Two anisotropic fourth-order partial differential equations for image inpainting

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Abstract: In this study, the authors propose two fourth-order partial differential equations (PDEs) to inpaint the image. By analysing those anisotropic fourth-order PDEs and comparing their diffusion images, the authors confirm they are forward diffusion or backward diffusion. A numerical algorithm is presented using a finite-difference method and analyse the stability of discretisation. Finally, they show various experimental results and conclude that the proposed new models are better than the second-order and third-order PDEs, especially for weakening the blocky effects.

1 Introduction

Image inpainting means to restore a damaged or corrupted image in which part of the information has been lost. Such degradations of an image may have different origins, such as image transmission problems or degradation of the real images because of storage conditions or manipulation. Inpainting may also be an useful tool for graphics people who artificially need to remove some parts of an image such as overlapping texts or to implement tricks used in special effects. It is important to restore the missing parts of an image so that the final image looks unaltered to the naked eye.

Image inpainting has been studied for the last 20 years. To solve the problem, several approaches have been proposed, which differ between a variety of mathematical models such as partial differential equations (PDEs) as well as differing in the various applications from textured images to movies. Image inpainting methods can be classified to two main classes: PDEs and texture synthesis.

The PDEs method mainly deals with geometric images, that is, without fine texture content. As we will see, most inpainting PDE methods are based on the simultaneous interpolation of isophotes and grey-level intensities. The PDEs can be derived from variational principles [1–4] and constructed according to the principles of inpainting. We emphasise the second origin to motivating the proposed models in this paper. Inspired by the basic techniques used by art conservator to inpaint while restoring real paintings, Bertalmio et al. [5] in 2000 introduced a third-order PDE that propagates the level lines arriving at the hole. Chan et al. [6, 7] also introduced an approach based on total variation. Following total variation minimisation inpainting, they proposed a new PDE model, a curvature-driven diffusion approach [8]. In 2005, Tschumperlé and Deriche [9] obtained good results with a newly defined PDE based on the respect of a coherent anisotropic smoothing method that preserves the global features of vector images. To avoid the blocky effects, a lot of fourth-order PDEs were introduced for image inpainting. In 2007, Bertozzi et al. [10, 11] used a modified fourth-order Cahn–Hilliard equation to inpainting binary images. In 2009, Burger et al. [12] discussed the stationary state of this model and introduced a generalisation for grey value images of bounded variation. In 2012, Roi et al. [13] proposed the use of an anisotropic diffusion inpainting method specifically devised for hyperspectral images, show some extreme examples and discuss its convenience. In the same year, Qin et al. [14] proposed a PDE-based image inpainting method using anisotropic heat transfer model, which can simultaneously propagate the structure and texture information.

The second category is texture synthesis, which tries to fill the missing regions by copying content from the existing part of the image. The texture synthesis method is the pioneering seminal work of Efros and Leung [15]. This method can be divided into two, Wei and Levoy’s pixel-level method [16] and a pitch-level method studied by Criminisi and Perez [17], Cheng et al. [18] and Ignácio and Jung [19]. In 2007, Komodakis and Tziritas [20] proposed an example-based technique that considered the image completion problem as a discrete global optimisation based on a Markov random field. Different from previous inpainting methods that synthesised structural and textural information from a target
image, Hay and Efes [21] and Li et al. [22] proposed a novel inpainting method using a huge image database, which includes millions of target images. In 2010, Bugeau et al. [23] combine the three techniques (copy-and-paste texture synthesis, PDEs and coherence amonge neighbouring pixels) in a variational model and provide a working algorithm for image inpainting. In 2011, Arias et al. [24] proposed a general variational framework for non-local image inpainting and obtained perfect effects. In 2012, Zhang and Lin [25] proposed a novel exemplar-based inpainting algorithm based upon the colour distribution analysis. Similarly, Lee et al. [26] also proposed a exemplar-based image inpainting algorithm using region segmentation in 2012. Vizireanu and Udrea [27, 28] have also done a lot of works that are similar to image inpainting.

Inspired by You and Kaveh [29] and Marquina and Osher [30], we propose several anisotropic fourth-order PDEs for image inpainting. Since the anisotropic diffusion PDE was first introduced by Perona and Malik [31], the technique of anisotropic diffusion has been further developed using second-order PDEs for image processing. Owing to the blocky effects, You and Kaveh [29] introduced a fourth-order PDE, and in 2011, Hajiaboli [32] improved this filter and obtained a noticeable improvement in the quality of de-noised images evaluated subjectively and quantitatively. On this basis, we introduce two new anisotropic fourth-order PDEs for image inpainting. We also analyse the diffusion directions of the two fourth-order PDEs. In the end, we present experimental results, which show that the proposed fourth-order models are better than second- and third-order models.

