Regularity criteria for the 3D magneto-micropolar fluid equations in Besov spaces with negative indices

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\textbf{Abstract}

We consider the Cauchy problem of the magneto-micropolar fluid equations in three space dimensions. It is proved that if the velocity, magnetic field and the micro-rotational velocity belong to some critical Besov space with negative indices, then the solution is in fact smooth.

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\section{1. Introduction}

We consider the magneto-micropolar fluid (MMF) equations in $\mathbb{R}^3$:

\begin{equation}
\begin{aligned}
\partial_t u + u \cdot \nabla u - (\mu + \chi) \Delta u - \nabla b + \nabla (p + b^2) - \chi \nabla \times \omega &= 0, \\
\partial_t \omega - \gamma \Delta \omega - \kappa \nabla \operatorname{div} \omega + 2\chi \omega + u \cdot \nabla \omega - \chi \nabla \times u &= 0, \\
\partial_t b - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(x, 0) = u_0(x), \quad \omega(0, x) = \omega_0(x), \quad b(0, x) = b_0(x).
\end{aligned}
\end{equation}

Here $u = u(x, t)$ represents the velocity field, $b = b(x, t)$ represents the magnetic field, $\omega = \omega(x, t)$ represents the micro-rotational velocity; $p$ denotes the hydrodynamic pressure; $\mu > 0$ is the kinematic viscosity, $\chi > 0$ is the vortex viscosity, $\kappa > 0$ and $\gamma > 0$ are spin viscosities, $1/\nu$ (with $\nu > 0$) is the magnetic Reynold; while $u_0, b_0, \omega_0$ are the corresponding initial data with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$.

System (1.1) was first proposed by Galdi and Rionero [5]. The existence of global (in time) weak solutions were established by Rojas-Medar and Boldrini [12], while the local strong solutions and global strong solutions for the small initial data were considered, respectively, by Rojas-Medar [11] and Ortega-Torres and Rojas-Medar [13]. However, whether the weak solution is regular or the unique strong solution can exist globally is unknown. Thus there are a lot of literatures devoted to find sufficient conditions to ensure smoothness, see [2,8,9,14,18] for the Navier–Stokes equations ($\omega = b = 0$ in (1.1)), and [6,19] for the MHD equations ($\omega = 0$ in (1.1)).

Very recently, Gala [4] and Zhang et al. [17] considered system (1.1) and showed that if $u$ or $\nabla u$ belongs to some critical Besov space, then the solution is actually regular. Our motivation is then to lower the regularity of $u$ to ensure smoothness also, but as a compensation, we need $\omega$ and $b$ have some (also rough) regularity. Our result seems to be more helpful in the regularity theory of system (1.1) since the smoothness of $u, \omega$ and $b$ are always the same.

The main result now reads:

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**Theorem 1.1.** Let $u_0, \omega_0, b_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, and the triple $(u, \omega, b)$ be the strong solution on $(0, T)$ of system (1.1) with initial data $(u_0, \omega_0, b_0)$. If additionally,

$$u, \omega, b \in L^\infty_t(0, T; B_{\infty, \infty}^{-\alpha}), \quad 0 < \alpha < 1,$$

(1.2)

then the solution $(u, \omega, b)$ can be extended smoothly beyond $t = T$.

**Remark 1.1.** Checking the proof of Bernstein Lemma (see [3]), it follows that the Riesz transform $R_j(1 \leq j \leq 3)$ is bounded in $B_{p, q}^{1-\alpha}$ for all $s \in \mathbb{R}$, $1 \leq p < q \leq \infty$. Thus by Theorem 1.1, we have the condition

$$\nabla \times u, \quad \nabla \times \omega, \quad \nabla \times b \in L^\infty_t(0, T; B_{\infty, \infty}^{-\alpha}), \quad 0 < \alpha < 1$$

is enough to ensure the smoothness. This is a Beal–Kato–Majda type criteria (see [1, 10]).

**Remark 1.2.** Our result covers the one in [7] for the Navier–Stokes equations. We would also like to mention that the result in [16] is an immediate corollary of the one in [7] in view of the boundedness of $R_j$ in $B_{p, q}^1$.

