) 如果  $F$  在  $K$  上维数是  $n$ ,  $V$  是  $m$  维  $F$ -线性空间, 证明  $V$  作为  $K$  线性空间是  $mn$  维的.

**Exercise 12** Given an  $\mathbb{R}$ -linear space  $V$  and a linear transform  $\mathcal{P} : V \rightarrow V$ . If  $\mathcal{P}^2 = \mathcal{P}$ , we say  $\mathcal{P}$  is a projection.

► 翻译: 对于一个  $\mathbb{R}$ -线性空间  $V$  和一个线性变换  $\mathcal{P} : V \rightarrow V$ . 如果  $\mathcal{P}^2 = \mathcal{P}$ , 我们说  $\mathcal{P}$  是一个投影.

(1) Prove that  $V = \ker \mathcal{P} \oplus \text{im } \mathcal{P}$ , and  $V$  acts as identity over  $\text{im } \mathcal{P}$ .

Can you give a geometry interpretation of these transform? Hint  $x = (x - \mathcal{P}x) + \mathcal{P}x$ .

► 翻译: (1) 证明  $V = \ker \mathcal{P} \oplus \text{im } \mathcal{P}$ , 并且  $V$  在  $\text{im } \mathcal{P}$  上作用是恒等映射. 你能给这类变换一个几何解释吗?

(2) Let  $\mathcal{P}' = I - \mathcal{P}$ , show that

$$\begin{cases} (\mathcal{P}')^2 = \mathcal{P}' \\ \mathcal{P}\mathcal{P}' = \mathcal{P}'\mathcal{P} = 0 \end{cases} \quad \begin{cases} \ker \mathcal{P} = \text{im } \mathcal{P}' \\ \ker \mathcal{P}' = \text{im } \mathcal{P} \end{cases}$$

► 翻译: 令  $\mathcal{P}' = I - \mathcal{P}$ , 证明

$$\begin{cases} (\mathcal{P}')^2 = \mathcal{P}' \\ \mathcal{P}\mathcal{P}' = \mathcal{P}'\mathcal{P} = 0 \end{cases} \quad \begin{cases} \ker \mathcal{P} = \text{im } \mathcal{P}' \\ \ker \mathcal{P}' = \text{im } \mathcal{P} \end{cases}$$

(3) Prove that the decompositions  $V = V_1 \oplus V_2$  are one-to-one corresponding to the projections  $\mathcal{P}$  over  $V$  through

$$V_1 = \ker \mathcal{P} \quad V_2 = \text{im } \mathcal{P}$$

Hint Construct  $\mathcal{P}$  exactly.

► 翻译: 证明分解  $V = V_1 \oplus V_2$  和  $V$  上的投影  $\mathcal{P}$  通过

$$V_1 = \ker \mathcal{P} \quad V_2 = \text{im } \mathcal{P}$$

一一对应.

**Exercise 13** Given a finite dimensional  $\mathbb{R}$ -linear space  $V$  and a linear transform  $\mathcal{S} : V \rightarrow V$ . If  $\mathcal{S}^2 = I$ , we say  $\mathcal{S}$  is a reflection.

► 翻译: 对于一个  $\mathbb{R}$ -线性空间  $V$  和一个线性变换  $\mathcal{S} : V \rightarrow V$ . 如果  $\mathcal{S}^2 = I$ , 我们说  $\mathcal{S}$  是一个反射.

(1) Let  $\text{Fix } \mathcal{S} = \{v \in V : \mathcal{S}v = v\}$ . Show that  $V = \text{Fix } \mathcal{S} \oplus \text{Fix}(-\mathcal{S})$ . Can you give a geometry interpretation of these transform? Hint  $v = \frac{1}{2}(v + \mathcal{S}v) + \frac{1}{2}(v - \mathcal{S}v)$ .

► 翻译: (1) 令  $\text{Fix } \mathcal{S} = \{v \in V : \mathcal{S}v = v\}$ . 证明  $V = \text{Fix } \mathcal{S} \oplus \text{Fix}(-\mathcal{S})$ . 你能给这类变换一个几何解释吗?

(2) If  $\dim \text{Fix}(-\mathcal{S}) = 1$ , then we say  $\mathcal{S}$  is a simple reflection. Prove that Each reflection is a product of simple reflection. We use the convention that product of nothing is identity. Hint Take basis of  $\text{Fix}(-\mathcal{S})$ .

► 翻译: 如果  $\dim \text{Fix}(-\mathcal{S}) = 1$ , 那么我们称  $\mathcal{S}$  是一个单反射. 证明每个反射是一些单反射的乘积. 我们采取空乘即恒等映射的约定.

(3) For a finite subset  $R$  of  $V \setminus \{0\}$  spanning  $V$ , and a fixed vector  $v \in R$ . Show that there are at most one simple reflection  $\mathcal{S}$  such that  $\mathcal{S}(R) \subseteq R$  and

$Sv = -v$ . Hint Let  $S$  and  $S'$  be such reflection serves both, then

$$(SS')(R) = R \quad (SS')(v) = v \quad (SS') \text{ acts as identity over } V/\mathbb{R}v$$

Thus the eigenvalue are all 1, but since  $R$  is finite and span whole  $V$ ,  $(SS')^n = I$ , thus it is diagonalizable.

► 翻译: 如果  $V \setminus \{0\}$  的有限子集  $R$  张成了  $V$ , 并固定一个  $v \in R$ . 证明至多只有一个单反射  $S$  使得  $S(R) \subseteq R$  且  $Sv = -v$ .

**Problem 14** Consider  $M_n(\mathbb{R})$  as an  $\mathbb{R}$ -linear space. The subset of the invertible matrix of  $M_n(\mathbb{R})$  is written by  $GL_n(\mathbb{R})$ . If we regard  $M_n(\mathbb{R})$  as  $\mathbb{R}^{n \times n}$ , then we can talk about open, closed subset over it.

► 翻译: 将  $M_n(\mathbb{R})$  视作一个  $\mathbb{R}$ -线性空间. 其中可逆矩阵组成的子集记为  $GL_n(\mathbb{R})$ . 如果我们将  $M_n(\mathbb{R})$  视作  $\mathbb{R}^{n \times n}$ , 那么我们可以谈论上面的开集和闭集.

(1) Prove that  $GL_n(\mathbb{R})$  is an open subset. Hint Show that  $\lim \det A_n = \det \lim A_n$ .

► 翻译: (1) 证明  $GL_n(\mathbb{R})$  是一个开集.

(2) Prove that  $GL_n(\mathbb{R})$  is dense, that is, any matrix is a limit of a sequence of invertible matrices. Hint It suffices to show that for any matrix  $A$ ,  $A + \lambda I$  is invertible for sufficiently small  $\lambda$ .

► 翻译: 证明  $GL_n(\mathbb{R})$  是稠密的, 即任何矩阵都是某个可逆矩阵列的极限.

**Fuxercise 15** Are there any nontrivial linear subspace  $U$  of  $M_n(\mathbb{R})$  such that all members of  $U$  are not invertible? What is the maximal dimension of them? And prove your statement.

► 翻译: 在  $M_n(\mathbb{R})$  中是否有非平凡子空间  $U$  使得任何  $U$  中的成员都不可逆? 他们的维数最大是多少? 证明你的结论.

*If you can give a right solution to this problem next time submitting your homework, you can skip homework for a month (So the hint is postponed to next week).*

► 翻译: 如果你能在下一次提交作业时给这个问题一个正确的解答, 你可以一个月不交作业 (所以提示下周见).

**Problem 16 (Fitting Lemma)** Given a finite-dimensional  $\mathbb{R}$ -linear space  $V$  and a linear transform  $\mathcal{A} : V \rightarrow V$ .

► 翻译: (Fitting 引理) 对于一个有限维  $\mathbb{R}$ -线性空间  $V$  和一个线性变换  $\mathcal{A} : V \rightarrow V$ .

(1) Show that

$$\begin{aligned} \ker \mathcal{A} &\subseteq \ker \mathcal{A}^2 \subseteq \dots \\ \operatorname{im} \mathcal{A} &\supseteq \operatorname{im} \mathcal{A}^2 \supseteq \dots \end{aligned}$$

► 翻译: (1) 证明

$$\begin{aligned} \ker \mathcal{A} &\subseteq \ker \mathcal{A}^2 \subseteq \dots \\ \operatorname{im} \mathcal{A} &\supseteq \operatorname{im} \mathcal{A}^2 \supseteq \dots \end{aligned}$$

(2) Show that one of inequality achieves equality, then so are all the inequalities after it.

► 翻译: (2) 证明如果上述不等式一个等号取到, 那么之后的所有不等号也都取到.

Let

$$\begin{aligned} \ker \mathcal{A}^\infty &= \ker \mathcal{A} \cup \ker \mathcal{A}^2 \cup \dots \\ \operatorname{im} \mathcal{A}^\infty &= \operatorname{im} \mathcal{A} \cap \operatorname{im} \mathcal{A}^2 \cap \dots \end{aligned}$$

► 翻译: 令

$$\begin{aligned} \ker \mathcal{A}^\infty &= \ker \mathcal{A} \cup \ker \mathcal{A}^2 \cup \dots \\ \operatorname{im} \mathcal{A}^\infty &= \operatorname{im} \mathcal{A} \cap \operatorname{im} \mathcal{A}^2 \cap \dots \end{aligned}$$

(3) Prove that they are all  $\mathcal{A}$ -invariant, that is,  $\mathcal{A}(\ker \mathcal{A}^\infty) \subseteq \ker \mathcal{A}^\infty$  and  $\mathcal{A}(\operatorname{im} \mathcal{A}^\infty) \subseteq \operatorname{im} \mathcal{A}^\infty$ .

► 翻译: (3) 证明他们都是  $\mathcal{A}$  不变的, 即  $\mathcal{A}(\ker \mathcal{A}^\infty) \subseteq \ker \mathcal{A}^\infty$  以及  $\mathcal{A}(\operatorname{im} \mathcal{A}^\infty) \subseteq \operatorname{im} \mathcal{A}^\infty$ .

(4) Check that  $\mathcal{A}|_{\ker \mathcal{A}^\infty} : \ker \mathcal{A}^\infty \rightarrow \ker \mathcal{A}^\infty$  is nilpotent, that is,  $(\mathcal{A}|_{\ker \mathcal{A}^\infty})^n = 0$  for some integer  $n \geq 0$ , and  $\mathcal{A}|_{\operatorname{im} \mathcal{A}^\infty} : \operatorname{im} \mathcal{A}^\infty \rightarrow \operatorname{im} \mathcal{A}^\infty$  is invertible.

► 翻译: (4) 验证  $\mathcal{A}|_{\ker \mathcal{A}^\infty} : \ker \mathcal{A}^\infty \rightarrow \ker \mathcal{A}^\infty$  是幂零的, 即  $(\mathcal{A}|_{\ker \mathcal{A}^\infty})^n = 0$  对某个正整数  $n$ , 以及  $\mathcal{A}|_{\operatorname{im} \mathcal{A}^\infty} : \operatorname{im} \mathcal{A}^\infty \rightarrow \operatorname{im} \mathcal{A}^\infty$  是可逆的.

(5) Prove that  $V = \ker \mathcal{A}^\infty \oplus \operatorname{im} \mathcal{A}^\infty$ . Hint Assume  $\operatorname{im} \mathcal{A}^\infty = \operatorname{im} \mathcal{A}^n$  and  $\ker \mathcal{A}^\infty = \ker \mathcal{A}^n$ . Firstly, if  $v \in \ker \mathcal{A}^\infty \cap \operatorname{im} \mathcal{A}^\infty$ , then  $\mathcal{A}^n v = 0$ , assume

$v = \mathcal{A}^n y$ , then  $y \in \ker \mathcal{A}^{2n} = \mathcal{A}^n$ , then  $x = Ay = 0$ . Lastly, for any  $v \in V$ ,  $v = (v - \mathcal{A}^n y) + \mathcal{A}^n y$ , one can pick  $y$  such that  $(v - \mathcal{A}^n y) \in \ker \mathcal{A}^n$ .

► 翻译: (5) 证明  $V = \ker \mathcal{A}^\infty \oplus \text{im } \mathcal{A}^\infty$ .

(6) If  $V = V_1 \oplus V_2$  such that  $\mathcal{A}(V_1) \subseteq V_1$  and  $\mathcal{A}(V_2) \subseteq V_2$ , and  $\mathcal{A}|_{V_1}$  is nilpotent, and  $\mathcal{A}|_{V_2}$  is invertible, then  $V_1 = \ker \mathcal{A}^\infty$ , and  $V_2 = \text{im } \mathcal{A}^\infty$ .

► 翻译: (6) 如果  $V = V_1 \oplus V_2$  使得  $\mathcal{A}(V_1) \subseteq V_1$  且  $\mathcal{A}|_{V_2}$  是可逆的,  $\mathcal{A}|_{V_1}$  是幂零的,  $\mathcal{A}|_{V_2}$  是可逆的, 那么  $V_1 = \ker \mathcal{A}^\infty$ ,  $V_2 = \text{im } \mathcal{A}^\infty$ .

**Problem 17 (inverse formula of binomial)** In this problem, we will get the famous ‘inverse formula of binomial’.

► 翻译: (二项反转公式) 在这个问题中, 我们要得到著名的 “二项反转公式”:

(1) Consider the linear space  $\mathbb{R}[X]_{\deg < n}$ , show that for any  $a \in \mathbb{R}$ ,

$$1, (X - a), (X - a)^2, \dots, (X - a)^{n-1}$$

forms a basis.

► 翻译: (1) 考虑线性空间  $\mathbb{R}[X]_{\deg < n}$ , 证明对任何  $a \in \mathbb{R}$ ,

$$1, (X - a), (X - a)^2, \dots, (X - a)^{n-1}$$

构成一组基.

(2) Let

$$(1, X, X^2, \dots, X^{n-1}) = (1, (X - a), (X - a)^2, \dots, (X - a)^{n-1})A(a)$$

Where  $A(a)$  is so-called ‘transition matrix’. Show that

$$A(a)A(-a) = I$$

Without calculate concisely. Hint Note that  $A(a)$  have nothing to do with  $X$ , so by replacing  $X$  by  $Y$ , the equality still holds, then by letting  $Y = X - a$ , the desired formula appears.

► 翻译: (2) 令

$$(1, X, X^2, \dots, X^{n-1}) = A(a) \cdot (1, (X - a), (X - a)^2, \dots, (X - a)^{n-1})$$

其中  $A(a)$  即所谓的“过渡矩阵”。不要通过具体计算, 证明

$$A(a)A(-a) = I$$

(3) Prove that

$$A(a) = \begin{pmatrix} \binom{0}{0} & \binom{1}{0}a & \cdots & \binom{n-1}{0}a^{n-1} \\ & \binom{1}{1} & \cdots & \binom{n-1}{1}a^{n-2} \\ & & \ddots & \vdots \\ & & & \binom{n-1}{n-1} \end{pmatrix} = \left( \binom{j}{i} a^{j-i} \right)_{0 \leq i, j \leq n-1}$$

Hint Let  $X = Y + a$ .

► 翻译: (3) 证明

$$A(a) = \begin{pmatrix} \binom{0}{0} & \binom{1}{0}a & \cdots & \binom{n-1}{0}a^{n-1} \\ & \binom{1}{1} & \cdots & \binom{n-1}{1}a^{n-2} \\ & & \ddots & \vdots \\ & & & \binom{n-1}{n-1} \end{pmatrix} = \left( \binom{j}{i} a^{j-i} \right)_{0 \leq i, j \leq n-1}$$

(4) Given a sequence  $\{x_n : n = 0, 1, \dots\}$  and  $\{y_n : n = 0, 1, \dots\}$ , prove that

$$x_k = \sum_{h=0}^k (-1)^h \binom{k}{h} y_h \iff y_h = \sum_{k=0}^h (-1)^k \binom{h}{k} x_k$$

This is known as ‘inverse formula of binomial’.

► 翻译: (4) 对于数列  $\{x_n : n = 0, 1, \dots\}$  和  $\{y_n : n = 0, 1, \dots\}$ , 证明

$$x_k = \sum_{h=0}^k (-1)^h \binom{k}{h} y_h \iff y_h = \sum_{k=0}^h (-1)^k \binom{h}{k} x_k$$

这被称为“二项反转公式”。

(5) For two finite sets  $A, B$ , assume  $|A| = a, |B| = b$ , compute the number of surjective maps from  $A$  to  $B$ . Hint Let the number to be  $S(a, b)$ ,

classification the image of  $f$ , gives the formula

$$\sum_{k=0}^b \binom{b}{k} S(a, k) = b^a$$

Then using (4) for  $\{b^a : b = 0, 1, \dots\}$  and  $\{(-1)^k S(a, k) : k = 0, 1, \dots\}$  gives

$$S(a, k) = \sum_{h=0}^k (-1)^{k-h} \binom{k}{h} h^a$$

► 翻译: (5) 对于两个有限集  $A, B$ , 假设  $|A| = a, |B| = b$ , 计算  $A$  到  $B$  的满射数目.

### 3 Week3 (11 Mar - 17 Mar)

► 翻译: 第三周 (3月11日 - 3月17日)

**Exercise 18** Give an example of a matrix that is not diagonalizable. Hint  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

► 翻译: 举一个不可对角化矩阵的例子.

**Exercise 19** Let the eigenvalues of  $A$  be  $\{\lambda_i\}_{i=1}^n$  (with multiplicity), show that  $\det A = \prod_{i=1}^n \lambda_i$ .

► 翻译: 令  $A$  带重数的特征值是  $\{\lambda_i\}_{i=1}^n$ , 证明  $\det A = \prod_{i=1}^n \lambda_i$ .

**Exercise 20** If  $\lambda$  is an eigenvalue of  $A$ , prove that  $f(\lambda)$  is an eigenvalue of  $f(A)$ , where  $f$  is an arbitrary polynomial.

► 翻译: 如果  $\lambda$  是  $A$  的特征值, 证明  $f(\lambda)$  是  $f(A)$  的特征值, 其中  $f$  是任意多项式.

**Exercise 21 (Naïve Lie Theorem)** Given two linear transforms of finite dimensional space  $\mathcal{A}, \mathcal{B}$ , if their Lie bracket  $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \lambda\mathcal{B}$ . Denote  $V_\mu(\mathcal{A}) = \{v : \mathcal{A}v = \mu v\}$  to be the eigensubspaces.

► 翻译: (朴素 Lie 定理) 给两个有限维线性变换  $\mathcal{A}, \mathcal{B}$ , 如果他们的李括号  $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \lambda\mathcal{B}$ . 记  $V_\lambda(\mathcal{A}) = \{v : \mathcal{A}v = \lambda v\}$  是特征子空间.