2 Preliminaries and background

Before introducing our proposed models, we first analyse the diffusion direction of fourth-order PDE, which introduced by You and Kaveh with different diffusivity functions.

2.1 From energy functional to PDEs

Let us consider the following equation, initially proposed by Perona and Malik [31]

$$\frac{\partial u}{\partial t} = \text{div}(\phi(|\nabla u|)\nabla u)$$

(1)

where $\phi(\cdot)$ is the diffusivity function. Equation (1) was associated with the following energy functional

$$E(u) = \int_{\Omega} f(|\nabla u|) \, d\Omega$$

(2)

where $\Omega$ is the image support, and $f(\cdot) \geq 0$ is an increasing function associated with the diffusion coefficient

$$c(s) = \frac{f'(s)}{s}$$

(3)

and we let $\phi(s) = f'(s)$, $\phi(s)$ is called the flux function.

One of the diffusivity functions defined by Perona and Malik is given by

$$c(|\nabla u|) = \frac{1}{1 + (|\nabla u|/k)^2}$$

(4)

where $k$ is the so-called contrast parameter. If the diffusivity function of (4) is used, then the energy function (2) becomes

$$E(u) = \int_{\Omega} \frac{k^2}{2} \ln(k^2 + |\nabla u|^2) \, d\Omega$$

(5)

We know that when there is no backward diffusion ($|\nabla u| < k$), a level image is the only minimum of the energy functional; so anisotropic diffusion will evolve towards the formation of a level image function. When there is backward diffusion ($|\nabla u| > k$), piecewise level image is global minimum of the energy functional. To resolve the blocky effects appear in image denoising, You and Kaveh [29] introduced a fourth-order PDE that is obtained by the minimisation of the potential function given by

$$E(u) = \int_{\Omega} f(|\Delta u|) \, d\Omega$$

(6)

The solution of the minimisation problem (6) after using Euler equation followed by gradient descent is given by

$$\frac{\partial u}{\partial t} = -\Delta (\phi(|\Delta u|)\Delta u)$$

(7)

which is solved by the forward Euler approximation of $\partial u/\partial t$.

2.2 Forward diffusion and backward diffusion

The principle of image inpainting is different from image denoising. The problem can be described as follows: given a domain image $\Omega$, a hole $D \subset \Omega$ and the intensity $u_0$ is known over $\Omega - D$. We want to find an image $u$, an inpainting of $u_0$, that matches $u_0$ outside the hole and that has ‘meaningful’ content inside the hole $D$. This can be achieved by examining the intensity around $D$ and propagating it in $D$ (see Fig. 1).

First, we introduce the local coordinate, two normalised and orthogonal vectors of $T$ and $N$ pointing in the direction of the gradient and level set are, respectively, given by

$$T = \frac{\left(-u_x, u_y\right)}{\sqrt{u_x^2 + u_y^2}} \quad \text{and} \quad N = \frac{\left(u_y, -u_x\right)}{\sqrt{u_x^2 + u_y^2}}$$

(8)

![Fig. 1](image)

Inpainting is to inpaint the missing region $D$ on an inpainting domain $\Omega$. 


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Based on the definition (8), one can derive the second-order derivatives of the image in the direction of the gradient and level set as

\[ u_{TT} = \frac{u_{xx}u^2_y - 2u_{xy}u_xu_y + u_{yy}u^2_x}{u_x^2 + u_y^2} \]  

(9)

and

\[ u_{NN} = \frac{u_{xx}u^2_y + 2u_{xy}u_xu_y + u_{yy}u^2_x}{u_x^2 + u_y^2} \]  

(10)

However, it can be simply shown that the sum of these second directional derivatives is equal to the Laplacian of the image

\[ \Delta u = u_{xx} + u_{yy} = u_{NN} + u_{TT} \]  

(11)

In image denoising, we would like to find models for removing the noise while preserving the edges. For example, we chose \( c(s) = (1/(1 + (s/k)^2)) \) in (1) to obtain

\[ \frac{\partial u}{\partial t} = \text{div} \left( \frac{k^2}{k^2 + |\nabla u|^2} \nabla u \right) = c(|\nabla u|)u_{TT} + \phi'(|\nabla u|)u_{NN} \]  