Let us now introduce the function spaces appeared in Theorem 1.1. Take $\psi \in S(\mathbb{R}^3)$ be a radial function supported in $\{\xi \in \mathbb{R}^3; 3/4 \leq |\xi| \leq 8/3\}$ with

$$\sum_{j} \xi^j \psi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3 - \{0\}.$$

Let $h = F^{-1}\psi$, then we have the formal Littlewood–Paley decomposition

$$f = \sum_{j \in \mathbb{Z}} \Delta f_j = \sum_{j \in \mathbb{Z}} \psi(2^{-j}D)f = \sum_{j \in \mathbb{Z}} 2^{j/2} \int_{\mathbb{R}^3} h(2^j y)f(x-y)dy.$$

For $s \in \mathbb{R}$, $1 \leq p < q < \infty$, the homogeneous Besov space is defined as

$$B_{p, q}^s = \left\{ f \in S'(\mathbb{R}^3); \|f\|_{B_{p, q}^s} < \infty \right\},$$

where

$$\|f\|_{B_{p, q}^s} = \left\| \left\{ \| \Delta f_j \|_p \right\}_{j \in \mathbb{Z}} \right\|_{l^q}.$$

It is proved in [16] that

$$\|fg\|_{B_{p, q}^s} \leq C \left( \|f\|_{\dot{B}_{p, q}^{s+\gamma}}\|g\|_{\dot{B}_{p, q}^{s+\gamma}} + \|f\|_{\dot{B}_{p, q}^{s+\gamma}} \|g\|_{\dot{B}_{p, q}^{s+\gamma}} \right),$$

(1.3)

if $s, \gamma, \delta > 0$, $1 \leq p, q, r, \kappa_1, \kappa_2, \kappa_3, \kappa_4 < \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

Also, it is well known that

$$\dot{B}_{\infty, \infty}^s = \dot{H}^s, \quad \dot{B}_{2, 2}^s = H^s, \quad \forall s \in \mathbb{R}.$$

For more detailed properties of Besov spaces, see [15].

Through the proof in the next section, we shall frequently use the Young inequality

$$ab \leq \varepsilon a^p + C b^q, \quad \forall \varepsilon > 0, \quad 1 < p, q < \infty \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1$$

(1.4)

and its generalization

$$abc \leq \varepsilon a^p + \varepsilon b^q + C c^r, \quad \forall \varepsilon > 0, \quad 1 < p, q, r < \infty \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

(1.5)