(1) Show that  $v \in V_\mu(\mathcal{A}) \Rightarrow \mathcal{B}v \in V_{\mu+\lambda}(\mathcal{A})$ . Hint Since,  $\mathcal{A}v = \lambda v$ ,  $\mathcal{A}\mathcal{B}v = \mathcal{B}\mathcal{A}v + \lambda\mathcal{B}v = (\mu + \lambda)\mathcal{B}v$ .

► 翻译: (1) 证明  $v \in V_\mu(\mathcal{A}) \Rightarrow \mathcal{B}v \in V_{\mu+\lambda}(\mathcal{A})$

Now assume that the base field is  $\mathbb{C}$ .

► 翻译: 下面, 假设基域是  $\mathbb{C}$ .

(2) If  $\lambda = 0$ , show that  $\mathcal{A}$  and  $\mathcal{B}$  share some common eigenvector. Hint

Note that  $V_\lambda(\mathcal{A})$  is  $\mathcal{B}$ -invariant.

► 翻译: (2) 如果  $\lambda = 0$ , 证明  $\mathcal{A}, \mathcal{B}$  有公共的特征向量.

(3) In  $\lambda \neq 0$ , show that  $\mathcal{A}$  and  $\mathcal{B}$  share some common eigenvector. Hint

We can pick some eigenvector  $v$ , consider  $v, \mathcal{B}v, \mathcal{B}^2v, \dots$ , then for sufficiently large  $n$  we have  $\mathcal{B}^n v = 0$  (why?). Pick the last  $n$  such that  $\mathcal{B}^n v \neq 0$ , it is a common eigenvector.

► 翻译: (3) 如果  $\lambda \neq 0$ , 证明  $\mathcal{A}, \mathcal{B}$  有公共的特征向量.

(4) Show that  $v \in V_\mu(\mathcal{B}) \Rightarrow \mathcal{A}v \in V_\mu(\mathcal{B})$ . (DO NOT DWELL ON IT.)

► 翻译: 证明  $v \in V_\mu(\mathcal{B}) \Rightarrow \mathcal{A}v \in V_\mu(\mathcal{B})$ . (请勿沉湎于此.)

**Exercise 22** (1) What is the eigenvalues of upper matrix?

► 翻译: (1) 上三角矩阵的特征值是什么?

(2) Without utilization of Jordan cardinal form, using induction, prove that any matrix  $A \in \mathbb{M}_n(\mathbb{C})$  is similar to some upper matrix. Hint Consider

$A$  as a linear transform, so that it is equivalent to prove we can pick a suitable basis. More precisely, pick one of the eigenvalue, say  $\lambda_1$ , and  $e_1$  to be one of eigenvectors belonging to  $\lambda_1$ . Then under the basis  $\{e_1, \dots\}$ ,  $A$  is presented

by  $\begin{pmatrix} \lambda_1 & * \\ & A' \end{pmatrix}$ .

► 翻译: (2) 不利用 Jordan 标准型, 而用归纳法, 证明任何矩阵  $A \in \mathbb{M}_n(\mathbb{C})$  相似于某个上三角矩阵.

(3) Use the fact of (2) to prove Hamilton-Cayley theorem. Hint Now, knowing (2), without loss of generality, we can assume  $A$  to be upper matrix with the eigenvalues located in diagonal with multiplicity. Now prove  $f(A)e_i = 0$ .

► 翻译: 利用 (2) 的事实来证明 Hamilton-Cayley 定理.

(4) Show that if the eigenvalues of  $A$  is  $\{\lambda_i\}_{i=1}^n$ , then for any polynomial  $f$ , the eigenvalues of  $f(A)$  is  $\{f(\lambda_i)\}_{i=1}^n$ . Note that this argument also claim the correspondence of multiplicity, where we can grasp more information than 20. Hint Just calculate the upper matrix.

► 翻译: 证明如果  $A$  的特征值是  $\{\lambda_i\}_{i=1}^n$ , 那么对任意多项式  $f$ ,  $f(A)$  的特征值是  $\{f(\lambda_i)\}_{i=1}^n$ . 注意到这里我们还论证了重数的关系, 这比我们在习题20知道得多.

**Exercise 23** Given a linear space  $V$ , and a linear transform  $\mathcal{A} : V \rightarrow V$ .

► 翻译: 对于一个线性空间  $V$  和线性变换  $\mathcal{A} : V \rightarrow V$ .

(1) Let  $f, g$  be two relative prime polynomials. show that  $\ker f(\mathcal{A}) \cap \ker g(\mathcal{A}) = \mathcal{O}$ . Hint You need to know the so-called Bézout theorem, it claims there exists polynomials  $a, b$  such that  $af + bg = 1$ .

► 翻译: (1) 令  $f, g$  是两个互质的多项式, 证明  $\ker f(\mathcal{A}) \cap \ker g(\mathcal{A}) = \mathcal{O}$ .

(2) And, in addition, assume  $h = fg$  satisfying  $h(\mathcal{A}) = \mathcal{O}$ , show that  $V = \ker f(\mathcal{A}) \oplus \ker g(\mathcal{A})$ . Hint Use Bézout theorem encore.

► 翻译: (2) 如果再假设  $h = fg$  使得  $h(\mathcal{A}) = \mathcal{O}$ , 证明  $V = \ker f(\mathcal{A}) \oplus \ker g(\mathcal{A})$ .

(3) What assumption should be made on  $f_1, \dots, f_k$  such that  $V = \ker f_1(\mathcal{A}) \oplus \dots \oplus \ker f_k(\mathcal{A})$ ?

► 翻译: (3) 应该对  $f_1, \dots, f_k$  作何种假设使得  $V = \ker f_1(\mathcal{A}) \oplus \dots \oplus \ker f_k(\mathcal{A})$ ?

(4) Assume the characteristic polynomial of  $\mathcal{A}$  is  $f = f_1^{n_1} \dots f_s^{n_s}$ , with  $f_1, \dots, f_s$  pairwise coprime, show that  $V = \ker f_1^{n_1}(\mathcal{A}) \oplus \dots \oplus \ker f_s^{n_s}(\mathcal{A})$ . Note that this argument prove Fitting lemma 16 again.

► 翻译: (4) 假设  $\mathcal{A}$  的特征多项式是  $f = f_1^{n_1} \dots f_s^{n_s}$ , 其中  $f_1, \dots, f_s$  两两互质, 证明  $V = \ker f_1^{n_1}(\mathcal{A}) \oplus \dots \oplus \ker f_s^{n_s}(\mathcal{A})$ . 注意到这又论证了一次 Fitting 引理16.

**Problem 24** Here are some problems on diagonalizable matrices. It can be considered as a successor of exercise 23. But the conclusion below can also be derived from the theory of  $\lambda$ -matrices.

► 翻译: 这里有一些关于可对角矩阵的问题. 这可看成是习题23的后续. 但是下面的结论也可由  $\lambda$ -矩阵理论得到.

(5) Prove that if there exists some polynomial  $f$  without multiple roots, such that  $f(A) = 0$ , then  $A$  is diagonalizable over  $\mathbb{C}$ . That is, the matrix of  $A$  is a diagonal matrix under some basis.

► 翻译: (5) 证明如果存在某个无重根的多项式  $f$  使得  $f(A) = 0$ , 那么  $A$  在  $\mathbb{C}$  上可对角化. 即,  $A$  在某组基下的矩阵是对角阵.

Note that, the most important examples, projections and reflections, are given in exercise 13 and exercise 12.

► 翻译: 注意到, 最重要的例子, 投射和反射, 已见于习题13和习题12.

(6) Does reverse of (5) still hold?

► 翻译: (6) 试问 (5) 的反面对吗?

(7) If a square matrix  $A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \dots & \\ & & & A_s \end{pmatrix}$ . Show that  $A$  is diagonalizable iff so are  $A_i$ 's.

► 翻译: (7) 假如方阵  $A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \dots & \\ & & & A_s \end{pmatrix}$ . 证明  $A$  可对角化当且仅当  $A_i$  都可对角化.

(8) If two complex diagonalizable matrices  $A, B$  satisfy  $AB = BA$ , show that they can be diagonalized simultaneously, i.e.

$\exists$  invertible  $P$ , such that  $PAP^{-1}, PBP^{-1}$  is diagonal matrix

Hint Assume  $A = \text{diag}(\lambda_1 I, \dots, \lambda_s I)$  with  $\lambda_i$ 's different, and compute the form of  $B$  and then use (7).

► 翻译: (8) 如果两个可对角复矩阵  $A, B$  满足  $AB = BA$ , 证明他们可以被同时对角化, 即

$\exists$ 可逆的  $P$ , 使得  $PAP^{-1}, PBP^{-1}$  是对角阵

(9) If a family of complex diagonalizable matrices  $\{A_i\}_{i \in I}$  satisfy  $(i, j \in I \Rightarrow A_i A_j = A_j A_i)$ , demonstrate that they can be diagonalized simultaneously, i.e.

$\exists$  invertible  $P$ , such that  $(i \in I \Rightarrow P A_i P^{-1}$  is diagonal matrix)

*Hint* Use induction on the size of  $A_i$ 's.

► 翻译: (9) 如果一族可对角复矩阵  $\{A_i\}_{i \in I}$  满足  $(i, j \in I \Rightarrow A_i A_j = A_j A_i)$ , 证明他们可以被同时对角化, 即

$\exists$  可逆的  $P$ , 使得  $(i \in I \Rightarrow P A_i P^{-1}$  是对角阵)

**Problem 25** Explain why the following plausible proof of Hamilton-Cayley theorem is not eligible

► 翻译: 解释为什么下述看似合理的 Hamilton-Cayley 定理的证明是不合理的

Proof. Let  $f$  be the characteristic polynomial. Note that  $f(\lambda) = \det(\lambda \mathcal{I} - \mathcal{A})$ , by taking  $\lambda = \mathcal{A}$ ,  $f(\mathcal{A}) = \det(\mathcal{A} \mathcal{I} - \mathcal{A}) = \det(O) = 0$ .

The proof is complete.  $\square$

► 翻译: 证明 令  $f$  是特征多项式. 注意到  $f(\lambda) = \det(\lambda \mathcal{I} - \mathcal{A})$ , 带入  $\lambda = \mathcal{A}$  得  $f(\mathcal{A}) = \det(\mathcal{A} \mathcal{I} - \mathcal{A}) = \det(O) = 0$ . 所证欲明.  $\square$

*Hint* Theoretically speaking, the reason that  $\lambda$  is a variable of “number” rather than “matrix” is not exact enough as  $\lambda$  is just a symbol, or more precisely, an indeterminate. The real mistake literally lies in the fact that the product of  $\lambda$  with  $\mathcal{I}$  is a scalar product rather than the product of matrix. Actually, if assume  $A = (a_{ij})$  is the matrix of  $\mathcal{A}$  under selection of basis

$x_1, \dots, x_n$  of  $V$ , then the relation  $Ax_i = \sum_{j=1}^n a_{ij}x_j$  gives

$$\begin{pmatrix} A - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & A - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & A - a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By taking the associated matrix appearing in the left side, the proof of  $H-C$  Theorem will get complete. Note that it is essentially a rewriting of the ordinary proof presenting by most of text books.

**Problem 26** In this problem, we will get some analytic information of diagonalizable matrices. We will regard  $M_n(\mathbb{C})$  as a  $\mathbb{C}$ -vector space with dimension  $n^2$  although this problem.

► 翻译: 在这个问题中, 我们要得到一些可对角矩阵的解析信息. 我们将在本问题下视  $M_n(\mathbb{C})$  为一个  $n^2$  维的  $\mathbb{C}$ -线性空间.

(1) Given a polynomial  $f \in \mathbb{C}[X_1, \dots, X_n]$ , show that if  $f = 0$  (as function) over some open subset of  $\mathbb{C}^n$ , then  $f = 0$ . Hint One can use the analytic method, that is, consider  $\frac{\partial f}{\partial X_i}$  etc. One can also use algebraic method, by induction. More exactly, for  $n = 1$ , it is easy by the finiteness of roots, and for  $n \geq 2$ , rearrange it as  $f = \sum_{i=0}^n f_i(X_1, \dots, X_{n-1})X_n^i$ , and use the case  $n = 1$  and induction assumption.

► 翻译: 给一个多项式  $f \in \mathbb{C}[X_1, \dots, X_n]$ , 证明如果在某个  $\mathbb{C}^n$  的开集上  $f = 0$  (作为函数), 那么  $f = 0$ .

(2) For a nonzero polynomial  $f$  in  $n$  variables taking its coefficients in  $\mathbb{C}$ , show that the subset  $\{(x_1, \dots, x_n) \in \mathbb{C}^n : f(x_1, \dots, x_n) \neq 0\}$  is dense in  $\mathbb{C}^n$ . Hint By a topological fortiori, this is equivalent to  $\{(x_1, \dots, x_n) : f(x_1, \dots, x_n) = 0\}$  has no interior, which is proved in (1).

► 翻译: 给一个非零  $n$  元复系数多项式, 证明子集  $\{(x_1, \dots, x_n) \in \mathbb{C}^n : f(x_1, \dots, x_n) \neq 0\}$  在  $\mathbb{C}^n$  中是稠密的.

(3) Using the fact that if the characteristic polynomial of a matrix has no multiple roots then it can be diagonalized to prove that the diagonalizable

matrices are dense in  $\mathbb{M}_n(\mathbb{C})$ . Hint Do you know any criterion on the existence of ‘multiple root’? For example, the resultants and discriminants?

► 翻译: 利用特征多项式没有重根则可以 diagonalized 的事实来证明可对角矩阵在  $\mathbb{M}_n(\mathbb{C})$  中稠密.

APPENDIX—hint to the ‘Bonus exercise’ last week 15.

By considering all the matrices  $A$  such that  $(1, 0, \dots, 0)A = 0$ , one see the maximal dimension  $\geq n(n-1)$ . To prove it achieves maximality, using induction. More precisely, pick a basis  $\mathcal{A}$  for  $U$ . Consider the rank of {first row of  $A \in \mathcal{A}$ }, if it has rank 0, the proof is complete. If of rank  $r \geq 1$ , then replace  $U$  by  $\{BXC : X \in U\}$  for some invertible  $B, C$ , one can assume, some  $A_1, \dots, A_r \in \mathcal{A}$  with first row  $(1, 0, \dots, 0), (0, 1, 0, \dots), \dots$ , and the other  $A_i$ ’s with first line and first  $r$  columns blank (WHY?). Then {delete first row and first column of  $A \in \mathcal{A}$ } can not have a invertible matrix (WHY, too?), so its dimension is no more than  $(n-1)(n-2)$ . By adding the first column, the dimension of the linear space spanned by ‘the other  $A_i$ ’s’ is no more than  $(n-1)(n-2) + n-1 = (n-1)^2$ . If  $r \leq n-1$ , then proof is complete. In fact, it is a must that one of lines fails to be full rank (WHY?).

## 4 Week4 (18 Mar - 24 Mar)

► 翻译: 第四周 (3月18日 - 3月24日)

**Exercise 27** Demonstrate that  $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$  is similar to  $\begin{pmatrix} A_2 & \\ & A_1 \end{pmatrix}$ . Where  $A_1, A_2$  are square matrix and generally not of same size.

► 翻译: 证明  $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$  与  $\begin{pmatrix} A_2 & \\ & A_1 \end{pmatrix}$  相似. 其中  $A_1, A_2$  是大小不必相同的方阵.

**Exercise 28** Given a finite-dimensional linear space  $V$  and a linear transform  $\mathcal{A} : V \rightarrow V$ , illustrate that all the eigenvalues of  $\mathcal{A}$  are 0 iff  $\mathcal{A}$  is nilpotent.

► 翻译: 给定一个有限维线性空间  $V$  和一个线性变换  $\mathcal{A} : V \rightarrow V$ , 证明  $\mathcal{A}$  的特征值都是 0 当且仅当  $\mathcal{A}$  是幂零的.

**Exercise 29 (Trace)** In this exercise, we will introduce the concept of trace. Although the conception of trace is seldom mentioned in advanced algebra, it abounds in several advanced algebras, and numbers of critical structural information is under the sway of it.

► 翻译: (迹) 在本习题中, 我们将要介绍迹这一概念. 尽管迹的概念在高等代数中很少提及, 但是它在各种高等代数中大量出现, 且大量结构性的信息受之控制.

Imitating what we have done for determinant, we define it firstly for matrix. Given an  $n \times n$  matrix  $A$ , the trace of  $A$  is defined to be sum of the indices lying in main diagonal. More precisely, assume  $A = (a_{ij})$ , then its trace is written by  $\text{tr } A = a_{11} + \dots + a_{nn}$ .

► 翻译: 仿照我们对行列式所做的, 我们先对矩阵定义. 对于一个  $n \times n$  矩阵  $A$ ,  $A$  的迹被定义为主对角线上元素的和. 即, 既记  $A = (a_{ij})$ , 记迹即  $\text{tr } A = a_{11} + \dots + a_{nn}$ .

(1) Demonstrate the following basic properties

$$\text{tr}(A + \lambda B) = \text{tr } A + \lambda \text{tr } B \quad \text{tr}(AB) = \text{tr}(BA)$$

And exploit them to illustrate that  $\text{tr}(PAP^{-1}) = \text{tr } A$ , so trace is stable under basis change, thus it can be defined for linear transform of finite-dimensional space.

► 翻译: 论证有如下基本性质

$$\text{tr}(A + \lambda B) = \text{tr } A + \lambda \text{tr } B \quad \text{tr}(AB) = \text{tr}(BA)$$

并利用它们来证明  $\text{tr}(PAP^{-1}) = \text{tr } A$ , 所以迹在基变换下不变, 故可对有限维线性空间的线性变换定义迹.

(2) Before proceeding the discussion, exemplify that

$$\text{tr}(ABC) \text{ not generally equals to } \text{tr}(ACB)$$

► 翻译: 在继续讨论之前, 请例证

$$\operatorname{tr}(ABC) \text{ 一般不等于 } \operatorname{tr}(ACB)$$

(3) Prove that for a matrix  $A$  of size  $n \times n$ , the characteristic polynomial of  $A$

$$\det(\lambda I - A) = \lambda^n + (-1)^{n-1}(\operatorname{tr} A)\lambda^{n-1} + \dots + (\dots)\lambda + \det A$$

Therefore,  $\operatorname{tr} A$  is the sum of the eigenvalue (with multiplicity).

► 翻译: 证明对于  $n$  阶方阵  $A$ , 其特征多项式为

$$\det(\lambda I - A) = \lambda^n + (-1)^{n-1}(\operatorname{tr} A)\lambda^{n-1} + \dots + (\dots)\lambda + \det A$$

所以,  $\operatorname{tr} A$  是特征值的和 (按重数计算).