(12)

and

\[ \phi'(|\nabla u|) = \frac{1 - \left(\frac{|\nabla u|}{k}\right)^2}{\left[1 + \left(\frac{|\nabla u|}{k}\right)^2\right]^2} \]  

\[ c(|\nabla u|) = \frac{1}{1 + \left(\frac{|\nabla u|}{k}\right)^2} \]  

(13)

Thus in the regions where the magnitude of the gradient of \( u \) is weak (\(|\nabla u| \rightarrow 0\)), we have \( c(|\nabla u|) \rightarrow 1 \) and \( \phi'(|\nabla u|) \rightarrow 1 \), (12) acts like the heat equation (forward diffusion), resulting in isotropic smoothing. Near boundaries of the region where the magnitude of gradient is large (\(|\nabla u| \rightarrow \infty\)), then \( c(|\nabla u|) \rightarrow 0 \), \( \phi'(|\nabla u|) \rightarrow 0 \), \( \phi(|\nabla u|) \rightarrow c(|\nabla u|)) \rightarrow -1 \), the regularization is ‘stopped’ and the edge is preserved (backward diffusion).

When we choose diffusion function

\[ c_1(s) = \frac{1}{1 + (s/k)} \]  

(14)

and

\[ c_2(s) = \frac{k}{s} \]  

(15)

then (1) become

\[ \frac{\partial u}{\partial t} = \frac{1}{1 + (|\nabla u|/k)} u_{TT} + \frac{(1/k)}{1 + (|\nabla u|/k)^2} u_{NN} \]  

(16)

and

\[ \frac{\partial u}{\partial t} = \frac{k}{|\nabla u|} u_{TT} \]  

(17)

the two PDEs are forward diffusion.

For (7), we obtain that

\[ \frac{\partial u}{\partial t} = -\Delta c(|\Delta u|)|\Delta u| \]  

\[ = -\left[ \Delta c(|\Delta u|)|\Delta u| + c(|\Delta u|)\Delta u + 2\nabla c(|\Delta u|) \nabla (\Delta u) \right] \]  

\[ = -c'(|\Delta u|)|\Delta u|(|\nabla u|)^2 + c'(|\Delta u|)|\Delta u|\Delta u \]  

\[ + c(|\Delta u|)\Delta u + 2c'(|\Delta u|)\nabla \frac{\Delta u}{|\nabla u|^2} \]  

\[ = -c'(|\Delta u|)|\Delta u| + 2c'(|\Delta u|) \frac{|\nabla u|^2}{|\Delta u|} \Delta u \]  

\[ = -c'(|\Delta u|)|\Delta u| + c(|\Delta u|)\Delta u \]  

\[ = -\phi'(|\Delta u|) \frac{|\nabla u|^2}{|\Delta u|} \Delta u - \phi(|\Delta u|)\Delta u \]  

(18)

When we choose the diffusion function as in (4), we know that

\[ \phi(s) = \frac{1}{1 + (s/k)^2} \]  

(19)

\[ \phi''(s) = \frac{2s(k^2)^2 - 3}{(1 + (s/k)^2)^3} \]  

(20)

Thus, when \(|\Delta u| < k\), \( \phi'(|\Delta u|) < 0 \), \( \phi'(|\Delta u|) > 0 \) and when \(|\Delta u| > \sqrt{3}k\), \( \phi'(|\Delta u|) > 0 \), \( \phi'(|\Delta u|) < 0 \). So, the model is forward diffusion for regions of the image with \(|\Delta u| < k\) and backward diffusion for regions with \(|\Delta u| > \sqrt{3}k\).

Now, we set \( c_1(s) = (1/(1 + (2/k))) \), at this moment

\[ \phi_1(s) = \frac{(1/k)}{(1 + (s/k)^2)^2} \]  

(21)

\[ \phi_1''(s) = \frac{-2(k^2)^2}{(1 + (s/k)^2)^3} \]  

(22)

The model is forward diffusion for all region of the image, because \( \phi_1(|\Delta u|) < 0 \) and \( \phi_1(|\Delta u|) < 0 \) for all region of \( u \). Owing to \( f_1(s) = s c_1(s) = (ks/(k + s)) \), then \( f_1(s) = ks - k^2 \ln(k + s) \), \( f_1(s) \) is a increasing and convex function. Therefore the evolution of (7) with diffusivity function \( c_1(s) \) is a process in which the image is smoothed more and more until it becomes a planar image.