2. **Proof of Theorem 1.1**

As in [4], applying $\partial_i$ to both sides of (1.1), and then multiplying both sides by $\partial_i u, \partial_i \omega, \partial_i b$, respectively, integration over $\mathbb{R}^3$, after suitable integration by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \| (\partial_t u, \partial_t \omega, \partial_t b) \|_{L^2}^2 + \sum_{j=1}^{3} \left[ (\mu + \chi) \| \partial_{ij}^2 u \|_{L^2}^2 + \gamma \| \partial_{ij}^2 \omega \|_{L^2}^2 + \nu \| \partial_{ij}^2 b \|_{L^2}^2 \right] + 2 \chi \| \partial_t \omega \|_{L^2}^2 + \kappa \| \nabla \text{div} \omega \|_{L^2}^2
\]
\[
\leq \| (\partial_t \cdot \nabla u, \partial_t u) \| + \| (\partial_t \cdot \nabla b, \partial_t u) \| + \| (\partial_t \cdot \nabla b, \partial_t u) \| + \| (\partial_t \cdot \nabla \omega, \partial_t \omega) \| + 2 \chi \| \nabla \times \partial_t u, \partial_t \omega \| + \| (\partial_t (u_j u_k), \partial_t u) \| + \| (\partial_t (u_j b_k), \partial_t b) \| + \| (\partial_t (b_j u_k), \partial_t b) \| + \| (\partial_t (u_j u_k), \partial_t \omega) \| + \| (\partial_t (u_j b_k), \partial_t \omega) \| + \| (\partial_t (b_j u_k), \partial_t \omega) \|
\]
\[
= \sum_{j=1}^{5} l_j, \tag{2.1}
\]
where we use the following facts:
\[
\nabla \cdot u = \nabla \cdot b = 0, \\
(b \cdot \partial_t \nabla b, \partial_t u) + (b \cdot \partial_t \partial_t u, \partial_t b) = 0,
\]
\[
\nabla \times \partial_t u, \partial_t \omega = (\nabla \times \partial_t \omega, \partial_t u).
\]
Using Young inequality, \( l_1 \) is easily estimated as
\[
l_1 \leq \frac{2}{2} \| \nabla \times \partial_t u \|_{L^2}^2 + 2 \chi \| \nabla \omega \|_{L^2}^2. \tag{2.2}
\]
For the second term \( l_2 \), invoking (1.3) and Young inequality, it follows that
\[
l_2 = \| (\Lambda^{-2} \partial_t (u_j u_k), \Lambda^{-2} \partial_t u) \| \leq \| u \otimes u \|_{B^2_{2,2}} \| u \|_{H^8}^2 \leq C \left( \left\| \frac{u}{\| u \|_{B^0_{2,2}}} \right\| \| u \|_{H^8}^2 \right) \| u \|_{H^8}^2 \| u \|_{H^8}^2
\]
\[
\leq C \| u \|_{B^0_{2,2}} \| u \|_{H^8}^2 + \varepsilon \| u \|_{H^8}^4. \tag{2.3}
\]
Here and thereafter, \( \varepsilon > 0 \) is to be determined later. Utilizing (1.5) with exponents
\[
\left( \frac{2}{1 - \alpha}, \frac{2}{2 + \alpha} \right),
\]
the third term \( l_3 \) is dominated as
\[
l_3 \leq \| b \otimes b \|_{B^2_{2,2}} \| u \|_{H^8} \| | u \|_{H^8}^2 \leq C \left( \left\| \frac{b}{\| b \|_{B^0_{2,2}}} \right\| \| u \|_{H^8}^2 \right) \| u \|_{H^8}^2 \| b \|_{H^8}^2
\]
\[
\leq C \| b \|_{B^0_{2,2}} \| u \|_{H^8}^4 + \varepsilon \| u \|_{H^8}^2. \tag{2.4}
\]
For \( l_4 \), using Young inequality with exponents \( (2 / (1 - \alpha), 2 / (1 + \alpha)) \) and (1.5) with exponents as in (2.4), we have
\[
l_4 \leq 2 \| u \otimes b \|_{B^2_{2,2}} \| b \|_{H^8} \| u \|_{H^8} \| b \|_{H^8}^2 \leq C \left( \left\| \frac{u}{\| u \|_{B^0_{2,2}}} \right\| \| b \|_{B^0_{2,2}} \| u \|_{H^8} \right) \| b \|_{H^8}^2 \| b \|_{H^8}^2
\]
\[
= C \left( \left\| \frac{u}{\| u \|_{B^0_{2,2}}} \right\| \| u \|_{H^8}^2 \right) \| b \|_{H^8}^2 \| b \|_{H^8} + \left( \left\| \frac{b}{\| b \|_{B^0_{2,2}}} \right\| \| b \|_{H^8}^2 \right) \| b \|_{H^8} \| u \|_{H^8} \| b \|_{H^8} \| u \|_{H^8} \leq C \left( \left\| \frac{u}{\| u \|_{B^0_{2,2}}} \right\| \| b \|_{B^0_{2,2}} \| b \|_{H^8} \| u \|_{H^8} \right) \| b \|_{H^8}^2 + \varepsilon \| u \|_{H^8}^4. \tag{2.5}
\]
The last term \( l_5 \) is treated the same way as the third, leading to
\[
l_5 \leq C \| u \|_{B^0_{2,2}} \| \omega \|_{H^8}^2 + \varepsilon \| u \|_{H^8}^2. \tag{2.6}
\]
Gathering (2.2) and (2.3), Eqs. (2.5)–(2.7), and substituting into (2.1), taking \( \varepsilon > 0 \) sufficiently small, we have
\[
\frac{d}{dt} \| (\nabla u, \nabla \omega, \nabla b) \|_{L^2}^2 \leq \| (u, \omega, b) \|_{B^0_{2,2}}^2 \| (\nabla u, \nabla \omega, \nabla b) \|_{L^2}^2.
\]
Gronwall inequality then implies the fact
\[
u, \omega, b \in L^{\infty}(0, T; H^4),
\]
which ensures the continuation of strong solutions beyond \( t = T \). The proof is complete. \( \square \)

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