**Problem 30** Synthesize the exercises above, demonstrate that a square matrix  $A$  is nilpotent iff  $\operatorname{tr} A^k = 0$  for all  $k \geq 1$ . Hint Let the nonzero eigenvalues of  $A$  to be  $\lambda_1, \dots, \lambda_s$ , and the multiplicity of  $\lambda_i$  is  $r_i$ , then  $\operatorname{tr} A^k = r_1\lambda_1^k + \dots + r_s\lambda_s^k = 0$  for all  $k \geq 1$ . That is

$$\underbrace{\begin{pmatrix} \lambda_1 & \dots & \lambda_s \\ \vdots & \ddots & \vdots \\ \lambda_1^s & \dots & \lambda_s^s \end{pmatrix}}_{\det(\cdot) \neq 0} \overbrace{\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}}^{\neq 0} = 0$$

which is the desired contradiction.

► 翻译: 综合上面习题, 证明一个方阵  $A$  幂零当且仅当  $\operatorname{tr} A^k = 0$  对任意  $k \geq 1$ .

**Exercise 31** Recollecting problem 21 concerning on Lie theorem. Under the condition of  $[A, B] = \lambda A$  with  $\lambda \neq 0$ , we can plausibly draw the reasonable conclusion that  $A$  is a nilpotent matrix.

► 翻译: 回忆关于 Lie 定理的问题 21. 在  $[A, B] = \lambda A$ , 我们可以得到  $A$  是幂零矩阵的合理结论, 其中  $\lambda \neq 0$ .

(1) Show that  $\text{tr}[X, Y] = 0$  for any couples of square matrices  $X, Y$ .

► 翻译: (1) 证明  $[X, Y] = 0$  对任何方阵对  $X, Y$ .

(2) Show that  $\text{tr} A^n = 0$  for all positive integers  $n$ . Hint Merely by virtue of the equality that  $\text{tr} A^n = \lambda^{-1} \text{tr} A^{n-1}[A, B] = \lambda^{-1} \text{tr}(A^n B - A^{n-1} B A) = \lambda^{-1}[\text{tr}(A^n B) - \text{tr}(A^{n-1} B A)] = 0$ .

► 翻译: (2) 证明  $\text{tr} A^n = 0$  对所有正整数  $n$ .

**Problem 32** For a subfield  $K$  of  $\mathbb{C}$ , given two square matrices  $A, B$  whose indices take value in  $K$ , demonstrate that the sufficient and necessary condition that  $A$  and  $B$  are similar over  $K$  is simply that  $A$  and  $B$  get similar over  $\mathbb{C}$ . Feeling frustrated and constrained, to facilitate, you can firstly confine yourself to the case  $K = \mathbb{R}$ . Hint Write a complex matrix  $P = P_1 + iP_2$  and prove that  $P$  is invertible, then some  $K$ -combination of  $P_1$  and  $P_2$  is invertible as well. Conspicuously, it is no more than the familiar technique utilizing determinant and attributes of polynomials. In the general case, select suitable  $x_1, x_2, \dots$ , and write  $P = x_1 P_1 + x_2 P_2 + \dots$ .

► 翻译: 对于  $\mathbb{C}$  之子域  $K$ , 两个  $K$  上方阵  $A, B$ , 证明  $A$  和  $B$  在  $K$  上相似的充分必要条件不过是  $A$  和  $B$  在  $\mathbb{C}$  上相似. 若感觉困难和受限, 便利起见, 可以假设  $K = \mathbb{R}$ .

**Exercise 33** In  $\mathbb{R}^2$ , classify all the linear transform  $\mathcal{A}$  satisfying the following properties

(i) takes integer points to integer points, that is,  $\mathcal{A}$  maps  $\mathbb{Z}^2$  to  $\mathbb{Z}^2$ , and

(ii) possess finite order, i.e. for some  $n > 0$ ,  $\mathcal{A}^n = \mathcal{I}$ .

Hint Assume the matrix of  $\mathcal{A}$  is  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , (i) implies  $a, b, c, d \in \mathbb{Z}$ . The characteristic polynomial of  $\mathcal{A}$  is  $f(\lambda) = \lambda^2 - 2(a+d)\lambda + (ad-bc)$ . So  $f(\lambda)$  shares a common factor with  $\lambda^n - 1$ . If the common factor is of degree 1, then

the case is trivial. Otherwise,  $f(\lambda)|\lambda^n - 1$ , so the choices of  $f$  is very limited by the theory of cyclotomic polynomials

$$X^2 - 1, X^2 + 1, X^2 + X + 1, X^2 - X + 1$$

So  $\mathcal{A}$  satisfies  $\text{tr } \mathcal{A} = 0$  with  $\det \mathcal{A} = \pm 1$ , or  $\det \mathcal{A} = 1$  with  $\text{tr } \mathcal{A} = 1$ . They are exactly the reflection or rotation under some basis.

► 翻译: 在  $\mathbb{R}^2$  上分类所有满足下列条件的线性变换  $\mathcal{A}$

(1) 将整点映到整点, 即是,  $\mathcal{A}$  将  $\mathbb{Z}^2$  映射到  $\mathbb{Z}^2$ ;

(2) 具有有限阶, 即, 存在某个  $n > 0$ ,  $\mathcal{A}^n = \mathcal{I}$ .

## 5 Week5 (25 Mar - 31 Mar)

► 翻译: 第五周 (3月25日 - 3月31日)

**Exercise 34** (1) Utilizing Jordan cardinal form, demonstrate that  $A$  is similar to  $A^\top$ , where  $*^\top$  stands for transposition. Hint It suffices to show in the condition that  $A$  is a Jordan matrix.

► 翻译: (1) 利用 Jordan 标准型证明  $A$  和  $A^\top$  相似, 其中  $*^\top$  代表转置.

(2) Applying theory of  $\lambda$ -matrices, illustrate the same fact.

► 翻译: (2) 利用  $\lambda$ -矩阵证明同样的事实.

**Exercise 35** (1) Utilizing Jordan cardinal form, demonstrate that that  $A$  is similar to  $B$  iff that  $\begin{pmatrix} A & \\ & A \end{pmatrix}$  is similar to  $\begin{pmatrix} B & \\ & B \end{pmatrix}$ .

► 翻译: (1) 利用 Jordan 标准型证明  $A$  和  $B$  相似当且仅当  $\begin{pmatrix} A & \\ & A \end{pmatrix}$  和  $\begin{pmatrix} B & \\ & B \end{pmatrix}$  相似.

(2) Applying theory of  $\lambda$ -matrices, illustrate the same fact.

► 翻译: (2) 利用  $\lambda$ -矩阵证明同样的事实.

**Exercise 36** Show that any matrices which commute with any matrices which commute  $A$  are exact polynomial in  $A$ . Hint After applying Jordan cardinal form, You will resort to violent computation.

► 翻译: 证明任何与任何与  $A$  可交换的矩阵可交换的矩阵正是  $A$  的多项式.

**Problem 37 (Jordan decomposition)** We will show that each complex square matrix  $A$  can be uniquely written as  $D + N$ , such that

$$D \text{ is diagonalizable} \quad N \text{ is nilpotent} \quad DN = ND$$

and, in fact,  $D$  and  $N$  are polynomials in  $A$ .

► 翻译: ( $Jordan$  分解) 我们将要证明每个复方阵  $A$  都可以唯一写成  $D + N$  使得

$$D \text{ 可对角化} \quad N \text{ 幂零} \quad DN = ND$$

并且, 事实上,  $D$  和  $N$  都是  $A$  的多项式.

(1) Prove the existence exploiting Jordan cardinal form.

► 翻译: 利用  $Jordan$  标准型证明存在性.

Denote  $V = \mathbb{R}^n$ ,  $V_\lambda = \{v \in V : n \gg 0 \Rightarrow (A - \lambda I)^n v = 0\}$ , where  $\gg$  means ‘be sufficiently large’. We have shown in Exercise 23 that  $V = \bigoplus_\lambda V_\lambda$ , with  $\lambda$  going through all the eigenvalues of  $A$ .

► 翻译: 记  $V = \mathbb{R}^n$ ,  $V_\lambda = \{v \in V : n \gg 0 \Rightarrow (A - \lambda I)^n v = 0\}$ , 其中  $\gg$  表示“充分大的”, 我们已经在习题23证明过  $V = \bigoplus_\lambda V_\lambda$ , 其中  $\lambda$  跑遍  $A$  所有特征值.

(2) Show that the projection from  $V$  to each  $V_\lambda$  is polynomial in  $A$ , i.e. the map  $[\sum_\mu v_\mu \mapsto v_\lambda]$  where  $v_\mu \in V_\mu$ . Hint That is, the linear transformation  $B$  acts as identity over  $V_\mu$  but zero over  $\bigoplus_{\mu \neq \lambda} V_\mu$ . Using Bézout’s identity,

$$f(X)(X - \lambda)^{n_i} + g(X)(\dots) = 1$$

by letting  $X = A$ .

► 翻译: (2) 证明  $V$  到  $V_\lambda$  的投影都是  $A$  的多项式. 即映射  $[\sum_\mu v_\mu \mapsto v_\lambda]$  其中  $v_\mu \in V_\mu$ .

(3) Illustrate that the desired  $D$  is a polynomial of  $A$ , so, so is  $N$ . Hint

Clearly,  $D = \sum \lambda P_\lambda$ , where  $P_\lambda$  is projection to  $V_\lambda$ .

► 翻译: (3) 证明想要的  $D$  是  $A$  的多项式, 于是,  $N$  也是.

(4) Prove the following nearly trivial conclusion that if  $N_1 N_2 = N_2 N_1$ ,

$$N_1, N_2 \text{ are both nilpotent} \Rightarrow N_1 + N_2 \text{ is nilpotent}$$

► 翻译: (4) 证明如下近乎显然的结论, 如果  $N_1 N_2 = N_2 N_1$

$$N_1, N_2 \text{ 皆幂零} \Rightarrow N_1 + N_2 \text{ 幂零}$$

(5) Demonstrate that the uniqueness. Hint Utilizing (4) and the argument of simultaneous diagonalization in Exercise 24. Note that, after proving  $D$  and  $N$  are polynomial in  $A$ , commutability with  $A$  is equivalent to commutability with both  $D$  and  $N$  constructed above.

► 翻译: 证明存在性.

**Exercise 38** Prove the multiplicative version of Jordan decomposition. Each complex invertible matrix  $A$  can be uniquely written as  $DU$ , such that

$$D \text{ is diagonalizable} \quad U \text{ is unipotent} \quad DU = UD$$

Where ‘ $U$  is unipotent’ means that  $U - I$  is nilpotent.

► 翻译: 证明 Jordan 分解的乘性版本. 每个可逆复矩阵  $A$  可被唯一地写成  $DU$ , 其中

$$D \text{ 可对角化} \quad U \text{ 是幂幺的} \quad DU = UD$$

其中 “ $U$  是幂幺的” 指的是  $U - I$  是幂零的.

Are  $D$  and  $U$  still polynomials in  $A$ ?

► 翻译: 现在  $D$  和  $U$  还一定是  $A$  的多项式吗?

**Exercise 39 (Exponent map)** In this exercise, we will define  $\exp$  for matrices whose indices are all complex number. Recollecting the fact that for any  $z \in \mathbb{C}$ ,

$$e^z = \exp z \triangleq 1 + z + \frac{1}{2}z^2 + \dots + \frac{1}{n!}z^n + \dots$$

(In this occasion,  $\triangleq$  represents ‘is defined to be’ rather than ‘heating’.)

► 翻译: (指数映射) 在本问题中, 我们要对复矩阵定义  $\exp$  映射. 回忆如下事实, 对任意  $z \in \mathbb{C}$ ,

$$e^z = \exp z \triangleq 1 + z + \frac{z^2}{2} + \dots + \frac{z^n}{n!} + \dots$$

(在这里,  $\triangleq$  表示 “被定义为” 而不是 “加热”.)

Since it involves infinite series, certain analysis is required to ensure the convergence. A well-known theorem is that different norms<sup>1</sup> in Euclidean space are all equivalent. Therefore, we will use this norm to proceed without loss of generality

$$\|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2} \quad A = (a_{ij})$$

► 翻译: 因为这涉及无穷级数, 我们需要一些分析来确保收敛性. 一个著名的定理是说欧式空间的不同的范数皆等价. 于是我们不是一般性地使用

$$\|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2} \quad A = (a_{ij})$$

来作为范数来继续.

(1) Show that  $\|AB\| \leq \|A\| \cdot \|B\|$ . Consequently,  $\|A^n\| \leq \|A\|^n$ . Hint

Merely because  $|\sum_i a_i b_j|^2 \leq (\sum_i |a_i|^2)(\sum_i |b_i|^2)$

► 翻译: (1) 证明  $\|AB\| \leq \|A\| \cdot \|B\|$ . 于是,  $\|A^n\| \leq \|A\|^n$ .

(2) Illustrate that for each complex matrix  $A$ ,  $\left\{ \sum_{k=0}^n \frac{A^k}{k!} \right\}_{n=1}^{\infty}$  is an Cauchy

sequence. Hint Since  $e^{\|A\|}$  is convergent,  $\left\| \sum_{k=n}^m \frac{A^k}{k!} \right\| \leq \sum_{k=n}^m \frac{\|A\|^k}{k!} < \epsilon$ .

► 翻译: 证明对每个复矩阵  $A$ ,  $\left\{ \sum_{k=0}^n \frac{A^k}{k!} \right\}_{n=1}^{\infty}$  是 Cauchy 列.

<sup>1</sup> We refer a norm of  $V$  to a map  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  such that  $\|a\| = 0 \iff a = 0$ ,  $\|a+b\| \leq \|a\| + \|b\|$  and  $\|\lambda a\| = |\lambda| \cdot \|a\|$ .

► 翻译: 我们说的范数指的是一个映射  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  使得  $\|a\| = 0 \iff a = 0$ ,  $\|a+b\| \leq \|a\| + \|b\|$  且  $\|\lambda a\| = |\lambda| \cdot \|a\|$ .

Now, we can define

$$e^A = \exp A := \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{A^k}{k!}$$

(In this occasion,  $:=$  represents ‘is defined to be’ rather than ‘said’.)

► 翻译: 现在, 我们可以定义

$$e^A = \exp A := \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{A^k}{k!}$$

(在这里,  $:=$  表示 “被定义为” 而不是 “说”.)

(3) Demonstrate the following basic properties

$$e^{PAP^{-1}} = Pe^AP^{-1} \quad AB = BA \Rightarrow e^Ae^B = e^{A+B} \quad e^{\lambda I} = e^{\lambda}I$$

Where  $\lambda \in \mathbb{C}$ .

► 翻译: (3) 证明如下基本性质

$$e^{PAP^{-1}} = Pe^AP^{-1} \quad AB = BA \Rightarrow e^Ae^B = e^{A+B} \quad e^{\lambda I} = e^{\lambda}I$$

其中  $\lambda \in \mathbb{C}$ .

(4) Compute  $e^{J(\lambda)}$ , where  $J(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$ . Hint Write

$J(\lambda) = \lambda I + J(0)$ . Note that  $J(0)$  is nilpotent.

► 翻译: (4) 计算  $e^{J(\lambda)}$ , 其中  $J(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$ .

(5) Show that

$$\det e^A = e^{\text{tr} A}$$

► 翻译: (5) 证明

$$\det e^A = e^{\text{tr} A}$$

(6) Exemplify that  $e^A e^B$  not in general equals to  $e^B e^A$ . Hint For instance,  $e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ ,  $e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ .

► 翻译: (6) 例证  $e^A e^B$  一般不等于  $e^B e^A$ .

(7) Prove that  $X(t) = [t \mapsto \exp tA]$  is a solution of the following differential equation

$$\frac{dX(t)}{dt} = X(t)A$$

by Calculating  $\frac{d \exp(tA)}{dt}$ .

► 翻译: (7) 证明  $X(t) = [t \mapsto \exp tA]$  是如下微分方程的解

$$\frac{dX(t)}{dt} = X(t)A$$

通过计算  $\frac{de^{tA}}{dt}$ .

(8) Assume  $X = (x_{ij})$  and  $e^X = (y_{ij})$ , by regarding  $y_{ij}$ 's as a function of  $x_{ij}$ 's, show that

$$\left. \frac{\partial y_{kh}}{\partial x_{ij}} \right|_{X=0} = \begin{cases} 1 & (i, j) = (k, h) \\ 0 & (i, j) \neq (k, h) \end{cases}$$

Hint This fact may be familiar to you, for  $f(x, y)$ , let  $f_0(x) = f(x, 0)$ ,

$$\frac{\partial f}{\partial x}(a, 0) = \frac{df_0}{dx}(a)$$

in other word, taking value and taking partial differential in different variables can exchange. Thus, in this case, you only need to calculate when  $X = x_{ij} E_{ij}$ , and (7) shows that the

$$\left. \frac{\partial \exp}{\partial x_{ij}} \right|_{X=0} = e^{x_{ij} E_{ij}} E_{ij} |_{X=0} = E_{ij}$$

$$\text{So, } \left. \frac{\partial y_{kh}}{\partial x_{ij}} \right|_{X=0} = (k, h)\text{-position of } E_{ij} = \begin{cases} 1 & (i, j) = (k, h) \\ 0 & (i, j) \neq (k, h) \end{cases}.$$

► 翻译: 假设  $X = (x_{ij})$  以及  $e^X = (y_{ij})$ , 通过视  $y_{ij}$  为变量  $x_{ij}$  的函数证明

$$\left. \frac{\partial y_{kh}}{\partial x_{ij}} \right|_{X=0} = \begin{cases} 1 & (i, j) = (k, h) \\ 0 & (i, j) \neq (k, h) \end{cases}$$

(9)(Weak Hausdorff Formula) Illustrate that

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}(XY - YX) + o(t^2)\right)$$

where  $o(t^2)$  means certain function  $f$  with  $\lim_{t \rightarrow 0} f/t^2 = 0$ . Hint Inverse map theorem combined with (8) ensures the existence of the required expression. This is essential mathematical analysis. You need to calculate  $H(t, s) = \exp(tX)\exp(sY)$ ,

$$\begin{aligned} H(0, 0) &= I \\ \frac{\partial H}{\partial t}(0, 0) &= X & \frac{\partial H}{\partial s}(0, 0) &= Y \\ \frac{\partial^2 H}{\partial t^2}(0, 0) &= X^2 & \frac{\partial^2 H}{\partial s \partial t}(0, 0) &= XY & \frac{\partial^2 H}{\partial s^2}(0, 0) &= Y^2 \end{aligned}$$

then by Taylor's formula

$$H(s, t) = I + tX + sY + \frac{1}{2}(t^2X^2 + 2stXY + s^2Y^2) + o(t^2 + s^2)$$

Therefore

$$\exp(tX)\exp(tY) = I + tX + tY + \frac{t^2}{2}(X^2 + 2XY + Y^2) + o(t^2 + s^2)$$

Then

$$\begin{aligned} &\exp(\mu_1 t + \mu_2 t^2 + o(t^2)) \\ &= I + \mu_1 t + \mu_2 t^2 + o(t^2) + \frac{1}{2}(\mu_1 t + \mu_2 t^2 + o(t^2))^2 + o(t^3) \\ &= I + \mu_1 t + (\mu_2 + \frac{1}{2}\mu_1^2)t^2 + o(t^2) \end{aligned}$$

Alors,

$$\mu_1 = X + Y \quad \mu_2 = \frac{1}{2}(X^2 + 2XY + Y^2) - \frac{1}{2}\mu_1^2 = XY - YX$$

The proof is complete.