From the expression of (12) and (18), we can conclude that the principle of diffusion of model (1) and model (7) is completely different. Model (1) is backward diffusion only in the direction of \( N \) in the regions of \(|\Delta u| > k\). However, model (7) is backward diffusion in all directions of \( T \) and \( N \) in the regions of \(|\Delta u| > \sqrt{3}k\). From the principle of inpainting, we know that the process of inpainting is to propagate the intensity around \( D \) into \( D \). We have know the principle of BSCB model is to propagate the Laplacian of \( u \) along \( |\nabla u|^{-1} \), and \( T = (\nabla u^{-1}/(|\nabla u|^{-2})) \). So, the diffusion in the direction of \( T \) is very important. Consequently, we need the inpainting model (7) is forward diffusion.
which \( g(s) = c(s) \cdot d(s) \), \( h(s) = c(s) \cdot \phi(s) \) where \( \phi(s) = s \cdot d(s) \). PDE (24) is the generalisation of the models proposed in You and Kaveh [29], Hajiaboli [32] and Burger et al. [12].

Now, we substitute diffusivity functions (14) and (15) for \( d(s) \) in (24), respectively.

First, we set \( d(s) = c_1(s) \) and obtain

\[
\frac{\partial u}{\partial t} = -\Delta \left[ c(|\nabla u|)(c_1(|\nabla u|)u_{TT} + c_1^2(|\nabla u|)u_{NN}) \right]
\]

(25)

Second, we set \( d(s) = c_2(s) \) to obtain

\[
\frac{\partial u}{\partial t} = -\Delta \left( c(|\nabla u|)c_2(|\nabla u|)u_{TT} \right)
\]

(26)

### 3.2 Improved model

In 1992, Rudin et al. [33] gave a model

\[
\frac{\partial u}{\partial t} = \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) - \lambda^* (j* u - u_0)
\]

(27)

However, the CFL-condition is very harsh in this model, and produces the blocky effect. To elevate the convergence rate and weaken the blocky effect of (27), Marquina and Osher [30] multiplied (27) by the magnitude of the image gradient, then the model reads as follows

\[
\frac{\partial u}{\partial t} = |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) - |\nabla u| \lambda^* (j* u - u_0)
\]

(28)

In 2000, Chan and Shen [6, 7] used this model (ITV) to deal with the inpainting problems. If the inpainted images have no noise, this model becomes

\[
\frac{\partial u}{\partial t} = |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) = u_{TT}
\]

(29)

Inspired by this technique, we let \( c(s) = (1/(d(s))) \), and (24) turn to be

\[
\frac{\partial u}{\partial t} = -\Delta \left( u_{TT} + \frac{\phi'(|\nabla u|)}{|\nabla u|} u_{NN} \right)
\]

(30)

where \( \phi(s) \) represents \( \phi_1(s) = s \cdot c_1(s) \) and \( \phi_2(s) = s \cdot c_2(s) \), respectively.

Accordingly, the models (25) and (26) can be reformulated

\[
\frac{\partial u}{\partial t} = -\Delta \left( c_1(|\nabla u|)u_{NN} + u_{TT} \right)
\]

(31)

\[
\frac{\partial u}{\partial t} = -\Delta (u_{TT})
\]

(32)

Then we obtain the proposed model I (31) and the proposed model II (32).
3.3 Analysis of diffusion in the proposed models

In this part, we analyse the diffusion direction of model (31) and (32). First, we give the energy function of \( u \) in a sub-region \( \Omega_1 \)

\[
E(u) = \frac{1}{2} \| u(x, t) \|^2_{L^2} = \frac{1}{2} \int_{\Omega_1} (u(x, t))^2 \, d\Omega \quad (33)
\]

To obtain the directions of diffusion, we need to know how the gradient of \( E(u) \) evolves with respect to \( t \) in sub-region \( \Omega_1 \). We just consider that the two models deal with an unprocessed image \( u \).

