Applied Mathematics and Computation 218 (2012) 10755–10758

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Regularity criteria for the 3D magneto-micropolar fluid equations in Besov spaces with negative indices

Congchong Guo [a], Zujin Zhang [b,*], Jialin Wang [b]

[a] Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, PR China
[b] School of Mathematics and Computer Science, Gannan Normal University, Ganzhou, 341000 Jiangxi, PR China

ARTICLE INFO

Keywords:
Magneto micropolar fluid equations
Regularity criteria
Besov spaces

ABSTRACT

We consider the Cauchy problem of the magneto-micropolar fluid equations in three space dimensions. It is proved that if the velocity, magnetic field and the micro-rotational velocity belong to some critical Besov space with negative indices, then the solution is in fact smooth.

1. Introduction

We consider the magneto-microploar fluid (MMF) equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + u\cdot\nabla u - (\mu+\chi)\Delta u - b\cdot\nabla b + \nabla(p+b^2) - \chi\nabla\times\omega = 0, \\ \partial_t\omega - \gamma\Delta\omega - \kappa\nabla\mathrm{div}\omega + 2\chi\omega + u\cdot\nabla\omega - \chi\nabla\times u = 0, \\ \partial_t b - \nu\Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad \omega(0,x) = \omega_0(x), \quad b(0,x) = b_0(x). \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ represents the velocity field, $b = b(x,t)$ represents the magnetic field, $\omega = \omega(x,t)$ represents the micro-rotational velocity; p denotes the hydrodynamic pressure; $\mu > 0$ is the kinematic viscosity, $\chi > 0$ is the vortex viscosity, $\kappa > 0$ and $\gamma > 0$ are spin viscosities, $1/\nu$ (with $\nu > 0$) is the magnetic Reynold; while u_0, b_0, ω_0 are the corresponding initial data with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$.

System (1.1) was first proposed by Galdi and Rionero [5]. The existence of global (in time) weak solutions were established by Rojas-Medar and Boldrini [12], while the local strong solutions and global strong solutions for the small initial data were considered, respectively, by Rojas-Medar [11] and Ortega-Torres and Rojas-Medar [13]. However, whether the weak solution is regular or the unique strong solution can exist globally is unknown. Thus there are a lot of literatures devoted to find sufficient conditions to ensure smoothness, see [2,8,9,14,18] for the Navier–Stokes equations ($\omega = b = 0$ in (1.1)), and [6,19] for the MHD equations ($\omega = 0$ in (1.1)).

Very recently, Gala [4] and Zhang et al. [17] considered system (1.1) and showed that if u or ∇u belongs to some critical Besov space, then the solution is actually regular. Our motivation is then to lower the regularity of u to ensure smoothness also, but as a compensation, we need ω and b have some (also rough) regularity. Our result seems to be more helpful in the regularity theory of system (1.1) since the smoothness of u, ω and b are always the same.

The main result now reads:

* Corresponding author.
E-mail addresses: guocongchong77@163.com (C. Guo), uia.china@gmail.com (Z. Zhang), jialinwang1025@hotmail.com (J. Wang).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2012.04.068

Theorem 1.1. *Let $u_0, \omega_0, b_0 \in \dot{H}^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, and the triple (u, ω, b) be the strong solution on $(0, T)$ of system (1.1) with initial data (u_0, ω_0, b_0). If additionally,*

$$u, \omega, b \in L^{\frac{2}{1-\alpha}}(0, T; \dot{B}^{-\alpha}_{\infty,\infty}), \quad 0 < \alpha < 1, \tag{1.2}$$

then the solution (u, ω, b) can be extended smoothly beyond $t = T$.

Remark 1.1. Checking the proof of Bernstein Lemma (see [3]), it follows that the Riesz transform $R_j (1 \leqslant j \leqslant 3)$ is bounded in $\dot{B}^s_{p,q}$ for all $s \in \mathbb{R},\ 1 \leqslant p,\ q \leqslant \infty$. Thus by Theorem 1.1, we have the condition

$$\nabla \times u, \quad \nabla \times \omega, \quad \nabla \times b \in L^{\frac{2}{1-\alpha}}(0, T; \dot{B}^{-1-\alpha}_{\infty,\infty}), \quad 0 < \alpha < 1$$

is enough to ensure the smoothness. This is a Beal–Kato–Majda type criteria (see [1,10]).