► 翻译: (9)(弱 Hausdorff 公式) 证明

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}(XY - YX) + o(t^2)\right)$$

其中  $o(t^2)$  表示某个使得  $\lim_{t \rightarrow 0} f/t^2 = 0$  的  $f$ .

**Problem 40 (Spectral radius)** In Exercise 39, norm is introduced. Now, let  $\|\cdot\|$  be a norm of complex matrix. Show that

$$\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = \max\{|\lambda| : \lambda \text{ is eigenvalue of } A\}$$

This result is known as ‘spectral radius formula’ which still holds over Banach algebra. This formula is known as Hint We can just check it for certain fixed specific norm by the equivalence of different norm. Then assume  $Av = \lambda v$ , then

$$|\lambda|^n \|v\| = \|\lambda^n v\| = \|A^n v\| \leq \|A^n\| \cdot \|v\|$$

Conversely, denote  $A_0 = A/(|\lambda_{max}| + \epsilon)$ , using Jordan form to show that  $A_0^n \rightarrow 0$ , then  $\|A^n\| \leq |\lambda_{max} + \epsilon|^n$ .

► 翻译: (谱半径) 在习题39中, 我们介绍了范数. 下面, 令  $\|\cdot\|$  是一个矩阵的范数. 证明

$$\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = \max\{|\lambda| : \lambda \text{ 是 } A \text{ 的特征值}\}$$

这一结果被称为“谱半径公式”, 这对 Banach 空间也成立.

## 6 Week6 (1 Apr - 7 Apr)

► 翻译: 第六周 (4月1日 - 4月7日)

**Exercise 41** Given a complex invertible matrix  $A$ , what condition should be made on  $A$  such that the group generated by  $A$  is bounded? That is,  $\{A^n : n \in \mathbb{Z}\}$  is a bounded subset of  $\mathbb{C}^{n \times n}$ . Hint Just compute.

► 翻译: 给一个可逆复矩阵  $A$ , 应该在  $A$  上加何种条件使得  $A$  生成的群是有界的? 即,  $\{A^n : n \in \mathbb{Z}\}$  是  $\mathbb{C}^{n \times n}$  中的有界集.

**Problem 42** Reading Different proofs of Jordan norm form.

► 翻译: 阅读《Jordan 标准型的几种证明》

**Problem 43 (Cyclic subspace)** Given a finite-dimensional  $\mathbb{C}$ -space  $V$ , and a linear transformation  $A$  over it. Let  $W$  be an  $A$ -invariant subspace of  $V$  such that there is a basis  $e_1, \dots, e_r$  such that

$$e_r \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} e_1 \xrightarrow{A-\lambda I} 0$$

we will call  $W$  an  $A$ -cyclic space belonging to  $\lambda$  of dimension  $r$ . Prove that  $W$  is  $A$ -invariant. Try to compute all the  $A$ -invariant subspace(s) of a cyclic subspace.

► 翻译: (循环子空间) 给定一个有限维  $\mathbb{C}$ -线性空间  $V$  和一个其上的线性变换  $A$ . Clearly, the theorem of Jordan norm form claims that  $V$  is direct sum of cyclic subspace. 令  $W$  是  $V$  的一个  $A$ -不变子空间使得存在一组基  $e_1, \dots, e_r$  使得

$$e_r \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} e_1 \xrightarrow{A-\lambda I} 0$$

我们称  $W$  是属于  $\lambda$  维数为  $r$  的  $A$ -循环子空间. 显然, Jordan 标准型定理断言  $V$  是循环子空间的直和. 尝试计算循环子空间的所有  $A$ -不变子空间.

**Problem 44** For what kind of linear transformation over finite-dimensional  $\mathbb{C}$ -space has only finite many invariant subspace? Hint The dimension of each eigenspace of each eigenvalue must be 1, this is sufficient.

► 翻译: 对怎样的有限维  $\mathbb{C}$ -线性空间的线性变换, 只有有限个不变子空间?

**Fuxercise 45** Given a finite-dimensional  $\mathbb{C}$ -space  $V$ , and a linear transformation  $A$  over it.

► 翻译: 给定一个有限维  $\mathbb{C}$ -线性空间  $V$  和一个其上的线性变换  $\mathcal{A}$ .

(1) Let  $W \subseteq V$  be an  $\mathcal{A}$ -invariant subspace. If under certain basis of  $W$  the matrix of  $\mathcal{A}$  is a Jordan block, show that it can be expanded to a basis of  $V$  under which the matrix of  $\mathcal{A}$  is Jordan norm form. Hint Firstly, let  $\{e_1, \dots, e_r\}$  be such base of  $W$  with

$$e_r \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} e_{r-1} \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} \dots \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} e_1 \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} 0$$

Extend it to  $\{e_1, \dots, e_s\}$  until  $(\mathcal{A} - \lambda\mathcal{I})x = e_s$  has no solution. Now, we can assume  $r = s$  without loss of generality. Or just follow the geometrical proof of Jordan norm form. Assume the basis of  $V_\lambda = \{v \in V : \exists n, (\mathcal{A} - \lambda\mathcal{I})^n v = 0\}$  under which the matrix is Jordan cardinal form is

$$\left\{ \begin{array}{cccccccc} f_{1r_1} & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & \dots & \dots & \dots & \dots & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & f_{11} & (\mapsto 0) \\ & & f_{2r_2} & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & \dots & \dots & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & f_{21} & (\mapsto 0) \\ & & & \vdots & \vdots & \vdots & \vdots & & \\ & & & f_{sr_s} & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & \dots & \xrightarrow{\mathcal{A}-\lambda\mathcal{I}} & f_{s1} & (\mapsto 0) \end{array} \right\}$$

Assume  $e_s = \sum \lambda_{ij} f_{ij}$ . Show that there exists some  $i = s$  such that  $\lambda_{is} \neq 0$  and  $r_i = s$ . Then replace the  $i$ -th row with  $e_i$ 's. The proof is complete.

► 翻译: 令  $W \subseteq V$  是一个  $\mathcal{A}$ -不变子空间. 如果在  $W$  某组基下,  $\mathcal{A}$  的矩阵是 Jordan 块, 证明可以将这组基扩充成  $V$  的一组基使得  $\mathcal{A}$  在这组基下的矩阵是 Jordan 标准型.

(2) Exemplify that the ‘Jordan block’ above cannot be replaced with ‘Jordan form’: Hint For example, assume  $\left\{ \begin{array}{ccc} e_2 & \xrightarrow{\mathcal{A}} & e_1 & (\mapsto 0) \\ & & f_1 & (\mapsto 0) \end{array} \right\}$ , then under the basis  $\{e_1 + f_1, e_1 + 2f_1\}$ ,  $\mathcal{A}$  over  $\text{span}(e_1, f_1)$  is  $I$ . But none of vector  $x$  such that  $\mathcal{A}x = e_1 + f_1$  or  $\mathcal{A}x = e_2 + 2f_1$ .

► 翻译: 例证上述 “Jordan 块” 不能替换为 “Jordan 型”.

**Fuxercise 46 (Structural theorem of invariant subspace)** Given a finite-dimensional  $\mathbb{C}$ -space  $V$ , and a linear transformation  $\mathcal{A}$  over it.

► 翻译: (不变子空间结构定理) 给定一个有限维  $\mathbb{C}$ -线性空间  $V$  和一个其上的线性变换  $\mathcal{A}$ .

(1) Show that each  $\mathcal{A}$ -cyclic subspace  $W$  belonging to  $\lambda$  of  $V$  is of form

$$W = (\mathcal{A} - \lambda\mathcal{I})^k U \quad k \in \mathbb{Z}_{\geq 0} \quad V = U \oplus U'$$

with  $U$   $\mathcal{A}$ -cyclic and  $U'$   $\mathcal{A}$ -invariant. Hint It is just the geometrical form of exercise 45.

► 翻译: 证明任何  $V$  的  $\mathcal{A}$ -循环子空间  $W$  都具有如下形式

$$W = (\mathcal{A} - \lambda\mathcal{I})^k U \quad V = U \oplus U'$$

其中  $U$  也循环子空间, 且  $U'$  是  $\mathcal{A}$  不变的.

(2) Show that each  $\mathcal{A}$ -invariant subspace  $W$  of  $V$  is of form

$$W = (\mathcal{A} - \lambda_1\mathcal{I})^{k_1} U_1 \oplus \dots \oplus (\mathcal{A} - \lambda_m\mathcal{I})^{k_m} U_m$$

Where

- Each  $\lambda_i \in \mathbb{C}$  and each  $k_i \in \mathbb{Z}_{\geq 0}$ .
- Each  $U_1$  is  $\mathcal{A}$ -cyclic and accepts an  $\mathcal{A}$ -complement  $U'_1$ , i.e. an  $\mathcal{A}$ -invariant subspace  $U'_1$  such that  $U_1 \oplus U'_1 = V$ .

► 翻译: 证明每个  $V$  的  $\mathcal{A}$ -不变子空间  $W$  都具有形式

$$W = (\mathcal{A} - \lambda_1\mathcal{I})^{k_1} U_1 \oplus \dots \oplus (\mathcal{A} - \lambda_m\mathcal{I})^{k_m} U_m$$

其中

- 每个  $\lambda_i \in \mathbb{C}$ , 每个  $k_i \in \mathbb{Z}_{\geq 0}$ .
- 每个  $U_1$  都是  $\mathcal{A}$  循环的且有  $\mathcal{A}$ -补充  $U'_1$ , 即一个  $\mathcal{A}$  不变子空间  $U'_1$  使得  $U_1 \oplus U'_1 = V$ .

**Exercise 47** Recollect the exercise 36 where violent computation is asked by hint. But now, we can show it more abstractly without resorting to violence. Let  $\mathcal{B}$  be a linear transformation commuting with all the linear transformation commuting with  $A$ .

► 翻译：回忆习题36当时提示要求进行暴力的计算。现在我们可以更抽象地证明，而不诉诸暴力。令  $\mathcal{B}$  是一个和所有  $\mathcal{A}$  可交换的线性变换可交换的线性变换。

(1) We firstly prove the weak form. That is

$$\forall v \in V, \exists f \in \mathbb{C}[X], \text{ such that } f(\mathcal{A})v = \mathcal{B}v$$

Which is equivalent to that any  $\mathcal{A}$ -invariant subspace is also  $\mathcal{B}$ -invariant.

Hint Using the structural theorem of invariant subspace in exercise 46 above. Let  $W$  be an  $\mathcal{A}$ -invariant subspace. Assume

$$W = (\mathcal{A} - \lambda_1 \mathcal{I})^{k_1} U_1 \oplus \dots \oplus (\mathcal{A} - \lambda_m \mathcal{I})^{k_m} U_m$$

Let  $\mathcal{P}_i$  be the projection from  $V = U_i \oplus U'_i$  to  $U_i$ . Clearly,  $\mathcal{A}\mathcal{P}_i = \mathcal{P}_i\mathcal{A}$ . So

$$\begin{aligned} \mathcal{B}(W) &= \mathcal{B}\left(\sum_i (\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \mathcal{P}_i(V)\right) \\ &\subseteq \sum_i \mathcal{B}(\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \mathcal{P}_i(V) && \because \mathcal{B}(v_1 + \dots + v_m) = \mathcal{B}v_1 + \dots + \mathcal{B}v_m \\ &= \sum_i (\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \mathcal{P}_i \mathcal{B}(V) && \because \begin{cases} \mathcal{A}\mathcal{P}_i = \mathcal{P}_i\mathcal{A} \\ (\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \mathcal{A} = \mathcal{A}(\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \end{cases} \\ &\subseteq (\mathcal{A} - \lambda_i \mathcal{I})^{k_i} \mathcal{P}_i(V) && \because \mathcal{B}(V) \subseteq V \\ &= W \end{aligned}$$

► 翻译：我们首先证明弱版本，

$$\forall v \in V, \exists f \in \mathbb{C}[X], \text{ such that } f(\mathcal{A})v = \mathcal{B}v$$

这等价于任何  $\mathcal{A}$ -不变子空间也是  $\mathcal{B}$ -不变的。

(2) Now deduce the conclusion by considering  $(V^\dagger, \mathcal{A}^\dagger)$  where

$$V^{\oplus n} \quad \mathcal{A}^\dagger : (v_1, \dots, v_n) \mapsto (\mathcal{A}v_1, \dots, \mathcal{A}v_n)$$

Hint Now, in language of matrix,  $\mathcal{A}^\dagger = \text{diag}(\mathcal{A}, \dots, \mathcal{A})$ . Hence, the matrices commuting with  $\mathcal{A}^\dagger$  are exactly  $(\mathcal{C}_{ij})$  with  $\mathcal{C}_{ij}\mathcal{A} = \mathcal{A}\mathcal{C}_{ij}$ . Thus  $\mathcal{B}^\dagger = \text{diag}(\mathcal{B}, \dots, \mathcal{B})$  commutes with all of them. Finally, note that  $f(\text{diag}(\mathcal{A}, \dots, \mathcal{A})) = \text{diag}(f(\mathcal{A}), \dots, f(\mathcal{A}))$ .

► 翻译: 现在通过考虑  $(V^\dagger, \mathcal{A}^\dagger)$  其中

$$V^{\oplus n} \quad \mathcal{A}^\dagger : (v_1, \dots, v_n) \mapsto (\mathcal{A}v_1, \dots, \mathcal{A}v_n)$$

证明结论.

*The above proof is a generation form of the proof due to Bourbaki of density theorem due to Jacobson and Chevally.*

► 翻译: 上述证明是 Jacobson 和 Chevally 稠密性定理 Bourbaki 证明的推广.

## 7 Week7 (8 Apr - 14 Apr)

► 翻译: 第七周 (4月8日 - 4月14日)

**Exercise 48** *Given an Hermitian matrix  $A$ , show that  $A + iI$  is invertible and  $i(A - iI)(A + iI)^{-1}$  is unitary. Give a geometrical explanation.*

► 翻译: 给一个 Hermite 矩阵  $A$ , 证明  $A + iI$  是可逆的且  $i(A - iI)(A + iI)^{-1}$  是酉矩阵. 给一个几何解释.

**Exercise 49 (Bessel Inequality)** *Given an inner product space  $V$ , if  $e_1, \dots, e_n$  are pairwise orthogonal unit vectors, show that for any  $x \in V$*

$$\langle x, x \rangle^2 \geq \sum_{i=1}^n \langle x, e_i \rangle^2$$

*Note that we do not make assumption that  $V$  is finite-dimensional.*

► 翻译: (Bessel 不等式) 对于一个内积空间, 如果  $e_1, \dots, e_n$  是两两正交的单位向量, 证明对任意  $x \in V$

$$\langle x, x \rangle^2 \geq \sum_{i=1}^n \langle x, e_i \rangle^2$$

注意我们没有假设  $V$  是有限维的.

**Exercise 50 (Orthogonal projection)** *Recollect that projection is introduced in exercise 12. Given an space  $V$  equipped with an inner product, show that for projection  $\mathcal{P}$  the following statements are equivalent*

- (a)  $\ker \mathcal{P} = \operatorname{im} \mathcal{P}^\perp$ .
- (b)  $\operatorname{im} \mathcal{P} = \ker \mathcal{P}^\perp$ .
- (c) the decomposition  $V = \ker \mathcal{P} \oplus \operatorname{im} \mathcal{P}$  is orthogonal.
- (d)  $\mathcal{P}$  is symmetric.
- (e)  $\mathcal{P}$  is normal, i.e.  $\mathcal{P}\mathcal{P}^\top = \mathcal{P}^\top\mathcal{P}$ .
- (f)  $\langle \mathcal{P}x, x \rangle \geq 0$  for all  $x \in V$ .

The projection satisfying the above condition is called orthogonal projection.

Hint The equivalence of (a), (b) and (c) is easy. (c) $\Rightarrow$ (d),

$$\langle \mathcal{P}(v_1 + v_2), w_1 + w_2 \rangle = \langle v_1, w_2 \rangle = \langle v_1 + v_2, \mathcal{P}(w_1 + w_2) \rangle$$

(d) $\Rightarrow$ (e) is trivial. (e)  $\Rightarrow$  (a), first note that if  $\mathcal{P}$  is normal,

$$\mathcal{P}x = 0 \iff \langle \mathcal{P}x, \mathcal{P}x \rangle = 0 \iff \langle \mathcal{P}^\top x, \mathcal{P}^\top x \rangle = 0 \iff \mathcal{P}^\top x = 0$$

Then using  $\langle x, \mathcal{P}y \rangle = \langle \mathcal{P}^\top x, y \rangle$ , one can prove  $\ker \mathcal{P}^\top = \ker \mathcal{P} = \operatorname{im} \mathcal{P}^\perp$ . (a)  $\Rightarrow$  (f) is easy. (f)  $\Rightarrow$  (a) is the most tricky, I recommend you to draw a figure to see what happens. Pick  $a \in \operatorname{im} \mathcal{P}^\perp, b \in \ker \mathcal{P}$ , we claim if  $a$  is not orthogonal to  $b$ , then there will be something wrong on some  $a + \lambda b$ . More exactly,  $0 \leq \langle \mathcal{P}(a + \lambda b), a + \lambda b \rangle = \langle a, a \rangle + \lambda \langle a, b \rangle$ ,  $\lambda$  is arbitrary, which is ridiculous.

► 翻译: (正交投影) 回忆投影已经在习题12被介绍. 给一个内积空间  $V$ , 证明对于投影  $\mathcal{P}$  下列条件是等价的

- (1)  $\ker \mathcal{P} = \operatorname{im} \mathcal{P}^\perp$ .
- (2)  $\operatorname{im} \mathcal{P} = \ker \mathcal{P}^\perp$ .
- (3) 分解  $V = \ker \mathcal{P} \oplus \operatorname{im} \mathcal{P}$  是正交的.
- (3)  $\mathcal{P}$  是对称的.

(4)  $\mathcal{P}$  是正规的, 即  $\mathcal{P}\mathcal{P}^\top = \mathcal{P}^\top\mathcal{P}$ .

(5)  $\langle \mathcal{P}x, x \rangle \geq 0$  对所有  $x \in V$ .

满足上述条件的投影被叫做正交投影.

**Exercise 51 (Orthogonal reflection)** *Recollect that projection is introduced in exercise 13.*

► 翻译: (正交反射) 回忆反射已经在习题13被介绍.