To model (31), we differentiate the energy \( E(u) \) with respect to \( t \) to obtain

\[
\frac{\partial}{\partial t} E(u) = \frac{1}{2} \frac{\partial}{\partial t} (\| u(x, t) \|^2_{L^2}) = -\int_{\Omega_1} \Delta(c_1(|\nabla u|)u_{NN} + u_{TT}) u \, d\Omega 
\]

\[
-\int_{\Omega_1} (c_1(|\nabla u|)u_{NN} + u_{TT}) \Delta u \, d\Omega \quad (34)
\]

When \( \text{sign}(c_1(|\nabla u|)u_{NN} + u_{TT}) = \text{sign}(|\Delta u|) \), we obtain that \( \frac{\partial}{\partial t} E(u) \leq 0 \). In other words, as \( t \) increases, the energy \( E(u) \) in sub-region \( \Omega_1 \) is decreasing, so the model will produce forward diffusion in this sub-region of \( u \). Oppositely, the model will become backward diffusion in this sub-region \( \Omega_1 \) as

\[
\text{sign}(c_1(|\nabla u|)u_{NN} + u_{TT}) \neq \text{sign}(|\Delta u|) \quad (35)
\]

The situation of (35) hold true if and only if \( \text{sign}(u_{NN}) \neq \text{sign}(u_{TT}) \), \( |u_{NN}| > |u_{TT}| \) and \( |c_1(|\nabla u|)u_{NN}| < |u_{TT}| \) in the sub-region \( \Omega_1 \) of \( u \). In this case, as the evolutionary process of (31) is continued (backward diffusion), \( u_{TT} \) is reduced and \( u_{NN} \) is increased. The process of increasing of \( u_{NN} \) can be divided into two stages. For the first stage, the increasing of \( u_{NN} \) cause the increasing of \( c_1(|\nabla u|)u_{NN} \); then \( c_1(|\nabla u|)u_{NN} \) becomes the sign of \( c_1(|\nabla u|)u_{NN} + u_{TT} \) and the backward diffusion is ceased as the condition in (35) is not satisfied. The increasing of \( u_{NN} \) does not mean that \( c_1(|\nabla u|)u_{NN} \) is always increased, since enhancement of the local image features can result in a dramatical reduction of \( c_1(|\nabla u|) \) which leads to decreasing of \( c_1(|\nabla u|)u_{NN} \). The second stage is when \( c_1(|\nabla u|)u_{NN} \) is decreasing as \( |u_{TT}| \). In this situation, the process of diffusion is stalled because of \( c_1(|\nabla u|)u_{NN} + u_{TT} = 0 \) which means the solution of (31) is a piecewise planar image. According to the argument, the backward diffusion just appear in the early stage of inpainting process in the sub-region \( \Omega_1 \).

To model (32), the analysis of diffusion direction is similar to model (31). However, model (32) is backward diffusion in the sub-region \( \Omega_1 \), which satisfied the conditions \( \text{sign}(u_{NN}) \neq \text{sign}(u_{TT}) \) and \( |u_{NN}| > |u_{TT}| \). So the proposed model II also evolves an observed image towards a piecewise planar image.

Although the process of inpainting by model (32) will be backward diffusion in some sub-regions of \( u \). The backward diffusion is allowable in model (32) like the model of (1). We will verify the fact that model (32) obtain good performance for inpainting in the experiments of Section 4.2.

Now, we verified the diffusion directions of fourth-order PDEs by experiments present in Fig. 3.

An example implementing the forward and backward diffusion is depicted in Fig. 3. We give an image, the grey value of which ranges from 0 (left) to 255 (right). Also, an inpainted region with grey value 100 is given. Hence the magnitude of image gradient on the edge of the inpainted domain varies from 0 to 102.88. We just give the early stages of inpainted images. The \((i, j)\) \((i = 1, 2, 3, 4, 5, 6, j = 1, 2, 3, 4)\) entry of Fig. 3 shows the inpainted image after \((i - 1) \times 1000\) iterations via \(j\)th fourth-order PDEs. The first fourth-order PDE model is (23) with diffusivity function (4) \((k = 4)\). The second model is (23) with diffusivity function (14) \((k = 1)\). The third model is (31). The fourth model is (32).

The first column of Fig. 3 lists the results from the experiment processed by model (23) of diffusivity function (4) \((k = 4)\). Through observation we can find that just the middle part of the inpainted region was diffused in the early stage of inpainting. It coincides with the conclusion that we deduced in the context of this section. We can conjecture that the extent of the forward diffusion part will enlarge as the contrast parameter \(k\) increases. We have verified this situation through experiments.

The second column is experimental results which are processed by model (23) of diffusivity function (14) \((k = 1)\). By comparison, we can conclude that this model diffuse everywhere no matter which value is assigned to the magnitude of the gradient when choosing the diffusivity function (14).