Remark 1.2. Our result covers the one in [7] for the Navier–Stokes equations. We would also like to mention that the result in [16] is an immediate corollary of the one in [7] in view of the boundedness of R_j in $\dot{B}^s_{p,q}$.

Let us now introduce the function spaces appeared in Theorem 1.1. Take $\psi \in \mathcal{S}(\mathbb{R}^3)$ be a radial function supported in $\{\xi \in \mathbb{R}^3;\ 3/4 \leqslant |\xi| \leqslant 8/3\}$ with

$$\sum_j \in \mathbb{Z}\psi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 - \{0\}.$$

Let $h = \mathcal{F}^{-1}\psi$, then we have the formal Littlewood–Paley decomposition

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f = \sum_{j\in\mathbb{Z}} \psi(2^{-j}D) f = \sum_{j\in\mathbb{Z}} 2^{3j} \int_{\mathbb{R}^3} h(2^{3j} y) f(x - y) dy.$$

For $s \in \mathbb{R},\ 1 \leqslant p,\ q \leqslant \infty$, the homogeneous Besov space is defined as

$$\dot{B}^s_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \left\{ 2^{js} \|\Delta_j f\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q}.$$

It is proved in [16] that

$$\|fg\|_{\dot{B}^s_{p,q}} \leqslant C\left(\|f\|_{\dot{B}^{s+\gamma}_{p_1,q_1}} \|g\|_{\dot{B}^{-\gamma}_{p_2,q_2}} + \|f\|_{\dot{B}^{-\delta}_{p_3,q_3}} \|g\|_{\dot{B}^{s+\delta}_{p_4,q_4}} \right), \tag{1.3}$$

if $s, \gamma, \delta > 0, 1 \leqslant p, q;\ p_1, q_1;\ p_2, q_2 \leqslant \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

Also, it is well known that

$$\dot{F}^s_{\infty,\infty} = \dot{B}^s_{\infty,\infty}, \quad \dot{B}^2_{2,2} = \dot{H}^s, \quad \forall s \in \mathbb{R}.$$

For more detailed properties of Besov spaces, see [15].

Through the proof in the next section, we shall frequently use the Young inequality

$$ab \leqslant \epsilon a^p + C_\epsilon b^q, \quad \forall \epsilon > 0,\ 1 < p, q < \infty \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{1.4}$$

and its generalization

$$abc \leqslant \epsilon a^p + \epsilon b^q + C_\epsilon c^r, \quad \forall \epsilon > 0,\ 1 < p, q, r < \infty \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1. \tag{1.5}$$

2. Proof of Theorem 1.1

As in [4], applying ∂_i to both sides of (1.1), and then multiplying both sides by $\partial_i u, \partial_i \omega, \partial_i b$, respectively, integration over $\mathbb{R}^3$, after suitable integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_i u,\partial_i\omega,\partial_i b)\|_{L^2}^2+\sum_{j=1}^{3}\Big[(\mu+\chi)\|\partial_{ij}^2 u\|_{L^2}^2+\gamma\|\partial_{ij}^2\omega\|_{L^2}^2+\nu\|\partial_{ij}^2 b\|_{L^2}^2\Big]+2\chi\|\partial_i\omega\|_{L^2}^2+\kappa\|\nabla\mathrm{div}\omega\|_{L^2}^2\\
&\leqslant|\langle\partial_i u\cdot\nabla u,\partial_i u\rangle|+|\langle\partial_i b\cdot\nabla b,\partial_i u\rangle|+|\langle\partial_i u\cdot\nabla b,\partial_i b\rangle|+|\langle\partial_i b\cdot\nabla u,\partial_i b\rangle|+|\langle\partial_i u\cdot\nabla\omega,\partial_i\omega\rangle|+2\chi|\langle\nabla\times\partial_i u,\partial_i\omega\rangle|\\
&=2\chi|\langle\nabla\times\partial_i u,\partial_i\omega\rangle|+|\langle\partial_{ij}(u_ju_k),\partial_i u_k\rangle|+|\langle\partial_{ij}(b_jb_k),\partial_i u_k\rangle|+\{|\langle\partial_{ij}(u_jb_k),\partial_i b_k\rangle|+|\partial_{ij}(b_ju_k),\partial_i b_k\}|+|\langle\partial_{ij}(u_ju_k),\partial_i\omega_k\rangle|\\
&=:\sum_{l=1}^{5}I_l, \qquad (2.1)
\end{aligned}$$