(1) *Given an space  $V$  equipped with an inner product, show that for reflection  $\mathcal{S}$  the following statements are equivalent*

(a)  $\text{Fix } \mathcal{S} = \text{Fix}(-\mathcal{S})^\perp$ .

(b)  $\text{Fix}(-\mathcal{S}) = \text{Fix } \mathcal{S}^\perp$ .

(c) *the decomposition  $V = \text{Fix } \mathcal{S} \oplus \text{Fix}(-\mathcal{S})$  is orthogonal.*

(d)  *$\mathcal{S}$  is orthogonal.*

(e)  *$\mathcal{S}$  is normal, i.e.  $\mathcal{S}\mathcal{S}^\top = \mathcal{S}^\top\mathcal{S}$ .*

(f)  $\langle \mathcal{S}x, x \rangle \geq \langle x, x \rangle$  for all  $x \in V$ .

*The reflection satisfying the above condition is called orthogonal reflection.*

Hint *Note that  $\mathcal{P} \mapsto \mathcal{I} - 2\mathcal{P}$  and  $\mathcal{S} \mapsto \frac{1}{2}(\mathcal{I} - \mathcal{S})$  give bijections between the projection and reflections.*

► 翻译: (1) 给一个内积空间  $V$ , 证明对于反射  $\mathcal{S}$  下列条件是等价的

(1)  $\ker \mathcal{P} = \text{im } \mathcal{P}^\perp$ .

(2)  $\text{im } \mathcal{P} = \ker \mathcal{P}^\perp$ .

(3) 分解  $V = \ker \mathcal{P} \oplus \text{im } \mathcal{P}$  是正交的.

(3)  $\mathcal{P}$  是对称的.

(4)  $\mathcal{P}$  是正规的, 即  $\mathcal{P}\mathcal{P}^\top = \mathcal{P}^\top\mathcal{P}$ .

(5)  $\langle Px, x \rangle \geq 0$  对所有  $x \in V$ .

满足上述条件的反射被叫做正交反射.

(2) Let  $V$  be a space equipped with an inner product. Let  $v \in V \setminus \{0\}$ , there is a unique orthogonal reflection  $S$  such that  $\text{Fix}(-S) = \mathbb{R}v$ , write it down and show the uniqueness. It is called simple orthogonal reflection by  $v$ .

**Hint**  $w \mapsto w - \frac{2\langle v, w \rangle}{\langle v, v \rangle} v$ .

► 翻译: 令  $V$  是内积空间. 令  $v \in V \setminus \{0\}$ , 则存在唯一的一个反射  $S$  使得  $\text{Fix}(-S) = \mathbb{R}v$ , 写下表达式并证明唯一性. 这被叫做沿着  $v$  的单正交反射.

(3) Show that in  $\mathbb{R}^n$  equipped with the standard inner product, the matrix of simple orthogonal reflection by  $v$  is  $I - 2\frac{v \cdot v^\top}{\langle v, v \rangle}$ .

► 翻译: 证明在赋予标准内积的  $\mathbb{R}^n$  上, 沿着  $v$  的单正交反射的矩阵是  $I - 2\frac{v \cdot v^\top}{\langle v, v \rangle}$ .

**Problem 52 (QR decomposition)** In this problem, we deal with the famous QR decomposition.

► 翻译: (QR 分解) 在这个问题中, 我们将要处理著名的 QR 分解

Prove that each real square matrix  $A$  can be written as

(1)  $A = QR$  with  $Q$  an orthogonal matrix and  $R$  an upper matrix.

(2)  $A = RQ$  with  $Q$  an orthogonal matrix and  $R$  an upper matrix.

(3)  $A = QL$  with  $Q$  an orthogonal matrix and  $L$  a lower matrix.

(4)  $A = LQ$  with  $Q$  an orthogonal matrix and  $L$  a lower matrix.

**Hint** Let  $A^\top = (v_1, \dots, v_n)$ , use the Gram-Schmidt process on  $(v_1, \dots, v_n)$ .

Another method is known as Householder transformation, we can pick a orthogonal reflection  $Q_1$  such that  $Q_1 A = \begin{pmatrix} \lambda & * \\ & * \end{pmatrix}$ . (1) implies (4) by consid-

ering  $A^\top$ . (1) implies (3) by considering  $PAP$  where  $P = \begin{pmatrix} & & 1 \\ & \dots & \\ 1 & & \end{pmatrix}$ . (1)

implies (2) by considering  $PA^\top P$ .

► 翻译: 证明任何一个矩阵  $A$  都可以写成

(1)  $A = QR$  其中  $Q$  是正交矩阵  $R$  是上三角矩阵.

(2)  $A = RQ$  其中  $Q$  是正交矩阵  $R$  是上三角矩阵.

(3)  $A = QL$  其中  $Q$  是正交矩阵  $L$  是下三角矩阵.

(4)  $A = LQ$  其中  $Q$  是正交矩阵  $L$  是下三角矩阵.

(5) If  $A$  is invertible, we assume more that the diagonal indices of  $R$  or  $L$  are positive, show that the decomposition is unique. Hint If  $Q_1R_1 = Q_2R_2$ , then  $Q_2^{-1}Q_1 = R_2R_1^{-1}$ , show that they both equals to  $I$ .

► 翻译: (5) 如果  $A$  可逆, 进一步假设  $R$  的对角元是正的, 证明分解是唯一的.

**Problem 53** Given a finite dimensional  $\mathbb{R}$ -space  $V$ .

► 翻译: 给定一个有限维实线性空间  $V$ .

(1) What kind of linear transformation is orthogonal for some inner product? Hint That is, the matrix of it is similar to a orthogonal matrix. It is well-known that any orthogonal matrix is orthogonally similar to

$$\text{diag} \left( 1, \dots, 1, -1, \dots, -1, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \dots, \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \right)$$

So the conclusion is the eigenvalue is of norm 1 and is diagonalizable.

► 翻译: (1) 对怎样的线性变换是某个内积下的正交变换?

(2) Given a finite set  $X$  of invertible linear transformation of  $V$ . Assume that  $X$  is a group, i.e.

$$\mathcal{I} \in X \quad \mathcal{A}, \mathcal{B} \in X \Rightarrow \mathcal{AB} \in X \quad \mathcal{A} \in X \Rightarrow \mathcal{A}^{-1} \in X$$

Show that under certain inner product, all members of  $X$  are orthogonal.

Hint Let  $\langle \cdot, \cdot \rangle$  be arbitrary inner product of  $V$ , put

$$\langle v, w \rangle' = \frac{1}{|X|} \sum_{\mathcal{A} \in X} \langle \mathcal{A}v, \mathcal{A}w \rangle$$

Without too much efforts, you can show  $\langle \cdot, \cdot \rangle'$  is an inner product with  $\langle \mathcal{A}v, \mathcal{A}w \rangle' = \langle v, w \rangle'$ .

► 翻译: 给一个由  $V$  的可逆线性变换组成的有限集  $X$ . 假设  $X$  是一个群, 即

$$\mathcal{I} \in X \quad \mathcal{A}, \mathcal{B} \in X \Rightarrow \mathcal{AB} \in X \quad \mathcal{A} \in X \Rightarrow \mathcal{A}^{-1} \in X$$

证明在某个内积下, 所有  $X$  的成员都是正交的.

**Problem 54** Given a unitary space  $V$ , show that a linear transformation  $\mathcal{A}$  is Hermitian iff  $\langle \mathcal{A}x, x \rangle \in \mathbb{R}$  for all  $x \in V$ . Hint Using the equality

$$\langle \mathcal{A}(x + \lambda y), x + \lambda y \rangle = \langle \mathcal{A}x, x \rangle + |\lambda|^2 \langle \mathcal{A}y, y \rangle + \bar{\lambda} \langle \mathcal{A}x, y \rangle + \lambda \langle \mathcal{A}y, x \rangle$$

By the assumption,

$$\lambda \langle \mathcal{A}y, x \rangle + \bar{\lambda} \langle \mathcal{A}x, y \rangle \in \mathbb{R}$$

Thus,

$$\begin{aligned} \lambda \langle \mathcal{A}y, x \rangle + \bar{\lambda} \langle \mathcal{A}x, y \rangle &= \overline{\lambda \langle \mathcal{A}y, x \rangle + \bar{\lambda} \langle \mathcal{A}x, y \rangle} \\ &= \bar{\lambda} \langle x, \mathcal{A}y \rangle + \lambda \langle y, \mathcal{A}x \rangle \\ &= \bar{\lambda} \langle A^H x, y \rangle + \lambda \langle A^H y, x \rangle \end{aligned}$$

Then by taking  $\lambda = 1, i$ , the coefficient of  $\bar{\lambda}$  coincides, that is

$$\langle \mathcal{A}x, y \rangle = \langle A^H x, y \rangle$$

The proof is complete.

► 翻译: 给定一个酉空间, 证明一个线性变换  $\mathcal{A}$  是 Hermite 的当且仅当  $\langle \mathcal{A}x, x \rangle \in \mathbb{R}$  对所有  $x \in V$ .

**Problem 55** (1) For any complex square matrix  $A$ , show that there exists a unitary matrix  $U$  such that  $UAU^{-1}$  is an upper matrix. Hint Using the same method in exercise 22. Or using the analogue of QR decomposition for unitary case.

► 翻译: (1) 对于任何复方阵  $A$ , 证明存在一个酉矩阵  $U$  使得  $UAU^{-1}$  是上三角.

(2) For any real square matrix  $A$  with all eigenvalue real, show that there exists an orthogonal matrix  $P$  such that  $PAP^{-1}$  is an upper matrix.

► 翻译: (2) 对任何所有特征值都是实数的方阵  $A$ , 证明存在一个正交阵  $U$  使得  $UAU^{-1}$  是上三角矩阵.

## 8 Week8 (8 Apr - 14 Apr)

► 翻译: 第八周 (4月8日 - 4月14日)

**Problem 56** For any invertible square complex matrix  $A$  and  $n \geq 1$ , show that there exists a complex matrix  $B$  such that  $B^n = A$ . Hint Clearly, it suffices to deal with Jordan blocks. Assume  $B$  is in Jordan block belonging to  $\lambda$ , show that we can choose  $\lambda$  such that  $A$  is similar to  $B^n$  by computation. Or, write  $A = \lambda I + N$  with  $N$  nilpotent, using the Taylor expansion of  $(\lambda + x)^{1/n}$ .

► 翻译: 对所有可逆复方阵  $A$  和正整数  $n \geq 1$ , 证明存在复矩阵  $B$  使得  $B^n = A$ .

**Exercise 57 (Riesz functional calculus)** Let  $V$  be a finite-dimensional  $\mathbb{C}$ -space, and  $\mathcal{A}$  be a linear transformation of  $V$ . Let  $f$  be an analytic function<sup>2</sup>. In this problem, we will define  $f(\mathcal{A})$  as much as possible.

► 翻译: (Riesz 泛函微积分) 令  $V$  是一个有限维  $\mathbb{C}$ -线性空间,  $\mathcal{A}$  是一个线性变换. 令  $f$  是一个解析函数. 在这个问题中, 我们将要尽可能地定义  $f(\mathcal{A})$ .

(1) If  $\mathcal{A} - \lambda I$  is nilpotent, and  $\lambda$  lies in the defined domain of  $f$ , then we can define

$$f(\mathcal{A}) := \sum_{k=0}^{\infty} \frac{f^{(k)}(\lambda)}{k!} (\mathcal{A} - \lambda I)^k$$

Show that this is well-defined. Hint It is a finite sum essentially!

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<sup>2</sup> We say a function  $f$  is analytic at  $p$  if there exists an  $\epsilon_p > 0$  such that

$$|z - p| < \epsilon_p \quad \Rightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!} (z - p)^k$$

We say a function  $f$  is analytic if the defined domain  $D$  is an open subset of  $\mathbb{C}$  and  $f$  is analytic all over the domain.

► 翻译: 我们说一个函数  $f$  在  $p$  点解析如果存在  $\epsilon_p > 0$  使得

$$|z - p| < \epsilon_p \quad \Rightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!} (z - p)^k$$

说  $f$  是解析的如果定义域  $D_f$  是  $\mathbb{C}$  的一个开集, 且定义域上点点解析.

► 翻译: (1) 如果  $\mathcal{A} - \lambda\mathcal{I}$  是幂零的, 并且  $\lambda$  在  $f$  的定义域里, 那么我们可以定义

$$f(\mathcal{A}) := \sum_{k=0}^{\infty} \frac{f^{(k)}(\lambda)}{k!} (\mathcal{A} - \lambda\mathcal{I})^k$$

证明这是良定义的.

(2) If all of eigenvalues of  $\mathcal{A}$  lie in the defined domain of  $f$ , then there exists a unique  $f(\mathcal{A})$  such that for any  $\mathcal{A}$ -invariant subspace  $W$

$$f(\mathcal{A}|_W) = \overline{f(\mathcal{A})}|_W$$

whenever  $\mathcal{A}|_W$  satisfies the condition in (1). Further more, actually,  $f(\mathcal{A})$  is a polynomial in  $\mathcal{A}$ .

► 翻译: (2) 如果  $\mathcal{A}$  的特征值均在  $f$  的定义域里, 那么存在唯一的  $f(\mathcal{A})$  使得任何  $\mathcal{A}$ -不变子空间  $W$

$$f(\mathcal{A}|_W) = f(\mathcal{A})|_W$$

只要  $\mathcal{A}|_W$  满足 (1) 的条件. 并且, 实际上  $f(\mathcal{A})$  是  $\mathcal{A}$  的多项式.

(3) Assume  $z_0$  and  $\epsilon > 0$  such that

$$|z - z_0| < \epsilon \quad \Rightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

If  $\mathcal{A} - \lambda\mathcal{I}$  is nilpotent, and  $|\lambda - z_0| < \epsilon$ , show that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} (\mathcal{A} - z_0\mathcal{I})^k = f(\mathcal{A}) \quad \text{defined in (1)}$$

Note that, this generalize the exponential map in Exercise 39. Hint Assume  $(\mathcal{A} - \lambda\mathcal{I})^n = 0$ . Then the expansion

$$[(\mathcal{A} - \lambda\mathcal{I}) + (\lambda - z_0)\mathcal{I}]^k = \sum_{h=0}^k \binom{k}{h} (\mathcal{A} - \lambda\mathcal{I})^h (\lambda - z_0)^{k-h}$$

has at most  $n - 1$  terms. So it is safe to change the order.

► 翻译: 假如  $z_0$  和  $\epsilon > 0$  使得

$$|z - z_0| < \epsilon \quad \Rightarrow \quad f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

如果  $\mathcal{A} - \lambda \mathcal{I}$  幂零, 且  $|\lambda - z_0| < \epsilon$ , 证明

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} (\mathcal{A} - z_0 \mathcal{I})^k = f(\mathcal{A}) \quad \text{defined in (1)}$$

注意到这推广了习题39的指数映射.

(4) What is the most difficult, show that for two analytic functions  $f, g$  with their composition  $g \circ f$  analytic, show that

$$(g \circ f)(\mathcal{A}) = g(f(\mathcal{A}))$$

whenever all the above notation is defined, that is, all the eigenvalues of  $\mathcal{A}$  lie in the domain of  $f$  and are mapped by  $f$  into the domain of  $g$ . Hint

By a transformation, it reduces to the case  $\mathcal{A}$  nilpotent and  $f(0) = 0$ . By an induction, one can show that  $(f \circ g)^{(n)}$  is a polynomial in  $\{f^{(k)}(0)\}_{k \leq n} \cup \{g^{(k)}(0)\}_{k \leq n}$ . Thus  $(g_{\leq n} \circ f_{\leq n})_{\leq n} = (g \circ f)_{\leq n}$ . Then by the definition,  $f(\mathcal{A}) = f_{\leq n}(\mathcal{A})$  for some sufficient large  $n$ .

► 翻译: 最难的是, 证明对两个解析函数  $f, g$  使得  $g \circ f$  解析, 证明

$$(g \circ f)(\mathcal{A}) = g(f(\mathcal{A}))$$

当上述记号被定义时, 即, 所有  $\mathcal{A}$  的特征值在  $f$  定义域上, 且被  $f$  映到  $g$  的定义域上.

(5) Show that  $\exp$  in Exercise 39 is surjection to  $\text{GL}_n(\mathbb{C})$ . Hint It requires some complex analysis. Elementarily, firstly show that any unipotent matrix is in the image by using the

$$\log(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \dots \quad |w| < 1$$

secondly, show that any diagonalizable matrix is in the image. Last, using Jordan decomposition in Exercise 38, note that if  $AB = BA$ ,  $f(A)g(B) = g(B)f(A)$ .

► 翻译: 证明习题39定义的  $\exp$  是到  $\mathrm{GL}_n(\mathbb{C})$  满射.

(6) Is  $\exp : \mathbb{M}_n(\mathbb{R}) \rightarrow \mathrm{GL}_n(\mathbb{R})$  a surjection? Hint  $\mathrm{GL}_n^+(\mathbb{R})$ .

► 翻译: 映射  $\exp : \mathbb{M}_n(\mathbb{R}) \rightarrow \mathrm{GL}_n(\mathbb{R})$  还是满射吗?

## APPENDIX—PROPERTIES OF ANALYTIC FUNCTIONS

► 翻译: 附录 — 解析函数的性质

1. If  $f$  is analytic at  $z$ , with the radius  $\epsilon$ , then  $f$  is analytic at every point of  $\{w : |w - z| < \epsilon\}$ . Further more, the radius of  $w$  is no less than  $\epsilon - |w - z|$ .

PROOF. Without loss of generality, assume  $z = 0$ . Let  $f = \sum a_k z^k$ . Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k \sum_{h=0}^k ((x-w) + w)^h \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^k a_k \binom{k}{h} w^{k-h} (x-w)^h \\ &\stackrel{?}{=} \sum_{h=0}^{\infty} \sum_{k=h}^{\infty} a_k \binom{k}{h} w^{k-h} (x-w)^h \\ &= \sum_{h=0}^{\infty} \left( \sum_{k=h}^{\infty} \binom{k}{h} a_k w^{k-h} \right) (x-w)^h \\ &= \sum_{h=0}^{\infty} \frac{f^{(h)}(w)}{h!} (x-w)^h \end{aligned}$$

If the change of order  $\stackrel{?}{=}$  above holds, the proof is complete. Note that

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{h=0}^k |a_k| \binom{k}{h} |w|^{k-h} |x-w|^h \\ &= \sum_{k=0}^{\infty} |a_k| (|w| + |x-w|)^k \\ &\leq \sum_{k=0}^{\infty} |a_k| |x|^k \\ &< \infty \qquad \qquad \qquad \because \text{radius formula} \end{aligned}$$

A mathematical analysis theorem ensures the exchange of order <sup>3</sup>

2. If  $f, g$  are analytic at  $a$ , then so is  $fg$ .

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<sup>3</sup>One know that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{nm}| < \infty \Rightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nm}$$

PROOF. Without loss of generality, assume  $a = 0$ . Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad g(x) = \sum_{h=0}^{\infty} b_h x^h$$

Then

$$\begin{aligned} (fg)(x) &= \sum_{k=0}^{\infty} a_k x^k \sum_{h=1}^{\infty} b_h x^h \\ &= \sum_{k=0}^{\infty} \sum_{h=1}^{\infty} a_k b_h x^{k+h} \\ &\stackrel{?}{=} \sum_{\ell=0}^{\infty} \sum_{\ell=k+h} a_k b_h x^{\ell} \\ &= \sum_{\ell=0}^{\infty} \left( \sum_{\ell=k+h} a_k b_h \right) x^{\ell} \end{aligned}$$

If the change of order  $\stackrel{?}{=}$  above holds, the proof is complete. Since  $f$  or  $g$  is absolutely convergent, the exchange is valid<sup>4</sup>.