The third column of Fig. 3 is the experimental results processed by model (31). We have known that model (31) is forward diffusion no matter how the value of \(|\nabla u|\). So, we can observe that model (31) diffuse everywhere in the inpainted region as the performance of second experiment results.

The fourth column of Fig. 3 is the experimental results inpainted by model (32). As the model is backward diffusion in some sub-regions, which satisfied the conditions \( \text{sign}(u_{NN}) \neq \text{sign}(u_{TT}) \) and \( |u_{NN}| > |u_{TT}| \). So, the inpainted images of different stages look like the first column and do not look like the second and third columns.

In the end, the experimental results of model (23), (31) and (32) coincide with the conclusion, which we have deduced in this section. We believe the fact that piecewise planar image is a better approximation to natural images. Therefore the processed image by the proposed models I and II will look less blocky and more natural.
4 Discretisation of the model and the results

4.1 Discretisation of the model

In practice, the processing of a digital image requires a discrete realisation of the proposed filter. A digital image in a spatial domain is naturally discrete with a constant grid size $\Delta x = \Delta y = 1$ in a rectangular support domain with size $M \times N$.

Thus, a straightforward replacement of the differential operator in the continuous setting by their central difference operator with a symmetric boundary condition seems to be a good approximation for the discrete model. These difference operators are given by

\[
\begin{align*}
& u_x \simeq D_x u_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2} \\
& u_y \simeq D_y u_{ij} = \frac{u_{i,j+1} - u_{i,j-1}}{2} \\
& u_{xx} \simeq D_{xx} u_{ij} = u_{i+1,j} + u_{i-1,j} - 2u_{ij} \\
& u_{yy} \simeq D_{yy} u_{ij} = u_{i,j+1} + u_{i,j-1} - 2u_{ij} \\
& u_{xy} \simeq D_{xy} u_{ij} = \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}}{4}
\end{align*}
\]

The norm of the gradient of $u$ in the discrete domain can be estimated as

\[
|\nabla u_{ij}| = \sqrt{\left(D_x u_{ij}\right)^2 + \left(D_y u_{ij}\right)^2} \tag{36}
\]

and the Laplacian of the image $u$ is, respectively

\[
\Delta u_{ij} = D_{xx} u_{ij} + D_{yy} u_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} \tag{37}
\]

From the expression of (9) and (10), we can denote that (see (38) and (39))

Therefore the discretisation model of (30) can be written as

\[
\begin{align*}
& u_{ij}^{n+1} = u_{ij}^{n} - \Delta t \left( \Delta \left( \frac{\nabla u_{ij}^n}{\nabla u_{ij}^n} \right) D_{xx} u_{ij}^n + D_{yy} u_{ij}^n \right) \tag{40}
\end{align*}
\]

where $\psi(s) = (\phi(s)/(\phi(s)))$, $\Delta t$ is time-step size and $n$ is the number of iteration. The process is started with the unpainted image as initial data ($u_0 = u_0$).

4.2 Stability

To guarantee the stability of (40), the time step size $\Delta t$ needs to be set to a small value. However, if $\Delta t$ is too small, the number of iterations for convergence of (40) will dramatically increase. So, we will find an optimal $\Delta t$ in order to reduce the convergence rate.

First, we give the fourth-order linear PDE $\left( \partial u/\partial t \right) = -\Delta u$ and its discrete setting is $u^{n+1} = u^n - \Delta t \cdot (P \ast u^n)$, where $P$ is a Laplacian kernel given by

\[
p = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

and symbol $\ast$ denotes the convolution operation. We have known from [32] that the CFL-condition of this discrete setting is $\Delta t < 0.0313$. We also have known the fact that the fourth-order linear equation is obtained when the diffusion strength of the proposed model I is maximum in the directions of $T$ and $N$, it means that when the time step size is $\Delta t < 0.0313$, the discrete setting of proposed model I is stable.