where we use the following facts:

$$\begin{aligned}
&\nabla\cdot u=\nabla\cdot b=0,\\
&\langle b\cdot\partial_i\nabla b,\partial_i u\rangle+\langle b\cdot\nabla\partial_i u,\partial_i b\rangle=0,\\
&\langle\nabla\times\partial_i u,\partial_i\omega\rangle=\langle\nabla\times\partial_i\omega,\partial_i u\rangle.
\end{aligned}$$

Using Young inequality, I_1 is easily estimated as

$$I_1\leqslant\frac{\chi}{2}\|\nabla\times\partial_i u\|_{L^2}^2+2\chi\|\nabla\omega\|_{L^2}^2. \qquad (2.2)$$

For the second term I_2, invoking (1.3) and Young inequality, it follows that

$$\begin{aligned}
I_2&=|\langle\Lambda^{-\alpha}\partial_{ij}(u_ju_k),\Lambda^{\alpha}\partial_i u_k\rangle|\leqslant\|u\otimes u\|_{\dot B^{2-\alpha}_{2,2}}\|u\|_{\dot H^{1+\alpha}}\leqslant C\Big(\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^2}\Big)\Big(\|u\|_{\dot H^1}^{1-\alpha}\|u\|_{\dot H^2}^{\alpha}\Big)=C\Big(\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^1}^{1-\alpha}\Big)\|u\|_{\dot H^2}^{1+\alpha}\\
&\leqslant C_\epsilon\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|u\|_{\dot H^1}^2+\epsilon\|u\|_{\dot H^2}^2. \qquad (2.3)
\end{aligned}$$

Here and thereafter, $\epsilon>0$ is to be determined later. Utilizing (1.5) with exponents

$$\left(\frac{2}{1-\alpha},\frac{2}{\alpha},2\right), \qquad (2.4)$$

the third term I_3 is dominated as

$$\begin{aligned}
I_3&\leqslant\|b\otimes b\|_{\dot B^{2-\alpha}_{2,2}}\|u\|_{\dot H^{1+\alpha}}\leqslant C\Big(\|b\|_{\dot B^{-\alpha}_{\infty,\infty}}\|b\|_{\dot H^2}\Big)\Big(\|u\|_{\dot H^1}^{1-\alpha}\|u\|_{\dot H^2}^{\alpha}\Big)=\Big(\|b\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^1}^{1-\alpha}\Big)\|u\|_{\dot H^2}^{\alpha}\|b\|_{\dot H^2}\\
&\leqslant C\|b\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|u\|_{\dot H^1}^2+\epsilon\|u\|_{\dot H^2}^2+\epsilon\|b\|_{\dot H^1}^2. \qquad (2.5)
\end{aligned}$$

For I_4, using Young inequality with exponents $(2/(1-\alpha),2/(1+\alpha))$ and (1.5) with exponents as in (2.4), we have

$$\begin{aligned}
I_4&\leqslant2\|u\otimes b\|_{\dot B^{2-\alpha}_{2,2}}\|b\|_{\dot H^{1+\alpha}}\leqslant C\Big(\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|b\|_{\dot H^2}+\|b\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^2}\Big)\|b\|_{\dot H^1}^{1-\alpha}\|b\|_{\dot H^2}^{\alpha}\\
&=C\Big(\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|b\|_{\dot H^1}^{1-\alpha}\Big)\|b\|_{\dot H^2}^{\alpha}+\Big(\|b_{\dot B^{-\alpha}_{\infty,\infty}}\|b\|_{\dot H^1}^{1-\alpha}\Big)\|b\|_{\dot H^2}^{\alpha}\|u\|_{\dot H^2}\leqslant C\Big(\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}+\|b\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\Big)\|b\|_{\dot H^1}^2+\epsilon\|b\|_{\dot H^2}^2+\epsilon\|u\|_{\dot H^2}^2. \qquad (2.6)
\end{aligned}$$