3. If  $f$  is analytic at  $a$ ,  $g$  is analytic at  $f(a)$ , then  $g \circ f$  is analytic at  $a$ .

Firstly, the convergence of  $\sum_{m=1}^{\infty} a_{nm}$  is clear. Secondly,

$$\begin{aligned} &\left| \sum_{m=1}^M \sum_{n=1}^{\infty} a_{nm} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \right| = \left| \sum_{n=1}^{\infty} \sum_{m=M+1}^{\infty} a_{nm} \right| \leq \sum_{n=1}^{\infty} \sum_{m=M+1}^{\infty} |a_{nm}| \\ &\leq \sum_{n=1}^N \sum_{m=M+1}^{\infty} |a_{nm}| + \sum_{n=N+1}^{\infty} \sum_{m=M+1}^{\infty} |a_{nm}| \\ &\leq \sum_{n=1}^N \sum_{m=M+1}^{\infty} |a_{nm}| + \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |a_{nm}| \end{aligned}$$

Given  $\epsilon > 0$ , pick  $N$  sufficient large such that the second term  $< \epsilon/2$ . And pick  $M$  sufficient large such that for all  $1 \leq n \leq N$ ,  $\sum_{m=M+1}^{\infty} |a_{nm}| < \epsilon/(2N)$ .

<sup>4</sup> Assume  $c_{\ell} = \sum_{\ell=k+h} a_k b_h$  and  $h = \sum_{\ell=1}^{\infty} c_{\ell} x^{\ell}$ . Then

$$\begin{aligned} |h_{\leq n}| &= \left| \sum_{i=0}^n a_i x^i g_{\leq n-i} \right| \\ &= \left| fg_{\leq n} - \sum_{i=0}^n a_i x^i (g - g_{\leq n-i}) \right| \\ &\leq |fg_{\leq n}| + \sum_{i=0}^n |a_i x_i| |g - g_{\leq n-i}| \\ &\leq |fg_{\leq n}| + \sum_{i=0}^a |a_i x_i| |g - g_{\leq n-i}| + \sum_{i=a}^n |a_i x_i| |g - g_{\leq n-i}| \end{aligned}$$

Given  $\epsilon > 0$ , pick  $N$  such that  $n > N$ ,  $|g - g_{\leq n}| < \epsilon$ , if  $n - a > N$ , then the third term

$$\sum_{i=a}^n |a_i x_i| |g - g_{\leq n-i}| < \sum_{i=a}^n |a_i x_i| \epsilon \leq \sum_{i=0}^n |a_i x_i| \epsilon$$

Let  $n \rightarrow \infty$ , we get the desired result.

PROOF. Without loss of generality, assume  $a = f(a) = 0$ . Let

$$f(x) = \sum_{k=1}^{\infty} a_k x^k \quad g(y) = \sum_{h=0}^{\infty} b_h y^h$$

Then

$$\begin{aligned} (g \circ f)(x) &= \sum_{h=0}^{\infty} b_h \left( \sum_{k=1}^{\infty} a_k x^k \right)^h \\ &= \sum_{h=0}^{\infty} b_h \sum_{K=1}^{\infty} A_{hK} x^K \\ &= \sum_{h=0}^{\infty} \sum_{K=1}^{\infty} b_h A_{hK} x^K \\ &\stackrel{?}{=} \sum_{K=1}^{\infty} \left( \sum_{h=1}^K b_h A_{hK} \right) x^K \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{h=0}^N \sum_{K=1}^M |b_h A_{hK} x^K| \\ &\leq \sum_{h=0}^{\infty} |b_h| \left| \sum_{k=1}^{\infty} |a_k| |x|^k \right|^h \quad \because \begin{array}{l} A_{hk} \text{ is polynomial in } a_k \text{'s} \\ \text{with positive coefficients.} \end{array} \end{aligned}$$

Since  $x \mapsto \sum_{k=1}^{\infty} |a_k| |x|^k$  is continuous, we can pick  $x$  sufficient small so that it lies in the convergent circle of  $\sum_{h=0}^{\infty} |b_h| y^h$  when the sum above  $< \infty$ .

## Exercises seminar(11 Apr)

► 翻译: 习题课 (4月11日)

**Exercise 58** Show that

$$\mathfrak{sl}_n(\mathbb{R}) = \text{span}\{[A, B] = AB - BA : A, B \in \mathbb{M}_n(\mathbb{R})\}$$

is of dimension  $n^2 - 1$ , a set of basis can be taken as

$$\{E_{11} - E_{ii}\}_{i=2}^n \quad \sqcup \quad \{E_{ij}\}_{i \neq j}$$

and  $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathbb{M}_n(\mathbb{R}) : \text{tr } A = 0\}$ . Hint Note that the  $[\cdot, \cdot]$  is linear, so can calculate at  $E_{ij}$ . Then  $E_{11} - E_{ii} = E_{1i}E_{i1} - E_{i1}E_{1i}$ ,  $E_{ij} = E_{i1}E_{1j} - E_{1j}E_{i1}$ .

► 翻译: 证明

$$\mathfrak{sl}_n(\mathbb{R}) = \text{span}\{AB - BA : A, B \in \mathbb{M}_n(\mathbb{R})\}$$

具有维数  $n^2 - 1$ , 且一组基可取作

$$\{E_{11} - E_{ii}\}_{i=2}^n \cup \{E_{ij}\}_{i \neq j}$$

并且  $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathbb{M}_n(\mathbb{R}) : \text{tr } A = 0\}$ .

**Fuxercise 59** For a matrix  $A \in \mathbb{M}_n(\mathbb{Q})$ , we shall show that if  $\text{tr } A = 0$ , then  $A$  is similar to some matrix whose entries in diagonal vanish.

► 翻译: 对于一个矩阵  $A \in \mathbb{M}_n(\mathbb{Q})$ , 我们要证明如果  $\text{tr } A = 0$ , 那么  $A$  相似于某个对角元全是 0 的矩阵.

(1) Prove the argument for diagonal matrix. Hint Show that if  $a \neq b$ ,

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \text{ is similar to } \begin{pmatrix} 0 & ab \\ -1 & a+b \end{pmatrix}.$$

► 翻译: 对对角矩阵证明结论.

(2) Using rational norm form to deduce the conclusion. Hint Note that

the companion matrix of  $\lambda^n + a\lambda^{n-1} + \dots$  is 
$$\begin{pmatrix} 0 & * & \dots & * \\ * & 0 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & -a \end{pmatrix}.$$

► 翻译: 用有理标准型来得到结果.

(3) Does the statement hold for  $\mathbb{R}$  or  $\mathbb{C}$ ?

► 翻译: 结论对  $\mathbb{R}$  或  $\mathbb{C}$  还对吗?

**Fuxercise 60** Prove that the ‘span’ in exercise 58 can be omitted (how amazing!). More exactly, for any matrix  $X \in \mathbb{M}_n(\mathbb{R})$  with  $\text{tr } X = 0$ , there exist  $A, B \in \mathbb{M}_n(\mathbb{R})$  such that  $AB - BA = X$ . Hint Using the conclusion in exercise 59 and assume  $A = \text{diag}(1, 2, \dots, n)$ .

► 翻译: 证明习题58中的 “span” 是可以删掉的 (多神奇啊!). 具体来说, 对任意矩阵  $X \in \mathbb{M}_n(\mathbb{R})$  使得  $\text{tr } X = 0$ , 那么存在  $A, B \in \mathbb{M}_n(\mathbb{R})$  使得  $AB - BA = X$ .



eigenvalue, then exists some  $f$  such  $f(A) = 0$  but  $f(B)$  invertible. And if  $Av = \lambda v, B^T w = \lambda w$ , then  $X = vw^T$  serves. To be more meaningful, let

$$x = (x_{11}, x_{21} \dots, x_{n1}, x_{12}, x_{22}, \dots, x_{n2}, \dots, x_{1n}, x_{2n} \dots, x_{nn})^T$$

the equation becomes

$$\underbrace{\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}}_{=: I \otimes A} x = \underbrace{\begin{pmatrix} b_{11}I & b_{12}I & \dots & b_{1n}I \\ b_{21}I & b_{22}I & \dots & b_{2n}I \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}I & b_{n2}I & \dots & b_{nn}I \end{pmatrix}}_{=: B \otimes I} x$$

So it suffices to calculate the eigenvalue of  $I \otimes A - B \otimes I$ . First, let  $P$  and  $Q$  be matrices such that  $PAP^{-1}$  and  $QBQ^{-1}$  are upper. Then

$$(Q \otimes P)(I \otimes A - B \otimes I)(Q^{-1} \otimes P^{-1}) = I \otimes PAP^{-1} - QBQ^{-1} \otimes I$$

The eigenvalues of  $I \otimes A - B \otimes I$  are exactly the difference of  $A$ 's and  $B$ 's (with multiplicities).

► 翻译: 给两个方阵  $A, B$ , 证明矩阵方程  $AX = XB$  只有零解当且仅当  $A, B$  无公共特征值.

**Exercise 64** In this problem, we will deal with Hermitian matrices. Let  $A$  be an Hermitian<sup>5</sup> matrix, that is,  $AA^H = A^H A$ .

► 翻译: 在这个问题中, 我们将要处理 Hermite 矩阵. Assume the eigenvalues of  $A$  are  $\lambda_1, \dots, \lambda_n$ . 令  $A$  是一个 Hermite 矩阵, 即  $AA^H = A^H A$ . 假设  $A$  的特征值是  $\lambda_1, \dots, \lambda_n$ .

(1) Show that there exists unitary matrix  $U$  such that  $UAU^H$  is diagonal.

Hint Using the same method in exercise 22.

► 翻译: (1) 证明存在酉矩阵  $U$  使得  $UAU^H$  是对角矩阵.

(2) For any continuous function  $f$ , there exists a unique matrix  $f(A)$  whose eigenvalues are  $f(\lambda_1), \dots, f(\lambda_n)$  and  $Av = \lambda_i v \Rightarrow f(A)v = f(\lambda_i)v$ .

---

<sup>5</sup>the "H" is silent.

► 翻译: (2) 对任何连续函数  $f$ , 存在唯一的矩阵  $f(A)$  其特征值是  $f(\lambda_1), \dots, f(\lambda_n)$ , 且  $BA = AB \Rightarrow Bf(A) = f(A)B$ .

(3) Assume the sequence  $\{f_n\}$  converges to  $f$  uniformly in some neighborhood of  $\lambda_1, \dots, \lambda_n$ . Show that  $\lim_{n \rightarrow \infty} f_n(A) = f(A)$ .

► 翻译: 假设序列  $\{f_n\}$  一致收敛到  $f$  在  $\lambda_1, \dots, \lambda_n$  的某个邻域内. 证明  $\lim_{n \rightarrow \infty} f_n(A) = f(A)$ .

**Problem 65** Given a matrix  $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ , we can define the 2-norm as follow

$$\|A\|_2 = \sup_{x \in \mathbb{R}^n} \frac{|Ax|}{|x|}$$

where  $|(x_1, \dots, x_n \text{ or } m)| = \sqrt{|x_1|^2 + \dots + |x_n \text{ or } m|^2}$ .

► 翻译: 给一矩阵  $A \in \mathbb{M}_{n \times m}(\mathbb{C})$ , 我们可以定义其 2-范数如下

$$\|A\|_2 = \sup_{x \in \mathbb{C}^n} \frac{|Ax|}{|x|}$$

其中  $|(x_1, \dots, x_n \text{ or } m)| = \sqrt{|x_1|^2 + \dots + |x_n \text{ or } m|^2}$ .

(1) Show that

$$\begin{aligned} \|A\|_2 &= \sup_{|x|=1} |Ax| \\ &= \sup_{|x| \leq 1} |Ax| \\ &= \inf\{c \in \mathbb{R}_{>0} : |Ax| \leq c|x|\} \\ &= \inf\{c \in \mathbb{R}_{>0} : |Ax| < c|x|\} \end{aligned}$$

As a result, by analysis, the supremum achieves.

► 翻译: (1) 证明

$$\begin{aligned} \|A\|_2 &= \sup_{|x|=1} |Ax| \\ &= \sup_{|x| \leq 1} |Ax| \\ &= \inf\{c \in \mathbb{R}_{>0} : |Ax| \leq c|x|\} \\ &= \inf\{c \in \mathbb{R}_{>0} : |Ax| < c|x|\} \end{aligned}$$

于是, 根据分析, 上确界可取到.

(2) If  $U, V$  is orthogonal, show that  $\|UAV\|_2 = \|A\|_2$ .

► 翻译: 如果  $U, V$  是正交阵, 证明  $\|UAV\|_2 = \|A\|_2$ .

### **Exercise 66 (Singular value decomposition and polar decomposition)**

Given a matrix  $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ , we will show there exist orthogonal matrices  $P, Q$  such that

$$A = P \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \end{pmatrix} Q$$

The elements lie in diagonal  $\sigma_1, \dots, \sigma_n$  are unique counting with the multiplicities, and are called the singular values of  $A$ . The above decomposition is called singular value decomposition.

► 翻译: (奇异值分解和极分解) 给一个矩阵  $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ , 我们将要证明存在正交矩阵  $P, Q$  使得

$$A = P \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \end{pmatrix} Q$$

且, 对角线上的元素  $\sigma_1, \dots, \sigma_n$  是按重数唯一的, 这被叫做  $A$  的奇异值. 上述分解被称为奇异值分解.

(1) Show the uniqueness. Hint Note that  $AA^\top$  is self-adjoint and semipositive-defined.

► 翻译: 证明唯一性.

(2) If  $n = m$  and  $A$  is invertible, show that the decomposition exists.

Hint Use the cardinal form of self-adjoint matrix.

► 翻译: (1) 如果  $n = m$  且  $A$  可逆, 证明分解存在.

(3) Prove the decomposition exists for arbitrary matrix. Hint Wlog assume  $m \geq n$ . Assume  $P$  such that  $AA^\top = P \text{diag}(\sigma_1^2, \dots, \sigma_n^2) P^\top$ . Now, it suffices to find a orthogonal matrix  $Q = (q_1, \dots, q_m)^\top$  such that

$$P^\top A = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & 0 \end{pmatrix} Q = (\sigma_1 q_1, \dots, \sigma_n q_n, 0)^\top$$

Actually, it depends all  $q_i$  with  $\sigma_i \neq 0$ . The other  $q_i$  is derived from basis expansion.

► 翻译: (2) 证明这一分解对所有矩阵都成立.

(4) Point what is  $\|A\|_2$  defined in Exercise 65.

► 翻译: (4) 指出习题65中的  $\|A\|_2$  是什么?

(5) Using SVD deduce the polar decomposition that any square matrix  $A$  can be written as  $PC$  with  $P$  orthogonal and  $C$  semi-positive-defined. Explain why it is called 'polar decomposition'? Hint  $z = e^{i\theta} \rho$ .

► 翻译: (5) 利用 SVD 得到极分解, 即任何一个方矩  $A$  可以写成  $PC$  其中  $P$  正交且  $C$  半正定. 并且解释为何这被称为“极分解”?

**Exercise 67** We will concern on the topological properties of  $GL_n(\mathbb{R})$  and  $GL_m(\mathbb{C})$  encore.

► 翻译: 我们要再次关注  $GL_n(\mathbb{R})$  和  $GL_n(\mathbb{C})$  的拓扑性质.

(1) Show that  $GL_n(\mathbb{C})$  is connected. Hint For two matrix  $A, B$ , consider the 'complex line' (which is a 'real plane') going through them, say  $\{(1-z)A + zB : z \in \mathbb{C}\}$ . Note that only finite many matrices of them not in  $GL_m(\mathbb{C})$ . What is nearly trivial that  $\mathbb{C}$  is connected after delating finite many points.

► 翻译: (1) 证明  $GL_n(\mathbb{C})$  是连通的.

(2) Show that  $GL_n(\mathbb{R})$  is not connected. Hint Find two matrices  $A, B$  such that  $\det A < 0 < \det B$ .

► 翻译: (2) 证明  $GL_n(\mathbb{R})$  不是连通的.

(3) Show that  $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A > 0\}$  is connected. Hint  
Prove that elementary matrices can connected with  $I$  or  $\text{diag}(-1, I)$ .

► 翻译: (3) 证明  $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A > 0\}$  是连通的.

(4) Show that  $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$  is connected. Hint  
It suffices to show any  $A$  can be connected with  $I$  by path. Using polar decomposition, it reduces to two cases

$A$  is orthogonal       $A$  is self-adjoint and semipositive-defined

Then use the norm form of two types connecting  $A$  with  $I$ . Of course, the usage of SVD is also eligible.

► 翻译: (4) 证明  $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$  是连通的.

(5) Show that  $SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : \det A = 1\}$  is connected. Hint

Using the multiplicative version of Jordan decomposition in Exercise 38, and follow the proof above. The surjectivity of  $\exp$  proved in Exercise 57 reduce the problem to prove

$$\exp^{-1}(SL_n(\mathbb{C})) = \{A \in M_n(\mathbb{C}) : \operatorname{tr} A = 0\} = \mathfrak{sl}_n(\mathbb{C})$$

which is a linear space.

► 翻译: (5) 证明  $SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : \det A = 1\}$  是连通的.

## 10 Week10 (22 Apr - 28 Apr)

► 翻译: 第十周 (4月22日 - 4月28日)

**Exercise 68** Given a matrix  $A \in M_n(\mathbb{C})$ , denote

$$\operatorname{ad}_A : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \quad B \longmapsto [A, B] = AB - BA$$

be the adjoint map.

► 翻译: 给一个矩阵  $A \in M_n(\mathbb{C})$ , 记

$$\operatorname{ad}_A : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \quad B \longmapsto [A, B] = AB - BA$$

是伴随映射.