Then we analyse the stability of proposed model II. The discrete setting of proposed model II is

\[
u_{ij}^{n+1} = u_{ij}^{n} - \Delta t \left( \Delta \left( D_{TT} u_{ij}^n \right) \right) \tag{41}
\]

the discrete setting (41) is equivalent to

\[
u_{ij}^{n+1} = u_{ij}^{n} - \Delta t \left( (P \ast Q) \ast u_{ij}^n \right) \tag{42}
\]

where $P$ is Laplacian kernel and

\[
Q = \begin{bmatrix} -u_{ij} & u_{ij}^2 & u_{ij} \\ 2(u_{ij}^2 + u_{ij}^2) & u_{ij}^2 + u_{ij}^2 & 2(u_{ij}^2 + u_{ij}^2) \\ u_{ij}^2 & u_{ij}^2 + u_{ij}^2 & -u_{ij} \\ 2(u_{ij}^2 + u_{ij}^2) & u_{ij}^2 + u_{ij}^2 & 2(u_{ij}^2 + u_{ij}^2) \end{bmatrix}
\]

If $u$ is arranged in a column-wise order, the (42) can be state by

\[
u_{ij}^{n+1} = (I + S) \times u_{ij}^n \tag{43}
\]

\[
D_{TT} u_{ij} = \frac{D_x u_{ij} \left(D_x u_{ij}\right)^2 - 2D_{xy} u_{ij} D_x u_{ij} D_y u_{ij} + D_{yy} u_{ij} \left(D_y u_{ij}\right)^2}{\left(D_x u_{ij}\right)^2 + \left(D_y u_{ij}\right)^2} \tag{38}
\]

\[
D_{NN} u_{ij} = \frac{D_x u_{ij} \left(D_x u_{ij}\right)^2 + 2D_{xy} u_{ij} D_x u_{ij} D_y u_{ij} + D_{yy} u_{ij} \left(D_y u_{ij}\right)^2}{\left(D_x u_{ij}\right)^2 + \left(D_y u_{ij}\right)^2} \tag{39}
\]
Inpainting with the proposed model I is the state with the small convolution kernel of $0 \times 4^4 + (S_0 + leads to the stability of $\times (P^{*}Q)$ is given by (See equation at the bottom of the page)

For the inner part of $u$, the strictly row-wise diagonally dominant condition (46), for the part of $(2I + S)$ corresponding to the inner region of $u$ mean that if

$$|2 - 10d| > \left(21 + \left|\frac{4u_i u_j}{u_i^2 + u_j^2}\right|\right) dr$$

for the stability of model (32), we can deduce that $|2 - 10d| > 23d$ since $0 < \left|\frac{4u_i u_j}{u_i^2 + u_j^2}\right| < 2$. Therefore (47)

$u_i^{n+1} = -(I + S) \times u^n_i$  

(44)

However, the state system in (44) can be written in standard form of Jacobi solver given by

$u_i^{n+1} + (2I + S) - I)u^n_i = 0$  

(45)

Stability of the Jacobi solver in (45) can be obtained if matrix $(2I + S)$ is strictly row-wise diagonally dominant. By definition, the matrix $A$ is strictly row-wise diagonally dominant if

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}| \text{ for } i = 1, 2, \ldots , n$$

(46)

We divide the pixels of $u$ into two parts: inner part and near-boundary part. The inner part include the pixels that are not on the boundaries or adjacent to the boundaries of $u$, other pixels of $u$ belong to the near-boundary part.

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$|2 - 10d| > \left(21 + \left|\frac{4u_i u_j}{u_i^2 + u_j^2}\right|\right) dr$
equivalent to $2 - 10\|d\| > 23\|d\|$ for the reason of $d\| > 0$. So, we can conclude that model (32) is data-independently stable when $d\| < 0.0606$.

For the near-boundary part of $u$, the Neumann boundary condition can be considered as extending of the domain of $u$ with intensity of $u$ at its boundary. Since this extension is not considered in the formation of $u$, one can reflect the effect of the boundary condition on $S$. For the rows of $S$ corresponding to the pixels on boundaries of $u$, the coefficients of $S$ are modified leading to a more relax stability condition. For example, of $u_{1,1}$, the deduced process same as the pixels of inner part. We obtain that

$$\frac{\partial u}{\partial t} = \text{div} \left( \frac{|k|}{|\nabla u|} \nabla u \right)$$

where $k$ is the scalar curvature $\text{div} \left( \nabla u / (|\nabla u|) \right)$ and we let $f(s) = s$.

3. The third model is (23) with the diffusivity function (14). We set $k = 1$ and $d\| = 0.04$.
4. The proposed model I (31) and $c_1(s) = (1/(1 + (s/k)))$. We set $k = 1$ and $d\| = 0.03$.
5. The proposed model II (32) with $d\| = 0.06$.

We apply those models in four inpainting problems. PSNR is denoted by

$$\text{PSNR} = 10 \log_{10} \frac{MN255^2}{\sum_{ij} (u_{ij} - \pi_{ij})^2}$$

where $u$ is the original image, $\pi_{ij}$ is the compared image, $M \times N$ is the pixel of image $u$ and the unit of PSNR is decibel (dB).