The last term I_5, is treated the same way as the third, leading to

$$I_5\leqslant C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|\omega\|_{\dot H^1}^2+\epsilon\|\omega\|_{\dot H^2}^2+\epsilon\|u\|_{\dot H^1}^2. \qquad (2.7)$$

Gathering (2.2) and (2.3), Eqs. (2.5)–(2.7), and substituting into (2.1), taking $\epsilon>0$ sufficiently small, we have

$$\frac{d}{dt}\|(\nabla u,\nabla\omega,\nabla b)\|_{L^2}^2\leqslant\|(u,\omega,b)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|(\nabla u,\nabla\omega,\nabla b)\|_{L^2}^2.$$

Gronwall inequality then implies the fact

$$u,\omega,b\in L^\infty(0,T;H^1),$$

which ensures the continuation of strong solutions beyond $t=T$. The proof is complete. □

Acknowledgments

Congchong Guo and Zujin Zhang would like to express sincere gratitude to Professor Zheng-an Yao for enthusiastic guidance and constant encouragement. Jialin Wang was supported by National Natural Science Foundation of China, No. 11126294. All authors thank the constructive and helpful comments of the anonymous referees.

References

[1] J.T. Beale, T. Kato, A.J. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Commun. Math. Phys. 94 (1984) 61–66.
[2] H. Beirão da Veiga, A new regularity class for the Navier–Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. 16 (1995) 407–412.

[3] J.-Y. Chemin, Perfect Incompressible Fluids, Oxford University Press, New York, 1998.
[4] S. Gala, Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space, Nonlinear Differ. Equat. Appl. <http://dx.doi.org/10.1007/s00030-009-0047-4>.
[5] G.P. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the microploar fluid equations, Int. J. Eng. Sci. 15 (1997) 105–108.
[6] C. He, Z. Xin, On the regularity of weak solutions to the magnetiohydrodynamic equations, J. Differ. Equat. 213 (2005) 235–254.
[7] H. Kozono, Y. Shimada, Bilinear estimates in homogeneous Triebel–Lizorkin spaces and the Naiver–Stokes equations, Math. Nachr. 276 (2004) 63–74.
[8] I. Kukavica, M. Ziane, One component regularity for the Navier–Stokes equations, Nonlinearity 19 (2006) 453–469.
[9] I. Kukavica, M. Ziane, The Navier–Stokes equation with regularity in one direction, J. Math. Phys. 48 (2007), http://dx.doi.org/10.1063/1.2395919.
[10] F. Planchon, An extension of the Beale–Kato–Majda criteria for the Euler equations, Commun. Math. Phys. 232 (2003) 319–326.
[11] M.A. Rojas-Medar, Magneto-microploar fluid motion: existence and uniqueness of strong solutions, Math. Nachr. 188 (1997) 301–319.
[12] M.A. Rojas-Medar, J.L. Boldrini, Magneto-microploar fluid motion: existence of weak solutions, Rev. Mat. Complut. 11 (1998) 443–460.
[13] E.E. Ortega-Torres, M.A. Rojas-Medar, Magneto-microploar fluid motion: global existence of strong solutions, Abstr. Appl. Anal. 4 (1999) 109–125.
[14] J. Serrin, On the interior regularity of weak solutions of Navier–Stokes equations, Arch. Ration. Mech. Anal. 9 (1962) 187–195.
[15] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.
[16] B.Q. Yuan, B. Zhang, Blow-up criteria of strong solutions to the Naiver–Stokes equations in Besov spaces with negative indices, J. Differ. Equat. 242 (2007) 1–10.
[17] Z.J. Zhang, Z.A. Yao, X.F. Wang, A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces, Nonlinear Anal. Theory Methods Appl. 74 (2011) 2220–2225.
[18] Y. Zhou, Regularity criteria in terms of pressure for the 3-D Navier–Stokes equations in a generic domain, Math. Ann. 328 (2004) 173–192.
[19] Y. Zhou, Remarks on the regularities for the 3D MHD equations, Discrete Contin. Dynam. Syst. 12 (2005) 881–886.