Show that

- (1)  $A$  is diagonalizable  $\Rightarrow$  diagonalizable is  $\operatorname{ad}_A$
- (2)  $A$  is nilpotent  $\Rightarrow$  nilpotent is  $\operatorname{ad}_A$
- (3)  $A$  is diagonalizable  $\Leftrightarrow$  diagonalizable is  $\operatorname{ad}_A$
- (4)  $A$  is nilpotent  $\Leftrightarrow$  nilpotent is  $\operatorname{ad}_A$

Hint (1) is just compute under standard basis  $E_{ij}$ . (2) Note that

$$\operatorname{ad} A = (\text{produce } A \text{ by left}) - (\text{produce } A \text{ at right})$$

but they are commutable and nilpotent. To show (3) and (4), using the fact  $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$  show that  $AB = BA$  implies  $\text{ad } A \text{ ad } B = \text{ad } B \text{ ad } A$  and the uniqueness of Jordan decomposition in Exercise 37.

► 翻译: 证明

- (1)  $A$  是可对角化的  $\Rightarrow$  可对交化的是  $\text{ad}_A$
- (2)  $A$  是幂零的  $\Rightarrow$  幂零的是  $\text{ad}_A$
- (3)  $A$  是可对角化的  $\Leftrightarrow$  可对角化的是  $\text{ad}_A$
- (4)  $A$  是幂零的  $\Leftrightarrow$  幂零的是  $\text{ad}_A$

**Exercise 69** Given a bilinear function  $f(-, -)$  over some  $\mathbb{R}$ -linear space  $V$ . Show that

$$\forall x \in V, f(x, x) = 0 \iff \forall x, y \in V, f(x, y) = -f(y, x)$$

► 翻译: 给一个实线性空间  $V$  上的双线性函数  $f(-, -)$ . 证明

$$\forall x \in V, f(x, x) = 0 \iff \forall x, y \in V, f(x, y) = -f(y, x)$$

**Problem 70 (Pfaffian)** We will introduce Pfaffian for anti-symmetric matrices.

► 翻译: (Pfaff) 我们要对反对称矩阵引入 Pfaff.

Let  $X = \{x_{ij} : 1 \leq i < j \leq n\}$  be a set of indeterminants with  $i, j$  integers. We take  $x_{ii} = 0$  and  $x_{ji} = x_{ij}$  for convenience.

► 翻译: (1) 令  $X = \{x_{ij} : 1 \leq i < j \leq n\}$  是一套不定元, 其中  $i, j$  是整数. 我们方便起见视  $x_{ii} = 0, x_{ji} = x_{ij}$ .

(1) Show that  $\det(x_{ij}) = (f/g)^2$  for some polynomials  $f, g$  in  $X$ . Hint

In the field  $\{f/g\}$ , we can using the normal form of congruence deduce the result.

► 翻译: (1) 证明  $\det(x_{ij}) = (f/g)^2$  其中  $f, g$  是  $X$  的多项式.

(2) Prove that the  $g$  in (1) can be taken as 1, and the coefficients of  $f$  are all integers. Hint Compare it with the fact  $\sqrt{n} \in \mathbb{Q} \iff \sqrt{n} \in \mathbb{Z}$ .

► 翻译: (2) 证明 (1) 中的  $g$  可以取作 (1), 且  $f$  的系数都是整数.

Now, we fix a specific choice of  $f$ , and for anti-symmetric matrix  $A = (a_{ij})$  denote  $\text{Pfaff}A = f(a_{ij})$ . Clearly,  $(\text{Pfaff}A)^2 = \det A$ .

► 翻译: 现在, 我们固定一个特定的  $f$  的选择, 并且对反对称矩阵  $A = (a_{ij})$  记  $\text{Pfaff}A = f(a_{ij})$ . 显然,  $(\text{Pfaff}A)^2 = \det A$ .

(3) Let  $A$  be an anti-symmetric matrix, show that  $\text{Pfaff}(B^T A B) = \det B \cdot \text{Pfaff}A$ . Hint Note that  $f^2 = g^2$  implies  $f = \pm g$  (since  $f, g$  is polynomials rather than just functions). Take  $B = I$ .

► 翻译: (3) 令  $A$  是反对称矩阵, 证明  $\text{Pfaff}(B^T A B) = \det B \cdot \text{Pfaff}A$ .

(4) Let  $S$  be a symplectic matrix, that is, a matrix such that

$$S^T \begin{pmatrix} & I \\ -I & \end{pmatrix} S = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

Show that  $\det S = 1$ .

► 翻译: (4) 令  $S$  是一个辛矩阵, 即一个使得

$$S^T \begin{pmatrix} & I \\ -I & \end{pmatrix} S = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

的矩阵, 证明  $\det S = 1$ .

(5) Prove that the determinants of anti-symmetric matrices with integer coefficients are perfect squares. Hint Utilizing Pfaffian or repeat the process of (1) and (2).

► 翻译: (5) 证明反对称整系数矩阵的行列式是完全平方数.

**Exercise 71** Given a bilinear function  $f(-, -)$  over some  $\mathbb{R}$ -linear space  $V$ . Show that if

$$f(x, y) = 0 \iff f(y, x) = 0$$

then  $f$  is symmetric or anti-symmetric. Hint For any  $x, y$ , consider the process of ‘orthogonalization’  $x, y \mapsto x, y - \frac{f(x, y)}{f(x, x)}x$ . To avoid disturbing for

the denominator, we use  $x, f(x, x)y - f(x, y)x$  instead. Then  $f(x, f(x, x)y - f(x, y)x) - f(x, y)x = 0$  implies

$$f(f(x, x)y - f(x, y)x, x) = f(x, x)[f(y, x) - f(x, y)] = 0$$

This means

$$\forall x, y \in V, \quad (f(x, x) = 0) \wedge (f(x, y) = f(y, x)) \quad (*)$$

What we need to show is

$$(\forall x \in V, f(x, x) = 0) \wedge (\forall x, y \in V, (f(x, y) = f(y, x))) \quad (**)$$

Let

$$A = \{x \in V, f(x, x) = 0\} \quad B = \{x \in V : \forall y \in V, f(x, y) = 0\}$$

Note that

$$(*) \Leftrightarrow V = A \cup B \quad (***) \Leftrightarrow (V = A \text{ or } V = B)$$

If (\*\*\*) does not hold, pick  $a \in A \setminus B, b \in B \setminus A$ , then we shall consider the ‘reproduction’ of them, that is,  $a + \lambda b$ .

- If  $a + \lambda b \in A$ , that is,

$$\begin{aligned} 0 &= f(a + \lambda b, a + \lambda b) && \because a + \lambda b \in A \\ &= f(a, a) + \lambda f(a, b) + \lambda f(b, a) + \lambda^2 f(b, b) \\ &= 0 + 2\lambda f(a, b) + \lambda^2 f(b, b) && \because a \in A, b \in B \end{aligned}$$

The assumption  $b \notin A$  indicates  $f(b, b) \neq 0$ . As a result, at most one  $\lambda \neq 0$  such the above equality holds.

- If  $a + \lambda b \in B$ , that is,

$$\begin{aligned} 0 &= f(a + \lambda b, x) - f(x, a + \lambda b) && \because a + \lambda b \in B \\ &= f(a, x) - f(x, a) + \lambda[f(b, x) - f(x, b)] \\ &= f(a, x) - f(x, a) && b \in B \end{aligned}$$

But  $a \notin B$  means  $f(a, x) \neq f(x, a)$  for some  $x$ .

Combine the discussion above, we prove (\*\*).

► 翻译: 给一个实线性空间  $V$  上的双线性函数  $f(-, -)$ . 证明如果

$$f(x, y) = 0 \iff f(y, x) = 0$$

那么  $f$  是对称的或者反对称的.

**Problem 72 (Maximal modulus principle)** *In this problem, we will prove the famous ‘maximal modulus principle for polynomials. Let  $F(z)$  be a polynomial in  $z$ . We will show that for any  $z_0 \in \mathbb{C}$ ,  $r > 0$ , there exists some  $z$  with  $|z - z_0| = r$  such that*

$$|f(z_0)| < |f(z)|$$

*unless  $f$  is a constant.*

► 翻译: (最大模原理) 在这个问题里, 我们将要对多项式证明著名的“最大模”原理. 令  $F(z)$  是  $z$  的多项式. 我们将要证明对任意  $z_0 \in \mathbb{C}$ ,  $r > 0$ , 存在某个  $z$  使得  $|z - z_0| = r$  使得

$$|f(z_0)| < |f(z)|$$

除非  $f$  是一个常数.

(1) *Firstly, show the following orthogonal relation*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \overline{e^{im\theta}} d\theta = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

*where  $n, m$  are integers.*

► 翻译: 首先证明正交关系

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \overline{e^{im\theta}} d\theta = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

其中  $n, m$  是整数.

(2) Secondly, using the substitution  $z = z_0 + re^{i\theta}$  show the argument.

Hint Let  $F(z) = a_0 + a_1z + \dots + a_nz^n$ . Then

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) \overline{F(e^{i\theta})} d\theta = |a_0|^2 + r^2|a_1|^2 + \dots + r^{2n}|a_n|^2$$

Then it is impossible that  $F(z) < |a_0|$  for all  $z$  with  $|z| = r$  unless  $F$  is a constant.

► 翻译: 其次利用替换  $z = e^{i\theta}$  证明  $z_0 = 0, r = 1$  的情形.

(3) Let  $w_1, \dots, w_n$  be unit vectors on a plane, show that there are some unit vector  $z$  such that

$$|z - w_1| \cdot \dots \cdot |z - w_n| > 1$$

Hint Consider  $F(z) = (z - w_1) \dots (z - w_n)$ .

► 翻译: 令  $w_1, \dots, w_n$  是平面上的单位向量, 证明存在某个单位向量  $z$  使得

$$|z - w_1| \cdot \dots \cdot |z - w_n| > 1$$

**Problem 73 (Fisher's inequality)** Let  $k > 0$  and  $A_1, \dots, A_m$  be subsets of  $\{1, \dots, n\}$ . If  $|A_i \cap A_j| = k$  for any  $i \neq j$ , show that  $m \leq n$ . Hint Let  $e_i$  be characteristic function of  $A_i$ . The relation becomes  $\langle e_i, e_j \rangle = k$ . We can actually show that  $e_i$ 's are linearly independent. If  $L = \sum_{i \in X} \lambda_i e_i = \sum_{j \in Y} \mu_j e_j = R$ , with  $\lambda_i, \mu_j > 0$  and  $X \cap Y = \emptyset$ . Then consider  $\langle e_i, L \rangle = \langle e_i, R \rangle$  and  $\langle e_j, L \rangle = \langle e_j, R \rangle$ .

► 翻译: (Fisher 不等式) 令  $k > 0, A_1, \dots, A_m$  是  $\{1, \dots, n\}$  的子集. 如果  $|A_i \cap A_j| = k$  对任意  $i \neq j$ , 证明  $m \geq n$ .

## 11 Week11 (29 Apr - 5 May)

► 翻译: 第十一周 (4月29日 - 5月5日)

**Exercise 74** For two complex matrices  $A, B$ , if  $[A, [A, B]] = 0$ , show that  $[A, B]$  is nilpotent. Hint  $[A, B]^{n-1}[A, B] = [A, B]^{n-1}AB - [A, B]^{n-1}BA = A([A, B]^{n-1}B) - ([A, B]^{n-1}B)A$ .

► 翻译: 对两个复矩阵  $A, B$ , 如果  $[A, [A, B]] = 0$ , 证明  $[A, B]$  幂零.

**Exercise 75** For two real symmetric matrices  $A, B$ , if  $A$  is positive-defined, show that exists some invertible  $P$  such that  $P^TAP$  and  $P^TBP$  are diagonal simultaneously. Hint WLOG assume  $A = I$ .

► 翻译: 对于两个对称矩阵  $A, B$ , 如果  $A$  是正定的, 证明存在可逆矩阵  $P$  使得  $P^TAP$  和  $P^TBP$  同时是对角矩阵.

**Problem 76** Given two square matrices  $A, B$ , if  $AB = BA$ , show that there exists invertible  $P$  such that  $PAP^{-1}$  and  $PBP^{-1}$  are upper matrices. Hint Show they share a common eigenvector. Note that the eigenspace of  $A$  is  $B$ -invariant.

► 翻译: 给两个矩阵  $A, B$ , 如果  $AB = BA$ , 证明存在可逆矩阵  $P$  使得  $PAP^{-1}$  和  $PBP^{-1}$  都是上三角矩阵.

Can you generalize this conclusion to the case of  $k$ ?

► 翻译: 你能将之推广到  $k$  的情况吗?

**Problem 77** Given two real square matrices  $A, B$ , if  $AB = BA$ , show that  $\det(A^2 + B^2) \geq 0$ . Hint  $\det(A^2 + B^2) = \det(A + iB) \det(A - iB) = \det(A + iB) \overline{\det(A - iB)} \geq 0$ .

► 翻译: 给两个矩阵  $A, B$ , 如果  $AB = BA$ , 证明  $\det(A^2 + B^2) \geq 0$ .

**Fuxercise 78** Generalize Exercise 77 to the case of  $k$ . Hint Let

$$f(a_1, \dots, a_k) = a_1^2 + \dots + a_k^2$$

Let  $A_1, \dots, A_k$  be commutable matrices. Assume the eigenvalues of  $x_1A_1 + \dots + x_kA_k$ , is

$$\Lambda_1 = x_1\lambda_{11} + \dots + x_{k1}\lambda_{k1} \quad \dots \quad \Lambda_n = x_1\lambda_{1n} + \dots + x_{k1}\lambda_{kn}$$

Clearly, by the argument of simultaneous upperization,

$$\det f(A_1, \dots, A_k) = \prod_{i=1}^n f(\lambda_{1i}, \dots, \lambda_{ki})$$

For each  $i$  (if we assume  $x_\bullet \in \mathbb{R}$ ),

- If  $\Lambda_i \in \mathbb{R}$  for any choices of  $x_\bullet$ ,  $\lambda_{1i}, \dots, \lambda_{ki} \in \mathbb{R}$ . As a result,

$$f(\lambda_{1i}, \dots, \lambda_{ki}) \geq 0$$

- If some  $x_\bullet$  such that  $\Lambda_i \notin \mathbb{R}$ , then

$$\overline{\Lambda_i} = x_1 \overline{\lambda_{1i}} + \dots + x_{k1} \overline{\lambda_{ki}} = x_1 \lambda_{1i'} + \dots + x_k \lambda_{ki'} = \Lambda_{i'}$$

for some  $i \neq i'$ . Note that  $\{x_\bullet : \Lambda_i \notin \mathbb{R}\}$  is a nonempty open subset, thus, the above equality implies

$$\lambda_{1i} = \overline{\lambda_{1i'}} \quad \dots \quad \lambda_{ki} = \overline{\lambda_{ki'}}$$

As a consequence,

$$f(\lambda_{1i'}, \dots, \lambda_{ki'}) = \overline{f(\lambda_{1i}, \dots, \lambda_{ki})}$$

Of course, the technique of specific choice of  $x_\bullet$  can be avoided by using more algebra definitions.

By the above discussion,  $\det f(A_1, \dots, A_k) \geq 0$ . Actually, we proved that all the positive polynomials serve.

► 翻译: 将习题77推广到  $k$  的情况.

**Problem 79** Let  $v_1, \dots, v_n, w_1, \dots, w_n \in V$ . Assume that for every  $\mathbb{R}$ -linear map  $f : V \rightarrow \mathbb{R}$ ,  $(f(v_1), \dots, f(v_n))$  and  $(f(w_1), \dots, f(w_n))$  coincide up to permutation of the indices. Deduce that  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_n)$  coincide up to permutation of the indices. Hint For any bijection  $\{1, \dots, n\} \xrightarrow{\sigma} \{1, \dots, n\}$ , the linear functions  $f$  such that  $f(v_i) = f(w_{\sigma(i)})$  form a linear subspace, then use subspace avoidance in exercise 6.

► 翻译: 令  $v_1, \dots, v_n, w_1, \dots, w_n \in V$ . 假设对所有  $\mathbb{R}$ -线性映射  $f: V \rightarrow K$ ,  $(f(v_1), \dots, f(v_n))$  和  $(f(w_1), \dots, f(w_n))$  不计顺序意义下相同, 证明  $(v_1, \dots, v_n)$  和  $(w_1, \dots, w_n)$  不计顺序意义下相同.

**Problem 80** Describe the connected components of

$$\{A \in M_n(\mathbb{C}) : A^m = I\}$$

Hint Two matrices is connected iff they are similar. If  $B = PAP^{-1}$ , then take some path in  $GL_n(\mathbb{C})$  connected  $P$  and  $I$ . To show the converse, note that the characteristic polynomials of them are only of finite choices and determine the similar class (for diagonalizable matrices).

► 翻译: 描述下述集合连通的连通分支

$$\{A \in M_n(\mathbb{C}) : A^m = I\}$$

**Exercise 81** Let  $A, B$  be two matrices of size  $n \times m$  and  $m \times n$  respectively.

Hint 令  $A, B$  是尺寸分别为  $n \times m$  和  $m \times n$  的矩阵.

(1) Show that

$$\lambda^m \det(\lambda I - AB) = \lambda^n \det(I - BA)$$

In particular, the eigenvalues of  $AB$  and  $BA$  coincide except the number of zeros. Hint Note that

$$\det \begin{pmatrix} \lambda I_n - AB & \\ & \lambda I_m \end{pmatrix} = \det \begin{pmatrix} \lambda I_n & A \\ B & \lambda I_m \end{pmatrix} = \det \begin{pmatrix} \lambda I & A \\ & \lambda I_m - BA \end{pmatrix}$$

► 翻译: 证明

$$\lambda^m \det(\lambda I - AB) = \lambda^n \det(I - BA)$$

特别地,  $AB$  和  $BA$  的特征值除了零的数目都是一样的.

(2) Let  $g$  be the minimal polynomial of  $AB$  and  $h$  be the counterpart of  $BA$ , show that either

$$xh(x) = g(x) \quad h(x) = g(x) \quad \text{or} \quad h(x) = xg(x)$$

Hint If  $g(AB) = 0$ , then  $Bg(AB)A = g(BA)BA = 0$ .

► 翻译: 令  $g$  是  $AB$  的最小多项式,  $h$  是  $BA$  的, 证明下列情况必居之一

$$xh(x) = g(x) \quad h(x) = g(x) \quad h(x) = xg(x)$$

Last but not least, no matter whether feel frustrated or not, just enjoy it, because it is math!

► 翻译: 最后, 不论是否感到挫败, 享受这些, 因为这些是数学!

Here are the exam paper and the answer of middle term in 2019.