The data shown in Table 4.1 exhibit that our new proposed models can out perform the other models in term of the PSNR of the inpainting images.

In this numerical study, the performance of the following models are going to be compared:

1. The filter of ITV model (29) for inpainting introduced by Chan et al. [6, 8]. In this filter, the diffusivity function $c(s)$ given in (24) has been used with $s = |\nabla u|$ and $d\| = 0.05$.
2. In 2001, guided by the connectivity principle of human visual perception, Chan and Shen [7] introduced a third-order PDE model based on CDD for non-texture images. Accordingly, the expression of CDD model is

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Fig. 5
Remove text from images

a (top left): The image with superimposed text
b (top middle): Inpainting with ITV model
c (top right): Inpainting with CDD model
d (bottom left): Inpainting with model (23)
e (bottom middle): Inpainting with the proposed model I
f (bottom right): Inpainting with the proposed model II
We consider four experiments. For the first experiment shown in Fig. 4, we give an image (Lena) whose pixel is 256 × 256, and choose four inpainted regions that are parts A, B, C and D from left to right, respectively. The pixel of every part is 10 × 10. We name the experiment as the block inpainting. In the four parts, part B represents the smooth region, and in the regions of parts C, A and D, the magnitude of gradient of ramp edges becomes larger and larger. From the experimental results we know that the five models almost obtain the same effects in the smoothest part (part B). For the other parts, the experimental results of inpainted by second-order PDEs produce blocky effects. The blocky effect that is inpainted by the third model is obviously weaker. As the diffusivity function in (23) is a convex function (14), we can view from the result that the solution of (23) is a planar image. Moreover, the solution of

![Fig. 6 Inpaint a scratched image (Pepper 256 × 256)](image)

a (top left): Scratch image  
b (top middle): Inpainting with ITV model  
c (top right): Inpainting with CDD model  
d (bottom left): Inpainting with model (23)  
e (bottom middle): Inpainting with the proposed model I  
f (bottom right): Inpainting with the proposed model II

![Fig. 7 Remove the glasses of a man](image)

a (top left): Original image  
b (top middle): Removal glasses with ITV model  
c (top right): Removal glasses with CDD model  
d (bottom left): Removal glasses with model (23)  
e (bottom middle): Removal glasses with the proposed model I  
f (bottom right): Removal glasses with the proposed model II
proposed models I and II are piecewise planar images, and the block effects of the results almost disappeared. We also survey that as the order of equations become bigger and bigger, the blocky effects obtains weaker and weaker.

In Fig. 5, we remove text from images. Fig. 5a is the Baboon image (256 × 256) with superimposed text. In Fig. 5b is the inpainted image by ITV model is shown in which the formation of blocky effects on the inpainted regions is visible, especially around the eyes. The next image, as shown in Fig. 5c is the result of the CDD model, except the blocky effects, checkerboard effects is very obviously on the bridge of the nose of Baboon. Moreover, those effects are obviously weaken in the inpainted images (Figs. 5d–f) by the fourth-order models, especially Figs. 5e and f are inpainted by our proposed two anisotropic fourth-order models.

In Fig. 6, we inpaint a scratched image (Pepper 256 × 256), which often occurs in old pictures and photographs. The scratched images inpainted by these models almost obtain the same effects in the smooth regions, but we can still find the artificial traces at the boundary of inpainted images, which obtained by low-order models (ITV and CDD). We readily know that our new fourth-order models are better than ITV and CDD models from Table 4.1, in addition to the observation effects mentioned above.

In Fig. 7, we remove the glasses of a man. Clearly blocky effects occur in inpainted image by the ITV model, especially in the region of the bridge of the nose. Similarly, the inpainted image by the CDD model appears the checkerboard effects in the inpainted region, especially in the temples of the man. The inpainted image by the proposed model I also slightly appears the checkerboard effects. However, the two bad effects disappeared in the inpainted image obtained by our proposed model II.

5 Conclusion
In this paper, two anisotropic fourth-order PDE models for image inpainting have been proposed. We analysed the backward diffusion or forward diffusion of the proposed models in different sub-regions and verified this properties through experiments. Moreover, the comparison of the results obtained by our two proposed models with those of low-order models (ITV and CDD) has shown that the proposed models noticeably improved subjectively and quantitatively.

6 References