## 高等代数 期中测试

2019 年 4 月 24 日

**问题 1 (15 分).** 对于固定的  $n$ , 证明

$$\{X \in \mathbb{M}_n(\mathbb{R}) : \operatorname{tr} X = 0\} = \operatorname{span}\{AB - BA : A, B \in \mathbb{M}_n(\mathbb{R})\}$$

**问题 2 (15 分).** 如果两个实方阵  $A, B$  作为复矩阵相似, 证明他们作为实矩阵也相似.

**问题 3 (20 分).** 如果两个复方阵  $A, B$  满足  $AB = BA$ , 证明

$$A, B \text{ 分别可对角化} \Rightarrow A, B \text{ 可以同时对角化}$$

**问题 4.** 对于有限维复线性空间  $V$  上的线性变换  $\mathcal{A}$ , 我们要证明存在  $v \in V$  使得  $\{f(\mathcal{A})v : f \text{ 是多项式}\} = V$  当且仅当  $\mathcal{A}$  的每个特征值的 Jordan 块只有一块.

**(1)(10 分)** 记  $\mathcal{A}$  的所有不同特征值为  $\lambda_1, \dots, \lambda_m$ , 重数分别为  $n_1, \dots, n_m$ , 证明

$$V = \ker(\mathcal{A} - \lambda_1 \mathcal{I})^{n_1} \oplus \dots \oplus \ker(\mathcal{A} - \lambda_m \mathcal{I})^{n_m}$$

**(2)(10 分)** 如果有不变子空间  $W$ , 证明存在  $w \in W$  使得  $\{f(\mathcal{A})w : f \text{ 是多项式}\} = W$ . 提示 带余除法.

**(3)(10 分)** 证明结论.

**问题 5 (20 分).** 对于  $n$  维内积空间  $V$ , 一组向量  $e_1, \dots, e_r$ , 如果他们两两内积取负, 求  $r$  的最大值.

## 高等代数 期中测试解答

问题 1 (15 分). 原题. 习题课讲过. 第八周 Exercise 3. 习题58.

问题 2 (15 分). 原题. 第四周 Problem 6. 习题32.

问题 3 (20 分). 原题. 习题课讲过. 第三周 Problem 7 (8). 习题24.

问题 4. (1)(10 分) 原题. 习题课讲过. 第三周 Exercise 6 (4). 习题23.

问题 4. (2)(10 分) 任意不变子空间  $W$ , 选择次数最小的非零多项式  $f$  使得  $f(\mathcal{A})v \in W$ , 我们证明  $f(\mathcal{A})v$  就是要求的  $w$ . 任意  $x \in W$ , 假设  $x = g(\mathcal{A})v$ , 作带余除法  $g = df + r$ , 于是  $r(\mathcal{A})v = x - d(\mathcal{A})f(\mathcal{A})v$ , 因为  $r$  次数更小, 这迫使  $r = 0$ . 这样  $x = d(\mathcal{A})f(\mathcal{A})v = d(\mathcal{A})w$ .

问题 4. (3)(10 分) 先证明充分性. 按照 (1) 中的分解, 存在  $x_i \in V$  使得  $\ker(\mathcal{A} - \lambda_i \mathcal{I})^{n_i} = \{f(\mathcal{A})x_i\}$ . 将  $f$  在  $\lambda_i$  处展开, 可知空间由  $x_i, (\mathcal{A} - \lambda_i \mathcal{I})x_i, \dots, (\mathcal{A} - \lambda_i \mathcal{I})^{n_i-1}x_i$  张成. 每个  $\ker(\mathcal{A} - \lambda_i \mathcal{I})^{n_i}$  都可选择这样的生成元, 通过维数论证可知其成为一组基, 在这组基下,  $\mathcal{A}$  成为每个特征值只有一块的 Jordan 标准型. 反之, 说明  $\ker(\mathcal{A} - \lambda_i \mathcal{I})^{n_i}$  都是循环子空间, 假设他们由  $x_i, (\mathcal{A} - \lambda_i \mathcal{I})x_i, \dots, (\mathcal{A} - \lambda_i \mathcal{I})^{n_i-1}x_i$  张成. 那么  $x_1 + \dots + x_m$  就是满足条件的  $v$ . 因为 (1) 确保了对每个  $i$ , 都有多项式  $p_i$  使得  $x_i = p_i(\mathcal{A})v$ .

问题 5 (20 分). 如  $e_1, \dots, e_s$  线性相关, 将线性关系按照正负整理得到

$$L := \sum_{i \in A} \lambda_i e_i = \sum_{i \in B} \lambda_i e_i =: R \quad \lambda_i > 0, A \sqcup B \subseteq \{1, \dots, s\}$$

假如  $A, B$  非空, 这样  $0 < \langle L, L \rangle = \langle R, L \rangle < 0$ , 矛盾, 故  $A, B$  之中必有空集. 此时线性关系将形如  $\sum_{i=1}^s \lambda_i e_i = 0$  其中  $\lambda_i \geq 0$ . 这样带入  $s = r - 1$ , 将上式和  $e_r$  作内积得到  $\lambda_i = 0$ . 故实际上,  $e_1, \dots, e_{r-1}$  线性相关, 从而  $r \leq n + 1$ .

为了取到最大值, 可使用归纳法, 任意取  $e_{n+1}$ , 在其正交补空间中取  $e'_1, \dots, e'_n$  两两内积为负, 注意到

$$\langle e'_i - \epsilon e_{n+1}, e'_j - \epsilon e_{n+1} \rangle = \langle e'_i, e'_j \rangle + \epsilon^2 \langle e_{n+1}, e_{n+1} \rangle$$

只需要取  $0 < \epsilon < \sqrt{\max_{i \neq j} |\langle e'_i, e'_j \rangle|}$ .

Here are the exam paper and the answer of middle term in 2018.

## 高等代数 期中测试

2018年5月03日

**问题 1 (10 分).** 对于线性空间  $V$ , 若  $V = A \oplus B = C \oplus D$ , 且  $A \subseteq C$ , 求证:

$$C = A \oplus (B \cap C)$$

**问题 2 (10 分).** 证明: 两个复方阵  $A, B$  相似当且仅当  $\begin{pmatrix} A & \\ & A \end{pmatrix}, \begin{pmatrix} B & \\ & B \end{pmatrix}$  相似.

**问题 3 (10 分, Bessel 不等式).** 对于  $\mathbb{R}$ -内积空间  $V$  (不必有限维), 若有限个单位向量  $e_1, \dots, e_n$  两两正交, 求证, 对任何  $x \in V$ ,

$$\langle x, x \rangle^2 \geq \sum_{i=1}^n |\langle x, e_i \rangle|^2$$

**问题 4 (20 分, Schur 不等式).** 证明: 每个复矩阵  $A$  都酉相似到上三角形. 具体来说, 存在酉矩阵  $U$  (即  $U^H U = E$ ,  $\{\}^H$  表示共轭转置) 和上三角矩阵  $T$  使得

$$A = UTU^{-1}$$

**提示** 回忆“前 Jordan 时代”的我们曾经证明的结论——任何一个矩阵都复相似到上三角矩阵.

**问题 5 (25 分).** 对于  $\mathbb{C}$ -线性空间  $V$ , 其上有两个线性变换  $\mathcal{A}, \mathcal{B}$ , 且  $\mathcal{A}, \mathcal{B}$  可交换. 求证:

$$\mathcal{A}, \mathcal{B} \text{ 可对角化} \iff \mathcal{A}, \mathcal{B} \text{ 可同时对角化}$$

这里可 (同时) 对交换的意思是存在一组基, 在这组基下, (两) 线性变换对应的矩阵 (都) 是对角矩阵. 换言之, 存在一组由 (公共) 特征向量组成的基.

**问题 6(25 分, Jordan 分解).** 对于有限维  $\mathbb{C}$ -线性空间  $V$ ,  $\mathcal{A}$  是其上的线性变换, 本题的目的是为了证明如下的 Jordan 分解存在且唯一,

$$\mathcal{A} = \mathcal{D} + \mathcal{N}$$

其中  $\mathcal{D}, \mathcal{N}$  都是  $V$  上的线性变换, 且满足

- $\mathcal{D}$  可以对角化. 换言之, 存在一组基  $\epsilon_1, \dots, \epsilon_n$  和特征值  $\lambda_1, \dots, \lambda_i$  使得  $\mathcal{D}\epsilon_i = \lambda_i\epsilon_i$ , 对每一个  $1 \leq i \leq n$ . (可对角化条件)
- $\mathcal{N}$  是幂零的. 即存在  $m > 0$  使得  $\mathcal{N}^m = \mathcal{O}$ . (幂零条件)
- $\mathcal{A}$  分别与  $\mathcal{D}, \mathcal{N}$  可交换. 即  $\mathcal{A}\mathcal{D} = \mathcal{D}\mathcal{A}, \mathcal{A}\mathcal{N} = \mathcal{N}\mathcal{A}$ . (可交换条件)

**注意:** 本题如果不采用如下过程的思路, 完整地解决亦可得满分 (只证明存在性可得 5 分).

直觉使我们相信  $\mathcal{D}$  的对角线上应该按重数排列着  $\mathcal{A}$  的所有特征值, 但是这需要构造和证明. 为此, 假设  $\mathcal{A}$  的特征多项式为

$$f(X) = (X - \lambda_1)^{n_1} \dots (X - \lambda_k)^{n_k} \quad \lambda_1, \dots, \lambda_n \in \mathbb{C} \text{ 两两不同}$$

并且记属于特征值  $\lambda_i$  的根子空间

$$V_i = \ker [(\mathcal{A} - \lambda_i)^{n_i}] \subseteq V$$

(1)(5 分) 证明:  $V_i$  是  $\mathcal{A}$ -不变子空间.

(2)(5 分) 证明:

$$V = V_1 \oplus \dots \oplus V_k$$

**提示** 只需证明对两个互质的多项式  $f, g$  有  $V = \ker f(\mathcal{A}) \oplus \ker g(\mathcal{A})$ .

(3)(5 分) 证明: 对任意  $1 \leq j \leq n$ , 存在多项式  $f_j(X)$  使得

$$[f_j(\mathcal{A})](v_1 + \dots + v_k) = v_j \quad \forall v_i \in V_i$$

**提示** 只需要  $[f_j(\mathcal{A})](v_i) = v_i$  当  $i = j$ , 而  $= 0$  当  $i \neq j$ . 用多项式的 Bézout 定理.

既然已经构造出  $f_j(X)$  那么取

$$D(X) = \lambda_1 f_1(X) + \dots + \lambda_k f_k(X) \quad N(X) = X - D(X)$$

并令

$$D = D(\mathcal{A}) \quad N = N(\mathcal{A})$$

(4)(5 分) 证明: 如上的  $D, N$  满足 Jordan 分解的三条要求. 注意到, 此时,  $D, N$  甚至还是  $\mathcal{A}$  的多项式.

(5)(5 分, 唯一性) 若  $D', N'$  满足 Jordan 分解的三条要求, 则

$$D' = D, N' = N$$

**注意:** 在这里你可以使用问题 5 的结论, 即使你没有解决问题 5.

**提示** 需要注意到我们构造出的  $D$  是  $\mathcal{A}$  的多项式, 同时注意到两个可以交换的幂零变换的差还是幂零的.

## 高等代数 期中测试解答

**问题 1 (10 分).** 首先,  $A \cap (B \cap C) \subseteq A \cap B = \{0\}$ , 故  $A \cap B = \{0\}$ . 此外,  $\forall x \in C \subseteq V$ , 则  $x = a + b$ , 其中  $a \in A \subseteq C, b \in B$ , 则  $b = x - a \in C$ , 故  $b \in B \cap C$ , 故  $C = A + (B \cap C)$ . 综上所述  $C = A \oplus (B \cap C)$ .

**问题 2 (10 分).** 必要性显然. 对于充分性, 将  $A, B$  相似到 Jordan 标准型  $J_A, J_B$ , 则  $\begin{pmatrix} A & \\ & A \end{pmatrix}, \begin{pmatrix} B & \\ & B \end{pmatrix}$  分别相似于  $\begin{pmatrix} J_A & \\ & J_A \end{pmatrix}, \begin{pmatrix} J_B & \\ & J_B \end{pmatrix}$ , 后者也是 Jordan 标准型, 若  $\begin{pmatrix} A & \\ & A \end{pmatrix}, \begin{pmatrix} B & \\ & B \end{pmatrix}$  相似, 则  $\begin{pmatrix} J_A & \\ & J_A \end{pmatrix}, \begin{pmatrix} J_B & \\ & J_B \end{pmatrix}$  只相差一些 Jordan 块的排列, 而  $J_A$  和  $J_B$  的 Jordan 块正是这些 Jordan 块的一半, 故  $J_A$  与  $J_B$  相似. (也可使用初等因子法)

**问题 3 (10 分, Bessel 不等式).**  $\langle x, e_i \rangle$  正是  $x$  在  $e_i$  上的投影, 受此启发, 作  $y = x - \sum_{i=1}^n \langle x, e_i \rangle e_i$ , 此时  $y \perp e_i$  对任意  $1 \leq i \leq n$ . 故根据勾股定理

$$\|x\|^2 = \left\| \sum_{i=1}^n \langle x, e_i \rangle e_i \right\|^2 + \|y\|^2 \geq \left\| \sum_{i=1}^n \langle x, e_i \rangle e_i \right\|^2 = \sum_{i=1}^n |\langle x, e_i \rangle|^2$$

得证.

**问题 4 (20 分, Schur 不等式).** 根据代数基本定理,  $A$  总有特征向量, 通过单位化, 不妨假设其是单位向量, 将其扩充为全空间的一组单位正交基, 在这组基下  $A$  的矩阵形如

$$\begin{pmatrix} \lambda & * \\ 0 & A' \end{pmatrix}$$

即  $A$  酉相似到  $\begin{pmatrix} \lambda & * \\ 0 & A' \end{pmatrix}$ , 然后利用归纳法得证. (或利用任何矩阵都相似到上三角矩阵 (Jordan 标准型), 以及 QR 分解.)

选定一组基, 转述为矩阵的条件, 假设  $A, B$  在这组基下的矩阵是  $A, B$ ,

不妨通过选择恰当的基, 假定  $A = \begin{pmatrix} \lambda_1 E_1 & & \\ & \ddots & \\ & & \lambda_k E_k \end{pmatrix}$ , 其中  $\lambda_i$  互不相

同, 对  $B$  施以同样的分块  $B = \begin{pmatrix} B_{11} & \dots & B_{1k} \\ \vdots & \ddots & \vdots \\ B_{k1} & \dots & B_{kk} \end{pmatrix}$ , 验证  $AB = BA$  得到

$\lambda_i B_{ij} = \lambda_j B_{ij}$ , 因为假定  $\lambda_i$  两两不同, 故  $B_{ij} = O$  当  $i \neq j$ . 故  $B = \begin{pmatrix} B_{11} & & \\ & \ddots & \\ & & B_{kk} \end{pmatrix}$ , 则因为  $B$  可对角化,  $B_{ii}$  也可对角化, 设  $P_i$  使得  $P_i B_{ii} P_i^{-1}$

为对角阵, 则  $P = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{pmatrix}$ . 则  $PAP^{-1} = A$ , 且  $PbP^{-1}$  为对角阵.

问题 6(25 分, Jordan 分解). (1) 任意  $x \in V_i$ ,

$$(\mathcal{A} - \lambda_i)^{n_i}(\mathcal{A}x) = \mathcal{A}(\mathcal{A} - \lambda_i)^{n_i}x = \mathcal{A}0 = 0$$

(2) 首先, 先证明  $f, g$  互质, 且  $fg(\mathcal{A}) = \mathcal{O}$  时,  $V \cong \ker[f(\mathcal{A})] \oplus \ker[g(\mathcal{A})]$ . 根据多项式的 Bézout 定理, 存在  $u, v$  使得  $fu + gv = 1$ , 则  $\forall x \in V$ ,  $x = u(\mathcal{A})f(\mathcal{A})x + v(\mathcal{A})g(\mathcal{A})x$ , 且  $g(\mathcal{A})u(\mathcal{A})f(\mathcal{A})x = 0, f(\mathcal{A})v(\mathcal{A})g(\mathcal{A})x = 0$ , 故  $V = \ker[f(\mathcal{A})] + \ker[g(\mathcal{A})]$ . 若  $x \in \ker[f(\mathcal{A})] \cap \ker[g(\mathcal{A})]$ , 则  $f(\mathcal{A})x = 0, g(\mathcal{A})x = 0$ , 从而  $x = u(\mathcal{A})f(\mathcal{A})x + v(\mathcal{A})g(\mathcal{A})x = 0$ , 故是直和.

然后, 对于这里, 用于  $(X - \lambda_1)^{n_1}$  和  $f/(X - \lambda_1)^{n_1}$ , 然后在子空间  $\ker[f/(X - \lambda_1)^{n_1}]$  上归纳可得.

(也可直接用  $\frac{f(X)}{(X - \lambda_i)^{n_i}}$ , 方法是类似的, 但是要注意多个直和的充分必要条件不是两两交 0.)

(3) 考虑  $F_i = \frac{f(X)}{(X - \lambda_i)^{n_i}}$ , 他们互质, 利用 Bézout 定理, 存在  $u_i$  使得  $\sum_{i=1}^k u_i F_i = 1$ , 取  $f_i = u_i F_i$ . 注意到  $f_i(v_j) = 0$  当  $i \neq j$  时, 故容易验证  $f_i(v_j) = v_i$  当  $i = j$ , 而  $= 0$  当  $i \neq j$ .

(4) 每个  $i$  选  $V_i$  的一组基, 他们的并还是一组基 (因为直和). 且  $\mathcal{D}$  在每个  $V_i$  上都是数乘, 从而可对角化. 幂零是因为此时特征值全为 0, 特征多项式必须是  $X^n$ . 可以交换是因为是多项式.

(5) 假设  $\mathcal{A} = \mathcal{D}' + \mathcal{N}'$ , 且  $\mathcal{A}$  与  $\mathcal{D}', \mathcal{N}'$  交换, 则因为  $\mathcal{D}, \mathcal{N}$  是  $\mathcal{A}$  的多项式,  $\mathcal{D}, \mathcal{N}$  也与  $\mathcal{D}', \mathcal{N}'$  交换. 则  $\mathcal{D} - \mathcal{D}' = \mathcal{N}' - \mathcal{N}$ . 假设  $\mathcal{N}'^m = \mathcal{O}, \mathcal{N}^n = \mathcal{O}$ , 因为交换, 所以利用二项式展开可得  $(\mathcal{N}' - \mathcal{N})^{m+n} = \mathcal{O}$ , 故也幂零. 而  $\mathcal{D}, \mathcal{D}'$  因为可交换且可以对角化可得可以同时对角化, 因为幂零矩阵特征值都为 0, 而在某组基下  $\mathcal{D} - \mathcal{D}'$  是对角阵, 从而  $\mathcal{D} = \mathcal{D}'$  进而  $\mathcal{N} = \mathcal{N}'$ .